Available online at www.sciencedirect.com

ScienceDirect

Topology and its Applications 155 (2008) 2084–2089

Topology and its Applications

www.elsevier.com/locate/topol

A note on the pure Morse complex of a graph

R. Ayala, L.M. Fernández, A. Quintero, J.A. Vilches *

Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad de Sevilla, Apartado de Correos 1160, 41080 Sevilla, Spain

Received 21 September 2006; accepted 3 April 2007

Abstract

The goal of this work is to study the structure of the pure Morse complex of a graph, that is, the simplicial complex given by the set of all possible classes of discrete Morse functions (in Forman's sense) defined on it. First, we characterize the pure Morse complex of a tree and prove that it is collapsible. In order to study the general case, we consider all the spanning trees included in a given graph G and we express the pure Morse complex of G as the union of all pure Morse complexes corresponding to such trees.

MSC: 57M15; 57M20; 57N05

Keywords: Discrete Morse function; Pure complex of discrete Morse functions; Acyclic matching; Hasse diagram; Critical simplex

1. Introduction

In the papers [3–5] Forman has developed a whole theory based on the notion of discrete Morse function. Without any continuity, he assigns a single number to every simplex of a finite simplicial complex in a way compatible with the simplicial structure of the complex. This study is partially focused on the discrete vector field induced by a discrete Morse function. This is an extension of the notion of smooth vector field to the combinatorial context and, roughly speaking, it is equivalent to the concept of equivalence class of discrete Morse functions which we use in this work. The study of the set of all possible classes of discrete Morse functions defined on a given simplicial complex was initiated by Chari and Joswig in [1], by giving to this set the structure of a simplicial complex and they called it the discrete Morse complex of discrete Morse functions associated to the given simplicial complex. Kozlov studied in [7] the discrete Morse complex in the particular case of one-dimensional simplicial complexes, that is, graphs and he expressed it as rooted forests on graphs.

Following a combinatorial approach based on discrete Morse theory, different from the used by Kozlov, this paper is focused on the study of the set of all equivalence classes of discrete Morse functions defined on a graph. In this context, we are going to work only with pure simplicial complexes, so we restrict our study to the pure Morse complex of a graph. The first section of this paper is devoted to the exposition of all basic notions which we shall need later. In the second section we study the pure Morse complex of a tree, we get a characterization of such object and we conclude that it is collapsible. Later on, in the third section we pretend to understand the structure of the pure Morse

* Corresponding author.
E-mail addresses: rdayala@us.es (R. Ayala), lmfer@us.es (L.M. Fernández), vilches@us.es (J.A. Vilches).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2007.04.023

complex of a general graph by considering the union of all pure Morse complexes of every spanning tree contained in the given graph. Finally, we express the number of simplices of maximum dimension of the pure Morse complex of a graph in terms of the eigenvalues of the Laplacian matrix of G.

2. Preliminaries

We are going to focus our study on graphs (one-dimensional simplicial complexes). For terminology and background concerning these objects, we refer to [6,2]. The notion of discrete Morse function was introduced by R. Forman in [3–5]. Given a simplicial complex M, a *discrete Morse function* is a function $f : M \to \mathbf{R}$ such that, for any p-simplex $\sigma \in M$:

(M1) $\operatorname{card}\{\tau^{(p+1)} > \sigma / f(\tau) \leqslant f(\sigma)\} \leqslant 1$.
(M2) $\operatorname{card}\{\upsilon^{(p-1)} < \sigma / f(\upsilon) \geqslant f(\sigma)\} \leqslant 1$.

A p-simplex $\sigma \in M$ is said to be *critical* with respect to f if:

(C1) $\operatorname{card}\{\tau^{(p+1)} > \sigma / f(\tau) \leqslant f(\sigma)\} = 0$.
(C2) $\operatorname{card}\{\upsilon^{(p-1)} < \sigma / f(\upsilon) \geqslant f(\sigma)\} = 0$.

It can be deduced from the above definitions that $\sigma^{(p)}$ is a noncritical simplex if and only if it verifies one of the following conditions:

(NC1) There exists a simplex $\tau^{(p+1)} > \sigma^{(p)}$ such that $f(\tau^{(p+1)}) \leqslant f(\sigma^{(p)})$.
(NC2) There exists a simplex $\upsilon^{(p-1)} < \sigma^{(p)}$ such that $f(\upsilon^{(p-1)}) \geqslant f(\sigma^{(p)})$.

It is important to point out that both conditions cannot be verified simultaneously by a noncritical simplex.

In [4], Forman also established the discrete Morse inequalities for discrete Morse functions defined on a finite simplicial complex, namely given a discrete Morse function f defined on a finite simplicial complex M and let b_p be the pth Betti number of M, $p = 0, \ldots, \dim(M)$. Then:

(I1) $m_p(f) - m_{p-1}(f) + \cdots \pm m_0(f) \geqslant b_p - b_{p-1} + \cdots \pm b_0$;
(I2) $m_p(f) \geqslant b_p$;
(I3) $m_0(f) - m_1(f) + m_2(f) - \cdots \pm m_{\dim(M)}(f) = b_0 - b_1 + b_2 - \cdots \pm b_{\dim(M)}$,

where $m_p(f)$ denotes the number of critical p-simplices with respect to f. A discrete Morse function where the above inequalities became equalities is called *minimal*.

Since we are going to study the pure Morse complex of a graph, it is convenient to state the following basic result on discrete Morse functions on graphs whose proof can be found in [8].

Proposition 1. *Given a discrete Morse function on a graph G, it holds that every critical simplex is a local extreme of f on G; in particular, every critical vertex is a local minimum of f on G and every critical edge is local maximum of f on G. The converse is true only when G does not have any vertex of degree* 1.

Given a simplicial complex M, the Hasse diagram of M is the directed graph $H(M)$ in which every vertex corresponds to one simplex of M and there is a directed edge connecting two vertices v_0 and v_1, if they correspond to two simplices of consecutive dimensions $\sigma_0^{(p)}$ and $\sigma_1^{(p+1)}$ such that $\sigma_0^{(p)}$ is face of $\sigma_1^{(p+1)}$ and such edge is directed from v_1 to v_0.

Note that in the particular case in which M is a graph, $H(M)$ is a subdivision of M obtained by bisection.

If we have defined a discrete Morse function f on the simplicial complex M, we can consider a modification of $H(M)$, known as the Hasse diagram of M modified by f, $H_f(M)$, by reversing the direction of the edges of $H(M)$ such that the simplices $\sigma_0^{(p)}$ and $\sigma_1^{(p+1)}$ corresponding to its vertices verify that $f(\sigma_0^{(p)}) \geqslant f(\sigma_1^{(p+1)})$. It is interesting to point out that the set of edges in $H_f(M)$ with reversed direction is a matching in $H_f(M)$. Moreover, the above matching is acyclic in the sense that there are no directed cycles or loops in $H_f(M)$. By using $H_f(M)$, it is possible to characterize critical simplices of f on M, as those simplices whose corresponding vertices in $H_f(M)$ are not in any edge of the acyclic matching induced by f.

Now, given two different discrete Morse functions f and g, defined on the same simplicial complex M, we say that f and g are equivalent if $H_f(M) = H_g(M)$, that is, both functions induce the same acyclic matching. By using this equivalence, we can work with a class of functions instead of considering a particular one.

The notion of Morse complex of a simplicial complex M is introduced in [1] in order to study the set of all nonequivalent class of functions which can be defined on M. The Morse complex of M, denoted by $\mathcal{M}(M)$, is the simplicial complex whose vertices correspond to edges of $H(M)$ and any simplex of higher dimension $\sigma^{(p)} = (v_0, \ldots, v_p)$ corresponds to the acyclic matching in $H(M)$ given by the edges corresponding to $v_0, \ldots, v_p$.

In this paper we deal with a slightly different version of the notion of Morse complex, namely the pure Morse complex, denoted by $\mathcal{M}_{\text{pure}}(M)$, which is defined as the subcomplex of $\mathcal{M}(M)$ generated only by the simplices of maximum dimension. Geometrically, the main advantage of using the notion of pure Morse complex is that the obtained complex is always a pure simplicial complex. However, with the more general notion of Morse complex, we can easily get examples such that the obtained complex is not a pure simplicial complex. It is interesting to point out that, in terms of classes of discrete Morse functions, $\mathcal{M}_{\text{pure}}(M)$ can be considered as the set of all possible classes of minimal discrete Morse functions which can be defined on M. This is true due to the equivalence between simplices of maximum dimension in $\mathcal{M}(M)$, acyclic matchings with maximal cardinal in $H(M)$ and classes of discrete Morse functions defined on M.

3. The pure Morse complex of a tree

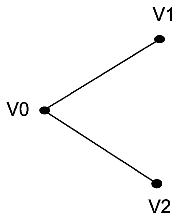
As a first step in our main objective, we devote this section to the study of the Morse complex of tree. As we shall prove in the next section, it will play a basic role in our research.

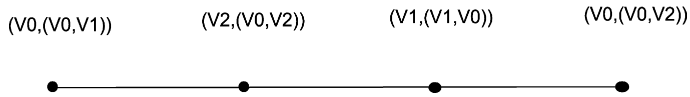
In the following result, we shall prove that the pure Morse complex of a tree is closely related to the original tree.

Proposition 2. *Let T be a tree with n vertices. Then, $\mathcal{M}_{\text{pure}}(T)$ is the simplicial complex of dimension $n-2$ obtained from T by substituting every vertex of T by a $n-2$-simplex and considering that two of such simplices have a common face of dimension $n-3$ if its corresponding vertices on T are connected by an edge.*

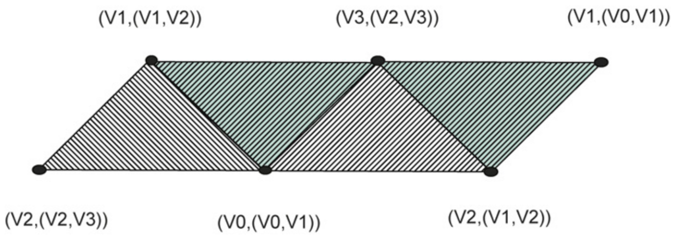
Proof. By definition, the pure Morse complex of a tree is the simplicial complex of dimension $n-2$ whose simplices of maximum dimension correspond to maximum matchings in $H(T)$, which obviously are acyclic. Since $H(T)$ is a subdivision of T by bisection, it is a tree and hence, the number of its vertices is $2n-1$ and the number of its edges is $2n-2$, so every maximum matching in $H(T)$ has $n-1$ edges. Then, we get that every simplex of maximum dimension of the pure Morse complex of T is generated by $n-1$ vertices and consequently the dimension of this complex is $n-2$. Moreover, since T is a tree, we get that every maximum matching in $H(T)$ is a set of edges which avoid a unique vertex of $H(T)$, whose corresponding simplex in T is a vertex. Thus every simplex of maximum dimension of $\mathcal{M}_{\text{pure}}(T)$ corresponds to a choice of a vertex of T and, in order to construct the pure complex, we put a simplex of dimension $n-2$ for every vertex of T. The way two of such simplices intersect is determined by the fact that two different maximum matchings in $H(T)$ must have, at least, one noncommon edge and in this case, since both matchings are maximum, these two noncommon edges are obtained as the bisection of the edge joining the two different chosen vertices of T which correspond to both maximum simplices of $\mathcal{M}_{\text{pure}}(T)$. But this implies that these simplices share a common face of dimension $n-3$. □

Example 3. Let us introduce some examples which illustrate the above proposition. The first one is a tree with three vertices and its corresponding pure Morse complex.

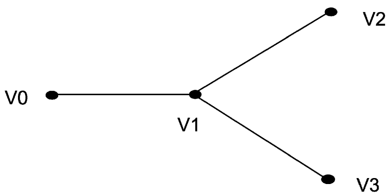


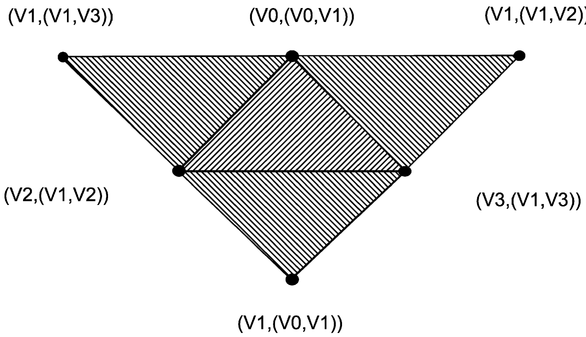


In the second example, we consider a tree with four vertices and its associated pure Morse complex.



Finally, the third example is another tree with four vertices (nonisomorphic to the tree considered in the previous example) and its corresponding pure Morse complex.





As a direct consequence of the above proposition we get the following result.

Corollary 4. *Let T be a tree. Then, $\mathcal{M}_{\text{pure}}(T)$ is a collapsible simplicial complex.*

Proof. Let us suppose that T has n vertices. It follows from the proof of the above proposition that every vertex of degree 1 in T corresponds to a simplex of dimension $n-2$ such that all of its faces of dimension $n-3$ are free excepting one, precisely, the unique face which joins this simplex to the rest of the complex. Once one of the above free faces of $\mathcal{M}_{\text{pure}}(T)$ has been located, we proceed by collapsing by this face and, since the pure complex has a treelike structure inherited from T, we continue collapsing by free faces until we finally finish in a vertex. Note that fixed a collapsing strategy in T, it gives us a not unique collapsing strategy in $\mathcal{M}_{\text{pure}}(T)$. □

4. The pure Morse complex of an arbitrary graph

In this section we shall study how maximum acyclic matchings in the Hasse diagram of a graph and in the Hasse diagram of any of its spanning trees are related.

Proposition 5. *Let G be a connected graph and let T be a spanning tree in G. Then, every maximum matching in $H(T)$ is an acyclic maximum matching in $H(G)$.*

Proof. Consider any maximum matching in $H(T)$. Since $H(T)$ is a subdivision of T by bisection, $H(T)$ is a tree, so this maximum matching is acyclic. Moreover, since T is a spanning tree of G, the number of edges in $G-T$ is $b_1 = \dim(H_1(G))$. Considered in $H(G)$, these edges correspond to those vertices of $H(G)$ which are not in T. Since T is a tree, it has $n+1$ vertices and n edges, and we have that every maximum matching in $H(T)$ has to avoid only one vertex corresponding to a vertex of G. So, the maximum matching contained in $H(G)$ considered above is a maximum acyclic matching. Note that any other matching in $H(G)$ with a greater number of edges it is not acyclic. □

The following result states the converse of the above proposition.

Proposition 6. *Let G be a connected graph. Then, every maximum acyclic matching in $H(G)$ is an maximum matching in $H(T)$, where T is a spanning tree in G.*

Proof. Consider any maximum acyclic matching in $H(G)$. Since G is connected, by applying discrete Morse inequalities (I2) we get that $m_0 \geqslant b_0 = 1$ and $m_1 \geqslant b_1$. Hence, every maximum matching in $H(G)$ avoids only one vertex corresponding to the vertices of G and avoids as many vertices corresponding to edges of G as b_1. Every of these edges is located in a basic cycle of $H_1(G)$, so if we consider the spanning tree T defined by removing these edges, we get that the considered maximum matching in $H(G)$ is a maximum matching in $H(T)$. □

We can join the above two results in the following theorem:

Theorem 7. *The set of maximum acyclic matchings in $H(G)$ is the union of all possible maximum matchings in $H(T)$, where T is in the set of spanning trees in G.*

This theorem can be rewritten in terms of the pure Morse complex in the following way:

Theorem 8. *Let G be a connected graph. Then,*

$$\mathcal{M}_{\text{pure}}(G) = \bigcup_{T_i \in Sp(G)} \mathcal{M}_{\text{pure}}(T_i),$$

where $Sp(G)$ is the set of spanning trees contained in G.

Proof. By definition of pure Morse complex, we know that every simplex of maximum dimension in $\mathcal{M}_{\text{pure}}(G)$ corresponds to a class of minimal discrete Morse functions defined on G or, equivalently, to a maximum acyclic matching in $H(G)$. □

In order to clarify how the unions of the above theorem are made, we shall prove the following result.

Proposition 9. *Let G be a connected graph with n vertices and let T_i and T_j be any different spanning trees contained in G. Then, $\mathcal{M}_{\text{pure}}(T_i) \cap \mathcal{M}_{\text{pure}}(T_j)$ is either empty or a union of simplices of dimension $n-2$.*

Proof. Since every simplex of maximum dimension in $\mathcal{M}_{\text{pure}}(T)$ corresponds to a maximum acyclic matching in $H(G)$, the intersection of two of such simplices corresponds to a common nonmaximum acyclic matching having exactly one edge less than both since, otherwise $\mathcal{M}_{\text{pure}}(G)$ would not be a pure simplicial complex, which it is a contradiction. □

We are going to conclude this section by calculating the number of simplices of maximum dimension which are contained in $\mathcal{M}_{\text{pure}}(G)$.

Proposition 10. *Let G be a connected graph with n vertices. Then, the number of simplices of maximum dimension of $\mathcal{M}_{\text{pure}}(G)$ is*

$$\prod_{i=1}^{n} \lambda_i(G),$$

where $\lambda_i(G)$ are the eigenvalues of the Laplacian matrix of G.

Proof. By definition of pure Morse complex, we know that every simplex of maximum dimension in $\mathcal{M}_{\text{pure}}(G)$ corresponds to a class of minimal discrete Morse functions defined on G. These kind of functions are determined by applying discrete Morse inequalities and hence, by $b_0 = 1$ and b_1. More precisely, these functions have a unique critical vertex and b_1 critical edges. We know that there is a bijection between edges which are not in a spanning tree in G and basic 1-cycles which generate $H_1(G)$. Hence, every class of minimal discrete Morse functions is equivalent to a spanning tree with a distinguished vertex and we can conclude that the total number of such classes of functions is equal to the product of the number of all possible spanning trees contained in G times n, the number of vertices of G. Finally, since we know that the number of spanning trees contained in G is

$$\frac{1}{n}\prod_{i=1}^{n} \lambda_i(G),$$

we can conclude that the number of classes of minimal discrete Morse functions defined on G is

$$\prod_{i=1}^{n} \lambda_i(G). \qquad \square$$

References

[1] M.K. Chari, M. Joswig, Complexes of discrete Morse functions, Discrete Math. 302 (2005) 39–51.
[2] R. Diestel, Graph Theory, Graduate Texts in Mathematics, vol. 173, Springer-Verlag, New York, 1997.
[3] R. Forman, Combinatorial Differential Topology and Geometry, in: New Perspectives in Geometric Combinatorics, vol. 38, MSRI Publications, 1999, pp. 177–206.
[4] R. Forman, Morse theory for cell complexes, Adv. Math. 134 (1) (1998) 90–145.
[5] R. Forman, Combinatorial vector fields and dynamical systems, Math. Z. 228 (1998) 629–681.
[6] F. Harary, Graph Theory, Addison–Wesley, Reading, MA, 1994.
[7] D. Kozlov, Complexes of directed trees, J. Combin. Theory Ser. A 88 (1) (1999) 112–122.
[8] J.A. Vilches, Funciones de Morse dicretas sobre complejos infinitos, Ph.D. thesis, Universidad de Sevilla, Sevilla, 2003.