Round Quadratic Forms

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INTRODUCTION AND TERMINOLOGY

Round forms were introduced by Witt to simplify proofs in the algebraic theory of quadratic forms (cf. Lorenz [14]). The structure of round forms has been examined in [2, 4, 9, 10, 15, 19], and a characterization could be given for several classes of fields. Although we are far away from a general characterization theorem there are some general results on decomposing round forms (cf. [2, 15]). In this paper we shall characterize the round forms for a wider class of fields including the so-called linked fields and fields with \( u \)-invariant \( \leq 4 \), so answering questions of Marshall [15] and Gentile [8].

From the geometric point of view, round forms are also interesting objects of research. The behaviour of several metric collineation groups with respect to transitivity can be expressed equivalently in terms of the algebraic theory of quadratic forms using round and semiround forms (cf. [1]).

In Section 1, we shall mainly deal with the case of dimension \( 2l \), \( (2, l) = 1 \). In Section 2, round forms over linked fields are characterized, and in Section 3, the results of Section 2 are extended to fields with \( u \)-invariant \( \leq 4 \) and to fields whose Witt rings can be constructed using the theory of abstract Witt rings. Moreover, special round forms are considered, in particular odd multiples of Pfister forms.

We use the standard terminology as is found in [18]. The fields occurring in this paper are commutative and of characteristic \( \neq 2 \). \( K \) usually denotes a field, \( W(K) \) the corresponding Witt ring, and \( W_e(K) \) or \( W_t \), the torsion part of this ring. The ideal of all even dimensional forms in \( W(K) \) is denoted by \( I(K) \) or \( I \). For forms \( \phi \) and \( \psi \), \( \phi \equiv \psi \) resp. \( \phi = \psi \) will express

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isometry resp. equality in $W(K)$, and we write $\varphi_a$ for the anisotropic part of $\varphi$. The set of nonzero elements of $K$ represented by $\varphi$ is denoted by $D(\varphi)$, and we shall write $SD(\varphi)$ for the semiring generated by $D(\varphi)$. If $D(\varphi) = K^* := K \setminus \{0\}$, then $\varphi$ is said to be universal. If $\varphi$ represents 1, then the derived form $\varphi'$ is defined by $\varphi \equiv \langle 1 \rangle \perp \varphi'$. Following [2, 19], $\varphi$ is called round if $x \cdot \varphi \equiv \varphi$ holds for all $x \in D(\varphi)$; i.e., $D(\varphi) = G(\varphi)$, where $G(\varphi) := \{x \in K^* \mid x\varphi \equiv \varphi\}$. Accordingly, an isotropic form is round if its anisotropic part is 0 or round and universal. Note that if $\varphi$ is a form over $K$, then $G(\varphi) = G(\varphi \perp \langle 1, -1 \rangle)$. Therefore, we can consider $G(\varphi)$ even if $\varphi$ is determined only up to equality in the Witt ring.

The well-known invariants of forms are abbreviated as usual: $\dim \varphi$, $\det \varphi$, $s_\alpha(\varphi)$, and $c_\alpha(\varphi)$ denote the dimension, the determinant, the Hasse-invariant, and the Witt-invariant, respectively, with respect to the universal symbol $\alpha: K^* \times K^* \to \mathbb{P}/\mathbb{P}$ that is defined in [18] (according to a theorem of Merkurjev, we could also use the quaternion symbol that is universal, too). If $P$ is an ordering of $K$, $\text{sgn}_P(\varphi)$ will denote the $P$-signature of $\varphi$. The field $K$ is called linked if the classes of quaternion algebras over $K$ form a subgroup of the Brauer group $\text{Br}(K)$ of $K$ (for equivalent conditions for a field $K$ to be linked see [5-7], where there are also lists of examples).

The basic theorem of Marshall ([15], independently proved by Becker and Köpping [2]) says that any anisotropic round form $\varphi$ of dimension $2^l l$, $l$ odd, has a decomposition $\varphi = l \times \psi + \rho$, where $\psi$ is a $v$-fold Pfister form and $\rho \in W_i$. We shall call any such decomposition a Marshall-decomposition (or for short: M-decomposition).

1. General Results

For Pythagorean fields, Marshall's theorem is a characterization theorem since $W_i(K) = \{0\}$. Moreover, in [15] Marshall determined the structure of round forms in the cases $u \leq 2$ and $(u \leq 4 \wedge (K \text{ linked}))$. In the latter case, the situation $\dim \varphi = 2l$, $l$ odd, remained open. For this situation, we intend to give a general result.

**Theorem 1.1.** Let $\varphi$ be an anisotropic and round form of dimension $2l$, $(2, l) = 1$ and $d := \det \varphi$. Then, $l \times \langle 1, d \rangle + \rho$, where $\rho := \varphi - l \times \langle 1, d \rangle$ is an M-decomposition of $\varphi$, and

(i) $2\rho = 0$;

(ii) $\dim \rho_a \leq 2l$;

(iii) if $\varphi \notin W_i(K)$, $\dim \rho_a < 2l$, and $l > 1$, then $K^* \neq D(\varphi) = D(\langle 1, d \rangle) = SD(\langle 1, d \rangle) \subset G(\rho)$;

(iv) if $\varphi \in W_i(K)$ and $\dim \rho_a < 2l$, then $l = 1$, $\rho = 0$, and $\varphi \equiv \langle 1, d \rangle$.‌
By [15, 1.1], we have $D(\rho) \subset D(\langle l, d \rangle)$. Since $d \in D(\varphi)(D(\varphi) \leq K^*)$, this yields for all orderings $P$ of $K$: $d \in P \iff \text{sgn}_P \varphi = 2l$. Therefore, the above decomposition is an $M$-decomposition.

(i) $2 \rho = 2 \times (\varphi - l \times \langle 1, d \rangle) = \varphi + \varphi - 2l \times \langle 1, d \rangle = \varphi + d \varphi - 2l \times \langle 1, d \rangle = 2 \times \langle 1, d \rangle - 2l \times \langle 1, d \rangle \otimes \varphi - 2l \times \langle 1, d \rangle = 2l \times \langle 1, d \rangle - 2l \times \langle 1, d \rangle = 0$, since $D(\varphi) \subset D(\langle 1, d \rangle)$ and $\langle 1, d \rangle$ is round.

(ii) This is clear from $D(\rho) \subset D(\langle 1, d \rangle)$, since $\rho = \varphi - l \times \langle 1, d \rangle$.

(iii) From $\varphi \notin W_1(K)$ it follows that $l \times \langle 1, d \rangle$ is anisotropic. Assume $\dim \rho_a = m < 2l$, $\rho_a = \langle a_1, \ldots, a_m \rangle$. Since $\dim(l \times \langle 1, d \rangle + \rho) = 2l$, we may assume $a_1, \ldots, a_{m/2} \in -SD(\langle 1, d \rangle)$, and, moreover, that $\langle a_{m/2} + 1, \ldots, a_m \rangle$ is a subform of $\varphi$. From $D(\varphi) \subset D(\langle 1, d \rangle)$ we obtain $a_{m/2} + 1, \ldots, a_m \in D(\langle 1, d \rangle)$. By (i), $\rho \cong -\rho$, so, for dimensional reasons, $\langle 1, d \rangle$ must be a subform of $\varphi$; i.e., $\varphi \cong \langle 1, d \rangle \perp \gamma$. Hence $D(\gamma) < D(\langle 1, d \rangle) = D(\varphi) \wedge G(\gamma) = D(\langle 1, d \rangle)$, and therefore, $D(\gamma) = D(\langle 1, d \rangle)$. This gives $D(2 \times \langle 1, d \rangle) \subset D(\langle 1, d \rangle)$; i.e., $D(\langle 1, d \rangle) = D(\varphi)$ is a semiring and $D(\varphi) \in G(\rho)$.

(iv) Assume first $l \times \langle 1, d \rangle$ is anisotropic. In case of $l > 1$, proceeding as in (iii) yields that $\langle 1, d \rangle$ is a subform of $\varphi$ and $D(\varphi) = D(\langle 1, d \rangle) = SD(\langle 1, d \rangle)$. Since $SD(\langle 1, d \rangle) = K$, $\varphi$ then is isotropic, a contradiction. This gives $l = 1$.

Now, we assume that $l \times \langle 1, d \rangle$ is isotropic, hence $l > 1$. Then, $l \times \langle 1, d \rangle = r \times \langle 1, d \rangle$, where $r < l$ or $l \times \langle 1, d \rangle = -s \times \langle 1, d \rangle$, where $s < l$. As in (iii) we obtain that $\rho$ has a subform $\langle a_1, \ldots, a_{m/2} \rangle$, where $a_1, \ldots, a_{m/2} \in D(\langle 1, d \rangle)$. Since $\rho \cong -\rho$ this gives $\dim(l \times \langle 1, d \rangle + \rho)_a < 2l$ in any case, a contradiction.

As a converse of Theorem 1.1, we show:

**Theorem 1.2.** Suppose $\varphi = l \times \langle 1, d \rangle + \rho$, $\dim \rho_a = 2l$, $(l, 2) = 1$. If $D(\langle 1, d \rangle)$ is a semiring with $D(\langle 1, d \rangle) \subset G(\rho)$, $\rho \in W_1(K)$, and $m := \dim \rho_a \leq 2l$, then $\varphi_a$ is round.

**Proof.** Let $\rho = \langle a_1, \ldots, a_m \rangle$. For dimensional reasons we may assume $a_1, \ldots, a_{m/2} \in -D(\langle 1, d \rangle)$. From $\rho \in W_1(K)$ we conclude $a_{m/2 + 1}, \ldots, a_m \in SD(\langle 1, d \rangle) = D(\langle 1, d \rangle)$ since $D(\langle 1, d \rangle)$ is a preordering, i.e., $D(\langle 1, d \rangle) = \bigcap_{P \in P} P$, and since for each ordering $P$ with $P \supset D(\langle 1, d \rangle)$ we obtain $a_{m/2 + 1}, \ldots, a_m \in P$ from $\text{sgn}_P \rho = 0$. This shows $D(\varphi) \subset D(\langle 1, d \rangle)$. Since $G(\varphi) \supset D(\langle 1, d \rangle)$, we are done.

**Remarks 1.3.** (1) We cannot eliminate the prerequisite $\dim \rho_a < 2l$ in Theorem 1.1(iv). This follows from a recent result by Merkurjev who constructed fields with $u$-invariant 6 where there are anisotropic forms of the type $\langle a, b, ab, -c, -d, -cd \rangle = (\langle a, b \rangle - \langle c, d \rangle)_a$ (see [13]). If $K$ is such a field and $K$ is not formally real, then forms of this type are round.
since 2-fold Pfister forms are universal. It is immediate from Springer's theorem [18, 6.2.6] that they remain round over \( K((t)) \) but they are no longer universal. So, not only do we have anisotropic round torsion forms with \( l > 1 \) but they need not even be universal. It is still not clear whether we need the condition \( \dim \rho_a < 2l \) in Theorem 1.1(iii).

(2) In the proof of Theorem 1.1 we essentially need \( D(\varphi) \subset D(\langle 1, d \rangle) \). Since in the case \( v \geq 2 \) this condition is not true in general, the proof provides no idea to handle the general case.

Finally, we want to prove a simple assertion which will turn out to be very useful in Section 2:

**Proposition 1.4.** Let \( \varphi = (l \times \psi + \rho)_a \) be an anisotropic round form of dimension \( 2^v l \), where \( \psi \) is a \( v \)-fold Pfister form (possibly in \( W \)). If \( \dim \rho_a < \dim \varphi \), then \( D(\psi) \subset D(\varphi) \).

Proof. For \( x \in D(\psi) \) we have \( \dim(\varphi - x\varphi)_a \leq 2 \dim \rho_a < 2 \dim \varphi \). Since \( D(\varphi) \leq K^* \), this means \( x \in D(\varphi) \). 

2. ROUND FORMS OVER LINKED FIELDS

In this section, \( K \) always denotes a linked field. An \( m \)-fold Pfister form \( \varphi \) and an \( n \)-fold Pfister form \( \psi \), where \( m, n \geq 1 \wedge m \leq n \), are said to be linked (\( \varphi \sim^l \psi \)) if there exist an \((m - 1)\)-fold Pfister form \( \sigma \) and Pfister forms \( \tau_1, \tau_2 \) such that \( \varphi \cong \sigma \otimes \tau_1 \wedge \psi \cong \sigma \otimes \tau_2 \) (for this general definition cf. Shapiro/Wadsworth [19]). The properties of forms over linked fields which are needed in the sequel can be found in [5, 7].

Note that by [7, 3.j], for \( a, b, c, d \in K^* \) there exists \( r \in K^* \) such that \( \langle a, b, c, d \rangle \cong \langle 1, 1, 1, r \rangle \). This means in particular that any semiring containing \( K^{*n} \) is generated by \( K^{*2} \) and one element. Hence, for linked fields Marshall's decomposition is immediate from this property.

In the sequel of this section, the dimension of the round form under consideration will be \( 2^v l \), where \( (2, l) = 1 \) and \( v \geq 1 \).

**Proposition 2.1.** Suppose \( \varphi \) is an anisotropic round form, and \( \varphi = l \times \psi + \rho \) is an M-decomposition. Then, \( \dim \rho_a \leq \dim \varphi \).

Proof. By the main theorem of Elman/Lam [5, 3.4], it suffices to consider the case \( l \leq 3 \wedge v \leq 2 \).

(i) Assume first \( v = 1 \). (Theorem 1.1 does not work since \( \det \varphi = \det \psi \) does not necessarily hold; the M-decomposition under consideration can be arbitrary.) Let \( d := \det \varphi \) and \( \langle 1, c \rangle = \psi \). In case of \( l = 1 \) we have \( \rho = \langle 1, d \rangle - \langle c, 1 \rangle \), hence \( \dim \rho_a \leq 2 \). In case of \( l = 3 \) we obtain
\(\varphi \cong \langle 1, d, 1, a, b, ab \rangle\) since the 5-dimensional form \(\varphi'\) represents its determinant \(d\) (cf. [7, 2(iii)]), and, consequently, \(\varphi \cong \langle 1, d \rangle \perp \gamma\), where \(\det \gamma = 1\) and \(1 \in D(\gamma)\) since \(D(\gamma) \subset D(\varphi) \subset D(\langle 1, d \rangle)\). If \(\varphi = 3 \times \langle 1, c \rangle + \rho\), then \(\rho = \varphi - 3 \times \langle 1, c \rangle\), and by linkage we may assume \(\langle 1, 1 \rangle \otimes \langle 1, c \rangle = \langle 1, a, f, af \rangle\). This gives \(\rho = \langle d, b, ab, -f, -af, -c \rangle\), hence \(\dim \rho_a \leq 6\).

(ii) If \(v = 2\) and \(\varphi = \psi + \rho\), then we have \(\rho = \varphi - \psi\), and hence \(\rho = \langle 1, -\det \varphi \rangle + \langle 1, a, b, ab \rangle - \langle 1, a, c, ac \rangle\), where \(\varphi - \langle \det \varphi, a, b, ab \rangle\) and \(\psi = \langle a, c \rangle\) (linkage!), and, moreover, \(\det \varphi \in D(\langle 1, 1, \rangle)\), by [15, 1.1]. Thus, \(\rho = \langle 1, -\det \varphi \rangle + \langle b, ab, -c, -ac \rangle\). Again by [15, 1.1], we obtain \(b \in D(\langle 1, -\det \varphi \rangle) = -D(\langle 1, -\det \varphi \rangle)\). This gives the lemma.

**Proposition 2.2.** Let \(\gamma, \tau\) be different 3-fold Pfister forms and \(\tau \in W_1\). Then, \(\varphi = (\gamma + \tau)_a\) is a 3-fold Pfister form and \(D(\varphi) = D(\gamma)\).

**Proof.** Since \(\gamma\) and \(\tau\) are linked, we obtain \(\gamma + \tau = \gamma - \tau = ax\), where \(a\) is a 3-fold Pfister form and \(a \in D(\gamma)\) (note that \(\tau\) is universal, by [5, 2.8]). Now, \(a \in D(\gamma) = G(\gamma) = G(\gamma + \tau) = G(ax)\) which gives the claim.

**Lemma 2.3.** Let \(\varphi\) be an anisotropic round form, where \(v = 1\) and \(l > 1\), \(d := \det \varphi\). Then, \(\varphi\) equals \(l \times \langle 1, d \rangle + \langle a, b \rangle + \langle e, f, g \rangle\), where \(D(\varphi) - D(\langle 1, d \rangle) = SD(\varphi), \langle a, b \rangle, \langle e, f, g \rangle \in W_1(K), D(\varphi) \subset D(\langle a, b \rangle)\), and \(2\langle a, b \rangle = 0\).

**Proof.** Since \(\rho := \varphi - l \times \langle 1, d \rangle \in W_1 \cap I^2\) (cf. [18, 2.12.10]), [7, 3(1)] yields \(\varphi = l \times \langle 1, d \rangle + \langle a, b \rangle + \langle e, f, g \rangle\), where \(\langle a, b \rangle, \langle e, f, g \rangle \in W_1(K)\) and, by [5, 2.7], \(\dim(\langle a, b \rangle + \langle e, f, g \rangle) \leq \{0, 4, 8\}\). By Proposition 2.1, \(\dim \rho_a < 2l\), and from Theorem 1.1(iii), (iv) we obtain \(D(\varphi) = SD(\varphi) = D(\langle 1, d \rangle)\). Since \(2\langle e, f, g \rangle = 0\) and \(2p = 0\) (by Theorem 1.1(i)) it follows \(2\langle a, b \rangle = 0\). Since \(\langle e, f, g \rangle\) is a round and universal form, we conclude \(D(\varphi) \subset G(\langle a, b \rangle) = D(\langle a, b \rangle)\).

**Remark 2.4.** In Lemma 2.3 we could have obtained the semiring property from the fact that every 5-dimensional form represents its determinant [7, 2(iii)] without applying Theorem 1.1: With an easy induction it follows from this fact that any form of dimension \(n \in 1 + 4N\) represents its determinant. Given \(\varphi\) as in Lemma 2.3, we have \(\dim \varphi' = 2l - 1\), where \(2l \in 2 + 4N\), hence \(2l - 1 \in 1 + 4N\). Thus, \(\varphi \cong \langle 1, d \rangle \perp \gamma\), where \(D(\gamma) \subset D(\varphi) \subset D(\langle 1, d \rangle) \subset D(\gamma) \subset D(\langle 1, d \rangle)\), and hence \(D(\gamma) = D(\langle 1, d \rangle) \cap D(\varphi) = SD(\varphi)\).

**Lemma 2.5.** Let \(\varphi\) be an anisotropic round form and \(d := \det \varphi\).
(i) If \( v = 2 \) and \( l > 1 \), then \( \varphi \) equals \( l \times \langle a, b \rangle + \langle t, w \rangle \), where \( D(\varphi) = SD(\varphi) = D(\langle a, b \rangle) \), \( \langle x, y, z \rangle \in W_i \), and \( 2 \times \langle t, w \rangle = 0 \).

(ii) If \( v = 2 \) and \( l = 1 \), then \( \varphi = \langle a, b \rangle + \langle 1, -d \rangle \), where \( D(\varphi) = D(\langle a, b \rangle) \).

Proof. (i) We have \( \langle d \rangle + \varphi' = l \times \langle a, b \rangle \pmod{l} \) (cf. [7, 2.2]). This gives \( \langle d \rangle + \varphi' = l \times \langle a, b \rangle + \rho \), where \( \rho \in I^2 \). Since for all orderings \( P \) we have \( \text{sgn}_P \rho \in \mathbb{Z} \), a comparison of signatures yields \( \rho \in W_1 \cap I^3 \); i.e., \( \rho \equiv \langle x, y, z \rangle \), by [5, 2.8]. By Proposition 2.2, \( \langle a, b, 1 \rangle - \langle x, y, z \rangle = \langle e, f, g \rangle \) and \( D(\langle e, f, g \rangle) = D(\langle a, b, 1 \rangle) \) for some \( e, f, g \in K^* \). In all, we have \( \varphi = l \times \langle a, b \rangle + \langle 1, -d \rangle + \langle x, y, z \rangle = (l - 2) \times \langle a, b \rangle + \langle 1, -d \rangle + \langle e, f, g \rangle \). Thus, \( \langle a, b \rangle \) is a subform of \( \varphi \). By linkage, we may assume \( e \in D(\langle a, b \rangle) \). As in the proof of Theorem 1.1(iii) we obtain \( D(\varphi) \supseteq SD(\langle a, b \rangle) \) and hence \( D(\varphi) = SD(\langle a, b \rangle) \). Now, we have \( \varphi = \langle a, b \rangle + (l - 1)/2 \times \langle a, b, 1 \rangle + \langle 1, -d \rangle + \langle x, y, z \rangle \), and from \( D(\varphi) \subseteq D(\langle 1, -d \rangle) \) we conclude \( D(\langle a, b, 1 \rangle) \subseteq G(\langle a, b \rangle) = D(\langle a, b \rangle) \), hence \( D(\langle a, b \rangle) = SD(\langle a, b \rangle) \). Clearly, \( 2 \times \langle 1, -d \rangle = 0 \).

(ii) We have \( \varphi \equiv \langle d, a, b, ab \rangle \), where \( d \in D(\langle 1, 1 \rangle) \). Thus, \( \varphi = \langle a, b \rangle + \langle 1, -d \rangle \). From \( D(\varphi) \subseteq D(\langle 1, -d \rangle) \) we obtain \( D(\varphi) \subseteq D(\langle a, b \rangle) \) since \( \varphi \) is round. Now, Proposition 1.4 gives the claim.

Lemma 2.6. Let \( \varphi \in W_i(K) \) be a round and anisotropic form, and \( v = 3 \), \( d := \det \varphi \). Then, \( \varphi \) equals \( l \times \langle a, b, c \rangle + \langle 1, -d \rangle + \langle t, w \rangle \), where \( D(\varphi) = SD(\varphi) = D(\langle a, b, c \rangle) \) and \( 2 \times \langle t, w \rangle = 0 \).

Proof. According to Marshall's theorem, [5, 2.8, 7, 3(1)], and Proposition 2.2, we may assume that \( \varphi \) has a decomposition \( \varphi = l \times \langle a, b, c \rangle + \langle t, w \rangle + \langle 1, -d \rangle \), where \( SD(\varphi) = SD(\langle a, b, c \rangle) = D(\langle a, b, c \rangle) \) (by [5, 2.4]) and \( \langle t, w \rangle \in W_i(K) \). By linkage, we may also assume that \( t = a \) and hence \( -w \in SD(\langle 1, a \rangle) \subseteq D(\langle a, b, c \rangle) \). From Proposition 1.4 it follows that \( D(\varphi) = D(\langle a, b, c \rangle) \). The second claim, to wit: \( 2 \times \langle t, w \rangle = 0 \), follows from \( D(\langle a, b, c \rangle) \subseteq G(\langle t, w \rangle) = D(\langle t, w \rangle) \) and \( -w \in D(\langle a, b, c \rangle) \).

Lemma 2.7. Let \( \varphi \) be an anisotropic and round form, \( v \geq 4 \) and \( d := \det \varphi \). Then, \( \varphi \) equals \( l \times \langle 1, \ldots, 1, r \rangle + \langle 1, -d \rangle + \langle t, w \rangle + \langle x, y, z \rangle \), where \( D(\varphi) = SD(\varphi) = D(\langle 1, \ldots, 1, r \rangle) \) and \( 2 \times \langle t, w \rangle = 0 \).

Proof. From Marshall's theorem and [7, 3(1)] we obtain \( \varphi = l \times \langle a_1, \ldots, a_v \rangle + \langle 1, -d \rangle + \langle t, w \rangle + \langle x, y, z \rangle \), where \( \langle t, w \rangle \),
\[ \langle x, y, z \rangle \in W_r(K) \text{ and } SD(\varphi) = D(\langle a_1, \ldots, a_m \rangle). \] By [7, 3.j], there is an \( r \in K^* \) such that \( \langle a_1, \ldots, a_m \rangle \cong \langle 1, \ldots, 1, r \rangle \). From Proposition 1.4 we obtain \( D(\varphi) = D(\langle 1, \ldots, 1, r \rangle) \) and hence \( D(\varphi) = SD(\varphi) \). The remaining claim follows as in the previous lemma.

### 2.8. Characterization of Round Torsion Forms.

Let \( \varphi \in W_r(K) \) be anisotropic and round, \( d := \det \varphi \). By Lemma 2.3, \( \dim \varphi \neq 6 \), and hence, by [5, 3.4], \( \dim \varphi \in \{2, 4, 8\} \).

(i) \( \dim \varphi = 2 \): Clearly, \( \varphi \cong \langle 1, d \rangle \).

(ii) \( \dim \varphi = 4 \): \( \varphi \cong \langle d \rangle \perp \langle a, b, ab \rangle = \langle 1, -d \rangle + \langle a, b \rangle \), where \( D(\varphi) = D(\langle a, b \rangle) \) (by Lemma 2.5(ii)) and \( \langle a, b \rangle \in W_r \).

(iii) \( \dim \varphi = 8 \): \( \varphi = \langle 1, -d \rangle + \langle t, w \rangle + \langle x, y, z \rangle \). By Proposition 1.4 (with \( \psi = \langle x, y, z \rangle \)), we have \( D(\varphi) = K^* \). From \( D(\varphi) \subseteq D(\langle 1, -d \rangle) \) and \( D(\langle x, y, z \rangle) = K^* \) we obtain \( K^* = D(\varphi) \subseteq D(\langle t, w \rangle) \), and hence \( 2 \langle t, w \rangle = 0 \).

**COROLLARY 2.9.** Let \( \varphi \) be a round and anisotropic form and \( \dim \varphi \geq 8 \). Then, \( D(\varphi) = SD(\varphi) \setminus \{0\} \).

Summarizing the above lemmata, we obtain the desired theorem. Furthermore, we provide a decomposition into simultaneously linked Pfister forms. If \( \psi \) and \( \varphi \) are Pfister forms we write \( \psi \mid \varphi \) in case of \( \varphi = \psi \otimes \tau \) for a suitable Pfister form \( \tau \).

**THEOREM 2.10.** Let \( \varphi \) be an anisotropic round form of dimension \( 2^v l > 1 \).

(i) There exist a \( v \)-fold Pfister form \( \psi \notin W_r(K) \setminus \{0\} \) and \( i \)-fold Pfister forms \( \rho_i \in W_i(K) \) \( (i = 1, \ldots, m; 2^m \leq \dim \varphi) \) such that \( \varphi = l \times \psi + \sum \rho_i \), where \( D(\varphi) \subseteq D(\psi) \cap D(\rho_i) \) and \( (l > 1 \Rightarrow D(\psi) = D(\varphi) = D(l \times \psi) \wedge 2 \rho_i = 0) \).

(ii) Given a decomposition according to (i), and \( r \in \mathbb{N} \) such that \( l \times 2^u \geq 2^v 2^u \geq 2^m \), w.l.o.g. we may assume that \( 2^r \times \psi \cong \langle a_1, \ldots, a_{r+v} \rangle \) and \( \langle a_1, \ldots, a_{r-1} \rangle \mid \rho_i \).

**Proof:** (i) follows from Lemmas 2.3, 2.5–2.7, and characterization 2.8.

(ii) \( v = 1 \). We may assume \( l \geq 3 \). In case of \( l = 3 \) we have \( \varphi = \langle 1, d \rangle + \langle 1, 1, d \rangle + \langle t, w \rangle \), and since \( \langle 1, 1, d \rangle \) and \( \langle t, w \rangle \) are linked there is nothing to prove. In case of \( l > 3 \) we have \( \varphi = (l-4) \times \langle 1, d \rangle + \langle 1, 1, d \rangle + \langle t, w \rangle + \langle x, y, z \rangle \). We may assume \( x, y \in D(\langle 1, 1, d \rangle) = D(\langle 1, d \rangle) \). Furthermore, since \( \langle t, w \rangle \sim \langle x, y \rangle \) we may also assume \( x = t \in D(\langle 1, d \rangle) \). From \( D(\langle 1, d \rangle) \subseteq D(\langle 1, 1, d \rangle) \) we conclude, using [18, 4.1.7], that there exist \( r, s \in K^* \) such that \( \langle 1, 1, d \rangle + \langle t, w \rangle + \langle x, y, z \rangle \cong \langle t, r, s \rangle \cong \langle t, w \rangle + \langle t, y, z \rangle \). Now, \( \langle r, s \rangle \sim \langle y, z \rangle \).
— $v = 2$. We may assume $l \geq 3$, and hence $\varphi = l \times \langle a, b \rangle + \langle 1, -d \rangle + \langle x, y, z \rangle$. Furthermore, $\langle a, b, 1 \rangle \sim \langle t, w \rangle + \langle 1, -d \rangle$ and $\langle a, b \rangle \sim \langle t, w \rangle$.

— $v = 3$. By Lemma 2.6, we have $\varphi = l \times \langle a, b, c \rangle + \langle t, w \rangle + \langle 1, -d \rangle$ and $\langle a, b, 1 \rangle \sim \langle x, y, z \rangle$, as desired.

— $v \geq 4$. We have $\varphi = l \times \langle a_1, ..., a_v \rangle + \langle 1, -d \rangle + \langle t, w \rangle + \langle x, y, z \rangle$. From [7, 3.5] we obtain $\langle a_1, ..., a_v \rangle \cong \langle 1, ..., 1, r \rangle$. Since $\langle 1, 1, r \rangle \sim \langle x, y, z \rangle$, as in the case $v = 1$ we may assume $x, y \in D(\langle 1, 1, r \rangle)$ and $t = x$. As $t \in D(\psi') = D(\langle 1, ..., 1, r \rangle')$ there exist $b_2, ..., b_v$ such that $\langle 1, ..., 1, r \rangle \cong \langle t, b_2, ..., b_v \rangle$. Since $\langle b_2, b_3 \rangle \sim \langle y, z \rangle$ we are done. 1

The property of simultaneous linkage can be used to construct round forms as the following theorem shows.

**Theorem 2.11.** Let $\psi$ be a $v$-fold Pfister form with $D(\psi) = SD(\psi)$, $l \in 2\mathbb{N} + 1$. Furthermore, let $\rho_i$ be $i$-fold Pfister forms $\in W_t(K)$ $(i = 1, ..., m)$, where $2^m < 2^v$, $D(\psi) \subset D(\rho_i)$, and $\rho_m \neq 0$. If there exists $r \in \mathbb{N}$ such that $2^v \geq 2^r \geq 2^m$, $2^r \cdot \psi \cong \langle a_1, ..., a_{r+e} \rangle$, and $\langle a_1, ..., a_{r+e} \rangle | \rho_i$, then $\varphi := (l \cdot \psi + \sum \rho_i)$ is a round form of dimension $2^v$, and $D(\varphi) = D(\psi)$. Moreover, $\dim(\varphi) = 2^v$, and, in case of $v \geq m$, $\dim(\psi + \sum \rho_i) = 2^v$.

**Proof.** We have $G(\varphi) \subset D(\psi)$. Since $SD(\psi) \subset D(\rho_i) \subset W_t \cap 2^r \times \psi \sim \rho_i$, it follows that $2\rho_i = 0$ $(i = 1, ..., m)$. We show $D(\varphi) \subset SD(\psi) = D(\psi) \cap \dim \varphi = 2^v$. There exist $r \in \mathbb{N}$ and $a_m \in K^*$ such that $\langle a_1, ..., a_m \rangle | 2^r \cdot \psi$ and $\rho_m \cong \langle a_1, ..., a_{m-1}, a'_m \rangle$. Now, we have $\langle a_1, ..., a_m \rangle + \rho_m = \langle a_1, ..., a_m \rangle - \langle a_1, ..., a_{m-1}, a'_m \rangle = \langle a_m, -a'_m \rangle \otimes \langle a_1, ..., a_{m-1} \rangle = \langle a_1, ..., a_{m-1}, -a_m a'_m \rangle$, since $a_m \in G(\langle a_1, ..., a_m \rangle + \rho_m)$. From $\rho_m \in W_t$ we conclude $a'_m \in -SD(\langle a_1, ..., a_{m-1} \rangle) \subset -D(\psi)$; i.e., $-a_m a'_m \in D(\psi)$. An easy induction yields $D((2^r \cdot \psi + \sum \rho_i) \subset D(\psi)$ and $\dim((2^r \cdot \psi + \sum \rho_i) = 2^v$. This establishes the first and the second claim. The other claims can be proved analogously.

**Remarks 2.12.** (1) It is possible to obtain more information on the sets $D(\rho_i)$ in Theorem 2.10 concerning universality. This will be provided in a forthcoming paper on round forms under algebraic extensions where it is needed.

(2) In the proof of Theorem 2.11, the linkage property has not been used. Therefore, the theorem holds for arbitrary fields. According to Theorem 2.10, every anisotropic round form $\varphi$ over a linked field with $\dim \varphi = 2^v$ and $l > 1$ can be constructed in this way.

(3) Implicitly, Theorem 2.11 contains the prerequisite that $K$ is formally real since we require $D(\psi) = SD(\psi)$. If $D(\psi) \cup \{0\} = SD(\psi) = K$ is required instead, $\varphi$ is $0$ or anisotropic, universal and round.
THEOREM 2.13. An anisotropic round form $\varphi$ of dimension $l2^v > 1$ has an (isometric!) decomposition $\varphi \cong l \times \gamma$ iff there is a decomposition according to Theorem 2.10(i), (ii) with $m \leq v$.

Proof. (i) Suppose there exists such a decomposition (according to Theorem 2.10(i)(ii)) with $m \leq v$. We may assume $l > 1$. By Theorem 2.10, we have $D(\varphi) = D(\psi) = SD(\psi)$. From Theorem 2.11 we obtain $\dim(\psi + \sum \rho_i) = 2^v$, and from $2 \sum \rho_i = 0$ it follows that $\varphi = l \times (\psi + \sum \rho_i)$, hence $\varphi \cong l \times (\psi + \sum \rho_i)$.

(ii) Now suppose there is such a decomposition with $m > v$. By Proposition 2.1, we have $l > 1$ and $l2^v > 2^m$. Since $\varphi = l \times (\psi + \sum \rho_i)$, it suffices to verify $\dim(\psi + \sum \rho_i) > 2^v$. From Theorem 2.11 we obtain $\dim(\psi + \sum \rho_i) = 2^m$, and so we can restrict ourselves to the case $v = m - 1$. Suppose $\dim(\psi + \sum \rho_i) = 2^v$. Then, for dimensional reasons, $-\psi$ is a subform of $(\sum \rho_i)$; i.e., $(\sum \rho_i) \cong -\psi \perp \beta$ with a suitable form $\beta$. Then, however, $\beta$ would be a subform of $\varphi$; i.e., $D(\beta) \subset D(\varphi) = D(\psi)$, a contradiction to the fact that $\psi \perp \beta$ is anisotropic. 

3. SPECIAL SITUATIONS AND GENERALIZATIONS

In [15], Marshall characterized the situation when a round form is an odd multiple of a Pfister form, where the field $K$ is linked and $u(K) < 4$. In order to generalize his assertions we shall only require that the field under consideration has the property that $l^3(K)$ is torsion free. If $u(K) \leq 4$, then $l^3(K)$ is torsion free as follows from the well-known theorem of Arason–Pfister.

A simple computation yields:

PROPOSITION 3.1. Let $l \in 2N + 1$, $a \in K^*$. Then,

$$s(l \times \langle 1, a \rangle) = \begin{cases} \langle 1, a \rangle ^3 = \sigma(a, a), & \text{if } l \in 3 + 4N; \\ 0, & \text{if } l \in 1 + 4N. \end{cases}$$

THEOREM 3.2. Let $l^3$ be torsion free, and let $\varphi$ be an anisotropic round form of dimension $2^vl$. Then, $\varphi$ is of the type $l \times \psi$, $\psi$ Pfister form, iff

(i) in case of $v = 1$: $(l \in 1 + 4N \land c(\varphi) = 0)$ or $(l \in 3 + 4N \land c(\varphi) = \sigma(-1, -\det \varphi));$

(ii) in case of $v = 2$: $\det \varphi = 1$ and $c(\varphi) \in \sigma(K^* \times K^*)$ (i.e., $\varphi$ is 2-fold Pfister form modulo $l^3$);

(iii) in case of $v \geq 3$: $\det \varphi = 1$ and $c(\varphi) = 0$.

Proof. $\Rightarrow$. In case of $v = 1$, the assertion follows from Proposition 3.1.
The cases \( u = 2 \) and \( u \geq 3 \) are trivial since \( c|_\beta \) is a homomorphism with kernel \( I^3 \) (cf. [18, pp. 83, 84]).

"\( = \)". We have to construct multiples of Pfister forms which have the same invariants as \( \varphi \) (the claim then follows from the well-known classification theorem of Elman/Lam). In case of \( u = 1 \) the form \( l \times \langle 1, \det \varphi \rangle \) has the same invariants as \( \varphi \) (use Proposition 3.1). Now, let \( v = 2 \) and \( \det \varphi = 1 \), and let \( c(\varphi) = \psi + I^3 \), where \( \psi \) is a 2-fold Pfister form. Then, \( \varphi \) and \( l \times \psi \) have the same invariants (cf. [18, 2.12.13]). Finally, let \( u \geq 3 \). According to Marshall's theorem there exists a \( v \)-fold Pfister form \( \psi \) such that \( \text{sgn}_P \varphi = \text{sgn}_P (l \times \psi) \) for any ordering \( P \). Since \( c|_\beta = 0 \) we are done.

Next, we shall examine the structure of round forms for fields with \( u(K) \leq 4 \). In [15], this case is considered on the additional condition that \( K \) is linked. These conditions are independent from each other: There are linked fields \( K \) with \( u(K) > 4 \) (e.g., \( C((t_1))((t_2))((t_3)) \)), and, on the other hand, there exist fields with \( u \)-invariant \( \leq 4 \) that are not linked (e.g., \( \mathbb{R}((t_1))((t_2)) \)). Again, at first we shall deal with the more general supposition "\( I^3 \) torsion free."

**Proposition 3.3.** Let \( I^3(K) \) be torsion free. Then, \( D(\langle a_1, ..., a_n \rangle) = SD(\langle a_1, ..., a_n \rangle) \setminus \{0\} \) if \( n \geq 2 \). If \( \varphi = l \times \psi + \rho \) is an \( M \)-decomposition of an anisotropic round form, then \( 2\rho = 0 \) in case of \( v \geq 2 \) (in case of \( v = 1 \) there is, by Theorem 1.1, a special \( M \)-decomposition with \( 2\rho = 0 \)).

**Proof.** The first assertion is trivial. Now, let \( d := \det \varphi \). Then, we have \( \langle d \rangle + \varphi' = l \times \psi + \rho + \langle 1, -d \rangle \). From \( \langle d \rangle + \varphi' \), \( l \times \psi \in I^2 \), it follows that \( 2 \times (\rho - \langle 1, -d \rangle) \in I^3 \cap W_1 = \{0\} \). Since \( 2 \times \langle 1, -d \rangle = 0 \) the proof is complete.

Now, we want to extend the results of Theorem 2.10(i) to fields with \( u \)-invariant \( \leq 4 \).

**Theorem 3.4.** If \( u(K) \leq 4 \), then the assertions of Theorem 2.10(i) are true.

**Proof.** Let \( \varphi \) be a round and anisotropic form of dimension \( 2^u l \). For \( v = 1 \) the assertions follow from Theorem 1.1 since \( \varphi = l \times \langle 1, \det \varphi \rangle + \rho \) implies \( \dim \rho < 2l \). Thus, assume \( v \geq 2 \), and let \( \varphi = l \times \psi + \rho \) be an \( M \)-decomposition of \( \varphi \). Then, \( \rho + \langle 1, -\det \varphi \rangle \in W_1 \cap I^2 \); i.e., \( \rho + \langle 1, -\det \varphi \rangle = \langle a, b \rangle \), so \( \rho = \langle a, b \rangle + \langle 1, -\det \varphi \rangle \). By Proposition 3.3, we have \( D(\psi) = SD(\psi) \setminus \{0\} \), and hence \( D(\varphi) \subset D(\psi) \subset D(\langle a, b \rangle) = K^* \). Now, Proposition 1.4 gives the claim.

Using Theorems 2.10(i) and 3.4 we can extend the results to a con-
siderably larger class of fields. In [16] the notion of an abstract Witt ring was introduced. In the category of abstract Witt rings we have group extensions and products to construct new rings from old ones (cf. [16], for a similar approach using so-called quadratic form schemes see [3, 11, 12]). According to a theorem of Kula [11, 12], a Witt ring constructed in this way is isomorphic to the Witt ring of a field if the original one(s) is (are). Now, let \( \mathcal{L} \) be the class of abstract Witt rings which can be constructed in finitely many steps using group extensions and products, starting with Witt rings of linked fields and of fields with \( u \)-invariant \( \leq 4 \). Then, by performing the usual induction proofs (for details see [1]), we obtain:

**Theorem 3.5.** Let \( K \) be a field such that \( W(K) \) is in the class \( \mathcal{L} \). Then, the assertions of Theorem 2.10(i) hold for \( K \).

**Remarks 3.6.** (1) According to a result of Carson and Marshall (cf. [17]), for any field \( K \) with \( \leq 32 \) square classes \( W(K) \) lies in \( \mathcal{L} \).

(2) As follows from Merkurjev's theorem on fields with \( u \)-invariant 6 (cf. Remark 1.3(1)), not every anisotropic round form has a decomposition like that given in Theorem 2.10(i).

**Note.** After the completion of this paper the author was sent a note on "Round Quadratic Forms" by E. Gentile. There, round quadratic forms over linked fields with \( u \)-invariant \( \leq 4 \) are determined. The characterization is different from that given in the above paper.

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**References**