Arc-disjoint spanning sub(di)graphs in digraphs

Jørgen Bang-Jensen a,*, Anders Yeo b

a Department of Mathematics and Computer Science, University of Southern Denmark, Odense DK-5230, Denmark
b Department of Computer Science, Royal Holloway, University of London, Egham Surrey TW20 0EX, United Kingdom

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We prove that a number of natural problems concerning the existence of arc-disjoint directed and “undirected” (spanning) subdigraphs in a digraph are NP-complete. Among these are the following of which the first settles an open problem due to Thomassé (see e.g. Bang-Jensen and Gutin (2009) [1, Problem 9.9.7] and Bang-Jensen and Kriesell (2009) [5,4]) and the second settles an open problem posed in Bang-Jensen and Kriesell (2009) [5].

• Given a directed graph D and a vertex s of D; does D contain an out-branching B s + rooted at s such that the digraph remains connected (in the underlying sense) after removing all arcs of B s +?

• Given a strongly connected directed graph D; does D contain a spanning strong subdigraph D′ such that the digraph remains connected (in the underlying sense) after removing all arcs of D′?

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1. Introduction

Notation not given below is consistent with [1]. Paths and cycles are always directed unless otherwise specified. For a digraph D we denote by V(D) and A(D), respectively, the set of vertices and the set of arcs of D. An (s, t)-path in a digraph D is a directed path from the vertex s to the vertex t. A digraph D = (V, A) is strongly connected (or just strong) if there exists an (x, y)-path and a (y, x)-path in D for every choice of distinct vertices x, y of D, and D is k-arc-strong if D − X is strong for every subset X ⊆ A of size at most k − 1. The underlying graph of a digraph D, denoted UG(D), is obtained from D by suppressing the orientation of each arc and replacing multiple edges by one edge. A digraph D is connected if UG(D) is a connected graph. If D = (V, A) is a digraph and X ⊆ V then we use the notation D(X) to denote the subdigraph of D induced by the vertices in X. We shall often use the shorthand notation i ∈ [m] for i ∈ {1, 2, . . . , m}.

An out-branching B s + in a digraph D = (V, A) is a connected spanning subdigraph of D in which each vertex x ≠ s has precisely one arc entering it and s has no arcs entering it. The vertex s is the root of B s +. The structure of digraphs with arc-disjoint out-branchings from the same root is well understood due to the following important result by Edmonds.

Theorem 1.1 (Edmonds [9]). A digraph D = (V, A) with a special vertex s has k-arc-disjoint out-branchings rooted at s if and only if there are k-arc-disjoint (s, v)-paths in D for every v ∈ V − s.

Using flows in networks, it is easy to check whether a given digraph D with special vertex s has k arc-disjoint (s, v)-paths for every v ∈ V − s (see e.g. [1, Section 5.5]) and thus checking whether D has k arc-disjoint out-branchings...
from s can be done efficiently. Furthermore, the proof of Theorem 1.1 by Lovász [11] implies that there is a polynomial algorithm for constructing a set of k arc-disjoint branchings when they exist (for details see [1, Section 9.3]). Similarly packing edge-disjoint spanning trees in undirected graphs is also well understood, namely there is a (more complicated) necessary and sufficient condition for the existence of k edge-disjoint spanning trees in a graph G.

**Theorem 1.2 (Tutte [13]).** A graph \( G = (V,E) \) has k edge-disjoint spanning trees if and only if, for every partition \( \mathcal{F} = \{X_1, X_2, \ldots, X_t\} \) of V into non-empty sets, the number \( e_\mathcal{F} \) of edges intersecting two of these sets is at least \( k(t-1) \).

Furthermore, it is a celebrated result due to Edmonds that using any algorithm for matroid partition, in polynomial time, one can check whether the condition above is satisfied and find k-edge-disjoint trees if it is. For details see e.g. [12].

Motivated by the fact that both the existence of arc-disjoint out-branchings from the same root in a digraph and the existence of edge-disjoint spanning trees in a graph can be decided in polynomial time and that both problems have good (polynomially verifiable) characterizations, Thomassé posed the following problem around 2005, a positive solution to which would be a first step for providing a link between Theorems 1.1 and 1.2. The problem is well known in the community and has been published on the Egres open problem list for several years.\(^1\)

**Problem 1.3 (Thomassé).** Find a good characterization of directed graphs D whose underlying undirected graph UG(D) has two edge-disjoint spanning trees such that one of these is an out-branching rooted at a given vertex in D.

Clearly the existence of such spanning trees is equivalent to the existence of an out-branching rooted at the given vertex s such that removing the arcs of this branching leaves a connected digraph. In the case where we replace “out-branching” by “a path with specified end vertices s, t” and “connected” by “existence of a path in the underlying graph between s and t” the problem is \( \mathcal{NP} \)-complete as was shown recently by the first author and Kriesell.

**Theorem 1.4 ([4]).** It is \( \mathcal{NP} \)-complete to decide for a given digraph and specified vertices s, t of D whether D contains a directed (s, t)-path P such that UG(D − A(P)) contains a path from s to t.

The proof of Theorem 1.4 does not generalize to the case of directed spanning trees. Furthermore, the fact that in Problem 1.3 we want spanning subdigraphs and that one of these does not have to respect the orientation of the arcs could indicate that there might be a nice characterization or at least a polynomial algorithm for testing the existence of a non-separating out-branching. However, we are going to prove the following which implies that such a characterization does not exist unless \( \mathcal{P} = \mathcal{NP} \). Our proof technique does not apply to the problem of Theorem 1.4 because we strongly use the fact that at least one of the two arc-disjoint digraphs we are looking for is a spanning subdigraph.

**Theorem 1.5.** It is \( \mathcal{NP} \)-complete to decide for a given digraph D = (V, A) and a vertex s ∈ V whether D contains an out-branching \( B^+\) such that UG(D − A(B^+)) is connected.

We shall also prove that a number of related problems are \( \mathcal{NP} \)-complete. In particular we prove the following. A digraph is k-regular if every vertex has precisely k-arcs out of it and k-arcs into it.

**Theorem 1.6.** It is \( \mathcal{NP} \)-complete to decide whether a 2-regular digraph D contains a spanning strong subdigraph D′ such that UG(D − A(D′)) is connected.

This result may be considered slightly surprising, given that if a positive solution exists, then the number of arcs in D′ and D − A(D′) is either n and n, or n + 1 and n − 1, where n is the number of vertices of the given digraph, that is, either D′ is a hamiltonian cycle or it has just one more arc than a hamiltonian cycle.

2. Main proofs

We shall use reductions from 3-CNF satisfiability (3-SAT) and Not-All-Equal 3-SAT (NAE-3-SAT). Recall that a boolean formula is in 3-conjunctive normal form, or 3-CNF, if it is expressed as an AND of clauses, each of which is an OR of exactly 3 distinct literals. In this paper, by 3-SAT we mean the problem of deciding whether a boolean 3-CNF formula \( \mathcal{F} \) is satisfiable (that is whether there exist a truth assignment \( t \) to the variables of \( \mathcal{F} \) such each clause of \( \mathcal{F} \) has at least one true literal). By NAE-3-SAT we mean the problem of deciding whether a boolean 3-CNF formula \( \mathcal{F} \) has a truth assignment such that for each clause there is at least one literal which is true and at least one literal which is false. Note that this is equivalent to saying that both \( \mathcal{F} \) and its negation (obtained by negating all literals in the clauses of \( \mathcal{F} \)) can be satisfied by the same truth assignment \( t \). It is well known that both 3-SAT and NAE-3-SAT are \( \mathcal{NP} \)-complete problems (see e.g. [10, p. 259]).

**Proof of Theorem 1.5.** The reduction used here uses the same type of variable gadget as the one used in the proof of Theorem 1 of [8].\(^2\) We shall show how to reduce 3-SAT to the problem of Theorem 1.5. Let \( H(r) \) be the digraph (the clause gadget) on 7 vertices \( \{a_{r,1}, a_{r,2}, a_{r,3}, b_{r,1}, b_{r,2}, b_{r,3}, c_{r}\} \) and arcs \( a_{r,1}b_{r,1}, a_{r,2}b_{r,2}, a_{r,3}b_{r,3}, c_{r}a_{r,1}, c_{r}a_{r,2}, c_{r}a_{r,3} \) \( 1 = 1, 2, 3 \) (see Fig. 1). Let \( W[u, v, p, q] \) be the digraph (the variable gadget) with vertices \( \{u, v, y_1, y_2, \ldots, y_p, z_1, z_2, \ldots, z_q\} \) and the arcs of the two \((u,v)\)-paths \( uy_1y_2\ldots y_pv, uz_1z_2\ldots z_qv \). Note that we allow \( \min[p, q] = 0 \) but \( p + q \geq 1 \) must hold.

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\(^2\) Which again uses ideas from another proof.
Let $\mathcal{F}$ be an instance of 3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$. We may assume that each variable $x$ occurs at least once either in the negated form or non-negated in $\mathcal{F}$. For each variable $x$ the ordering of the clauses $C_1, C_2, \ldots, C_m$ induces an ordering of the occurrences of the literal $x$ and the literal $\bar{x}$ in these. With each variable $x_i$ we associate a copy of $W[u_i, v_i, p_i, q_i]$ where the literal $x_i$ occurs $p_i$ times and the literal $\bar{x}_i$ occurs $q_i$ times in the clauses of $\mathcal{F}$. Identify end vertices of these digraphs by setting $v_1 = u_{i+1}$ for $i = 1, 2, \ldots, n - 1$. Let $s = u_1$ and $t = u_n$. Next, for each clause $C_j$ we take a copy $H_j = H(j)$ of the clause gadget and identify the vertices $a_{j,1}, a_{j,2}, a_{j,3}$ of $H_j$ with vertices in the chain we build above as follows: assume $C_j$ contains literals involving the variables $x_i, x_q, x_k$. If $C_j$ contains the literal $x_i$ and this is the $r$'th copy of the literal $x_i$ (in the order of the clauses that use literal $x_i$), then we identify $a_{j,1}$ with $y_{i,r}$ and if $C_j$ contains the literal $\bar{x}_i$ and this is the $k$'th occurrence of literal $\bar{x}_i$, then we identify $a_{j,1}$ with $z_{i,k}$. We make similar identifications for $a_{j,2}, a_{j,3}$. Finally we add all the arcs $tc_j$ for $j \in [m]$. This concludes the description of the digraph $D_F$ with special vertices $s, t$. Let $D'$ be the subdigraph induced by the union of all the vertices from $W[u_i, v_i, p_i, q_i], i \in [n]$. Recall that by the identifications above $D'$ contains all the vertices $a_{i,r}, j \in [m], r \in [3]$. See Fig. 2 for an example.

**Claim 1.** $D'$ contains an $(s, t)$-path $P$ which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$ if and only if $\mathcal{F}$ is satisfiable.

**Proof of Claim 1.** Suppose $P$ is an $(s, t)$-path which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$. By construction, for each variable $x_i, P$ traverses either the subpath $Q_i = u_i v_{i,1} v_{i,2} \ldots v_{i, p_i} v_i$ or the subpath $P_i = u_i a_{i,1} z_{i,2} \ldots z_{i, q_i} v_i$. Now define a truth assignment by setting $x_i$ false when $P$ traverses $Q_i$ and true if $P$ traverses $P_i$ for $i \in [n]$. This is a satisfying truth assignment for $\mathcal{F}$ since for any clause $C_j$ at least one literal is avoided by $P$ and hence becomes true by the assignment (the literals traversed become false and those not traversed become true). Conversely, given a truth assignment for $\mathcal{F}$ we can form $P$ by routing it through all the false literals in the chain of variable gadgets. \[\square\]
Claim 2. $D_F$ has an out-branching $B^+_F$ such that $D_F - A(B^+_F)$ is connected if and only if $D'$ contains an $(s, t)$-path $P$ which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$.

Proof of Claim 2. Suppose first that there exists $B^+_F$ such that $D - A(B^+_F)$ is connected. It follows from the structure of $D_F$ that the $(s, t)$-path $P$ in $B^+_F$ lies entirely inside $D'$ and since $t_j$ is the only arc entering $c_j$, all arcs of the form $t_jc_j$, $j \in [m]$ are in $B^+_F$. Now it follows that $P$ cannot contain any of $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for some clause $c_j$ because that would disconnect the vertices of $H_j$ from the remaining vertices in $D - A(B^+_F)$. Conversely, suppose that $D'$ contains an $(s, t)$-path $P$ which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$. Then we form an out-branching $B^+_F$ by adding the following arcs to $P$: all arcs of the form $t_jc_j$, $j \in [m]$ and for each clause $c_j$, $j \in [m]$ and $r \in [3]$ if $P$ contains the vertex $a_{j,r}$ we add the arc $a_{j,r}b_{j,r}$ and otherwise we add the arc $c_{j,r}b_{j,r}$. This clearly gives an out-branching $B^+_F$ of $D_F$. It remains to show that $D^* = D_F - A(B^+_F)$ is connected. First observe that $D' \cup (\{V(D')\})$ contains either all arcs of the subpath $u_iy_1y_2 \ldots y_{p_i}v_i$ or all arcs of the subpath $u_iz_1z_2 \ldots z_{q_i}v_i$ for each $i \in [n]$ and hence it contains an $(s, t)$-path which passes through all the vertices $u_1, u_2, \ldots, u_n$. By the description of $P$ above, for each clause $c_j$, $j \in [m]$ and $r \in [3]$, if $P$ contains the vertex $a_{j,r}$, then $D^*$ contains the arcs $c_{j,r}b_{j,r}$, $c_{j,r}a_{j,r}$ and if $P$ does not contain the vertex $a_{j,r}$ then $D^*$ contains the arcs $a_{j,r}b_{j,r}$, $a_{j,r}b_{j,r}$. Now it is easy to see that $D^*$ is connected and spanning. □

Theorem 1.5 now follows by combining Claims 1 and 2. □

In the proof of Theorem 1.6 we shall use the following result due to the second author (the result is mentioned in [1, Section 13.10] and in [7]). Since a proof has never appeared in print before and the proof of this result plays an important role in the proof below, we include a proof here for completeness (the proof is a refinement of the proof of Theorem 6.1.3 in [1]).

We recall from [1] that a k-path factor of a digraph $H$ is a collection of $k$ vertex disjoint paths that cover all vertices of $V(H)$.

Theorem 2.1. It is $\mathcal{NP}$-complete to decide whether a 2-regular digraph $D$ contains a pair of arc-disjoint hamiltonian cycles.

Proof. We will reduce the Not-All-Equal 3-SAT (NAE-3-SAT) problem to the problem of deciding whether a 2-regular digraph has two arc-disjoint hamiltonian cycles. Consider the following digraph $D(x, y, z)$

$$V(D(x, y, z)) = \{x_i, y_i, z_i : i = 1, 2, 3, 4, 5, 6\},$$

$$A(D(x, y, z)) = \{(x_i, y_{i+1}, z_{i+1}) : i = 1, 2, 3, 4, 5, 6\}$$

(see Fig. 3). It is easy to verify that the digraph $D(x, y, z)$ has the following properties:

(i) There is a unique hamiltonian path $P$ of $D(x, y, z)$ starting at $x_1(y_1, z_1$, respectively) and this terminates at $x_6(y_6, z_6$, respectively). Furthermore, $D(x, y, z) - A(P)$ has a unique 2-path factor $R \cup S$ and $R$ is a $(y_1, y_6)$-path and $S$ is a $(z_1, z_6)$-path. Similarly, when $P$ is a hamiltonian path from $y_1$ to $y_6$ or from $z_1$ to $z_6$.

(ii) Let $P \cup Q$ be a 2-path factor of $D(x, y, z)$ such that the path $P$ starts at $x_1$ and the path $Q$ starts at $y_1$ and both paths end in the set $\{x_6, y_6, z_6\}$. Then $P$ terminates at $x_6$ and $Q$ at $y_6$. Furthermore, $D(x, y, z) - A(P) - A(Q)$ is a hamiltonian path starting at $z_1$ and terminating at $z_6$. Similarly for the pairs $x_1, z_1$ and $y_1, y_1$.

(iii) Let $P \cup Q \cup R$ be a 3-path factor of $D(x, y, z)$ such that the paths $P, Q$ and $R$ start at $x_1, y_1$ and $z_1$, respectively and all three paths end in the set $\{x_6, y_6, z_6\}$. Then $P, Q$ and $R$ terminate at $x_6, y_6$ and $z_6$, respectively. Furthermore, after removing the arcs of $P \cup Q \cup R$ we obtain 6 vertex disjoint 3-cycles with no arcs between them.

That (iii) holds is obvious. To see that property (i) holds it suffices to check that the unique hamiltonian path starting in $x_1$ in $D(x, y, z)$ is $x_1y_1z_1z_2x_2y_2z_3x_3y_3x_4y_4z_5x_5y_5z_6x_6$ and that after deleting these arcs the unique 2-path factor of the remaining digraph consists of the paths $y_1z_2z_3x_4y_4x_5y_5z_6$ and $z_1x_2z_3x_4y_5z_5z_6$. We leave it to the reader to verify the (ii) holds (again the paths are unique and easy to construct).

We are going to use $D(x, y, z)$ as a building block in a bigger digraph below and since we will only connect the vertices $x_1, x_6, y_1, y_6, z_1, z_6$ to other parts of the digraph, we will use the names $x, x', y, y', z, z'$ for these below and denote the subdigraph by $H(x, x', y, y', z, z')$.

Consider an instance $I$ of NAE-3-SAT with variables $v_1, \ldots, v_k$ and clauses $C_1, \ldots, C_p$. Since we require that every clause contains both true and false literals in any satisfying truth assignment, we may assume that every variable and its
negation appear in $I$ as literals (otherwise we can add negated copies of some of the clauses). Construct a digraph $D_I$ as follows: start from a disjoint union $U = H_1 \cup H_2 \cup \cdots \cup H_p$, where $H_j = H(a_j, a'_j, b_j, b'_j, c_j, c'_j)$ and $a_j, b_j, c_j$ are the literals in $C_j, j \in [p]$.

For every variable $v$ the ordering of $C_1, \ldots, C_p$ induces an ordering $C_{v,1}, \ldots, C_{v,q_v}$ of the clauses containing the literal $v$ and an ordering $C_{\overline{v},1}, \ldots, C_{\overline{v},q_{\overline{v}}}$ of the clauses containing the literal $\overline{v}$. Based on this ordering we join the vertices of different pairs among $H_1, H_2, \ldots, H_p$ as follows (where we denote the clause gadget corresponding to the clause $C_{v,j}$ by $H_{v,j}$): for each variable $v$ and $r \leq p_v - 1$ we add an arc $\alpha \rightarrow \beta$ from $H_{v,r}$ to $H_{v,r+1}$ where $\alpha$ equals one of the vertices $a_{v,r}, b_{v,r}, c_{v,r}$ depending on whether the first, second or third literal in $C_{v,j}$ is equal to $v$ and $\beta$ equals one of the vertices $a_{\overline{v},r}, b_{\overline{v},r}, c_{\overline{v},r}$ depending on whether the first, second or third literal in $C_{\overline{v},q}$ is equal to $\overline{v}$. Similarly, for each variable $v$ and $r \leq q_v - 1$ we add an arc $\alpha \rightarrow \beta$ from $H_{\overline{v},r}$ to $H_{\overline{v},r+1}$ where $\alpha$ equals one of the vertices $a_{v,r}, b_{v,r}, c_{v,r}$ depending on whether the first, second or third literal in $C_{\overline{v},r}$ is equal to $\overline{v}$ and $\beta$ equals one of the vertices $a_{\overline{v},r}, b_{\overline{v},r}, c_{\overline{v},r}$ depending on whether the first, second or third literal in $C_{\overline{v},r}$ is equal to $\overline{v}$. See Fig. 4.

Next we add $2k$ new vertices $u_1, w_1, u_2, w_2, \ldots, u_k, w_k$ where the vertices $u_i, w_i$ correspond to the variable $v_i$ for $i \in [k]$. Each vertex $v_i$ dominates one vertex in each of $H_{v_i,1}$ and $H_{\overline{v}_i,1}$, namely one of the vertices $a_{v_i,1}, b_{v_i,1}, c_{v_i,1}$ depending on whether $v_i$ is the first, second or third literal in $C_{v_i,1}$ and one of the vertices $a_{\overline{v}_i,1}, b_{\overline{v}_i,1}, c_{\overline{v}_i,1}$ depending on whether $v_i$ is the first, second or third literal in $C_{\overline{v}_i,1}$. Each vertex $w_i$ is dominated by one vertex from each of $H_{v_i,\overline{p}_i}$ and $H_{\overline{v}_i,\overline{p}_i}$, namely one of the vertices $a'_{v_i,\overline{p}_i}, b'_{v_i,\overline{p}_i}, c'_{v_i,\overline{p}_i}$ depending on whether $v_i$ is the first, second or third literal in $C_{v_i,\overline{p}_i}$ and one of the vertices $a'_{\overline{v}_i,\overline{p}_i}, b'_{\overline{v}_i,\overline{p}_i}, c'_{\overline{v}_i,\overline{p}_i}$ depending on whether $v_i$ is the first, second or third literal in $C_{\overline{v}_i,\overline{p}_i}$. Finally, we add the arcs $w_i u_{i-1}, w_i u_{i+1}$ for every $i \in [k]$, where $u_0 = u_k$, $u_{k+1} = u_1$.

It is easy to verify that $D$ is 2-regular.

Suppose $I$ is a ‘yes’ instance of NAE-3-SAT and consider a satisfying truth assignment $t$. Note that the complementary truth assignment $\overline{t}$ (where we set a variable true if and only if it is false in $t$) is also a satisfying truth assignment for $I$. We will show how to construct arc-disjoint hamiltonian cycles $C, C'$ of $D_I$ based on the values of the variables in $t$. For each variable $v_i$ such that $v_i$ is true in $t$ we let $C$ contain the arc $w_i u_{i+1}$, the arc from $u_i$ to $H_{v_i,1}$, the arc from $H_{\overline{v}_i,\overline{p}_i}$ to $w_i$ and all the arcs from $H_{\overline{v}_i,1}$ to $H_{\overline{v}_i,\overline{p}_i}, r = 1, 2, \ldots, p_v - 1$ that were described above corresponding to the occurrences of $v_i$ in the clauses $C_{v_i,1}, \ldots, C_{v_i,\overline{p}_i}$. For each variable $v_i$ such that $v_i$ is false in $t$ we let $C$ contain the arc $w_i u_{i+1},$ the arc from $u_i$ to $H_{\overline{v}_i,\overline{p}_i}$, the arc from $H_{\overline{v}_i,\overline{p}_i}$ to $w_i$ and all the arcs from $H_{\overline{v}_i,1}$ to $H_{\overline{v}_i,\overline{p}_i}, r = 1, 2, \ldots, q_v - 1$ that were described above corresponding to the occurrences of $v_i$ in the clauses $C_{v_i,1}, \ldots, C_{v_i,\overline{p}_i}$. Similarly (except we use arcs from $w_i$ to $u_{i-1}$ for all $j \in [k]$). Since every clause is satisfied by $t$, the cycle $C$ uses vertices from each digraph from the disjoint union $H_1 \cup H_2 \cup \cdots \cup H_p$. By the properties (i) and (ii) of $H(x, y, z)$ above, if $s (1 \leq s \leq 2)$ literals are satisfied in a clause $C_{v,j}$ by $t$, all vertices of the corresponding digraph $H_j$ can be used in $C$ due to the existence of an appropriate $s$-path factor in $H_j$. Thus, $C$ is indeed hamiltonian. Similarly $C'$ is a hamiltonian cycle and it is arc-disjoint from $C$ by the way we constructed it.

Suppose now that $D_I$ has a pair of arc-disjoint hamiltonian cycles $C, C'$, it follows from (iii) above that none of the cycles $C, C'$ passes through any $H_t$ more than two times. Hence if we set $v_i$ true if $C$ uses the arc from $u_i$ to $H_{v_i,1}$ and false otherwise, then we obtain a truth assignment $t$ such that both $t$ and $\overline{t}$ satisfy all clauses of $I$ (a literal will be set to true if and only if $C$ uses the arcs of $D_I$ that correspond to this literal). \(\square\)
Lemma 2.2. The digraph \(Q\) in Fig. 5 has the following properties.

(i) \(Q\) has no hamiltonian cycle.

(ii) \(Q\) contains the strong spanning subdigraph induced by the arcs \(\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_2v_6, v_6v_1\}\) which is arc-disjoint from the connected spanning subgraph of \(UC(D)\) formed by the arcs \(\{v_1v_6, v_6v_5, v_5v_4, v_4v_3, v_3v_2\}\).

(iii) There is no strong spanning subdigraph \(Q'\) of \(Q\) such that \(Q - A(Q')\) is connected and both of the arcs \(v_4v_3, v_4v_5\) are in \(Q'\).

Proof. We leave it to the reader to make the easy check that \(Q\) has no hamiltonian cycle. That (ii) holds is easy to verify so it only remains to prove (iii). Let \(Q'\) be any spanning subdigraph of \(Q\) which contains both of the arcs \(v_4v_3, v_4v_5\) and assume that \(Q - A(Q')\) is connected. Then precisely one of the arcs \(v_3v_4, v_4v_5\) is in \(Q'\). Note that, by (i), \(Q'\) must contain at least 7 arcs and \(Q - A(Q')\) must be a spanning tree of \(Q\). Therefore all vertices in \(Q'\) except one have out-degree one and all vertices in \(Q'\) except one have in-degree one. Suppose first that \(Q'\) contains \(v_3v_4\). Then \(v_3v_2 \notin A(Q')\) and \(v_1v_2 \in A(Q')\). Suppose first that \(Q'\) contains \(v_4v_5\). Then \(v_5v_1 \notin A(Q')\) and \(v_6v_1 \in A(Q')\). Therefore this is a contradiction to \(Q - A(Q')\) is connected. If \(V\) is a source (has in-degree 0) and hence is the only vertex from which an out-branching can start in \(D_f\). Thus the proof of Theorem 1.5 shows that the following holds.

Theorem 3.2. It is \(NP\)-complete to decide for a given strongly connected digraph \(D\) whether \(D\) contains some vertex \(s\) and an out-branching \(B^*_s\) such that \(D - A(B^*_s)\) is connected.

It is easy to check that the proof of Theorem 1.5 still works if we add the arc \(ts\) in which case the digraph \(D_f\) becomes strongly connected. Hence we have shown the following.

Theorem 3.3. It is \(NP\)-complete to decide for a given strongly connected digraph \(D\) and a specified vertex \(s\) of \(D\) whether \(D\) contains an out-branching \(B^*_s\) such that \(D - A(B^*_s)\) is connected.
Clearly, by Theorem 1.1, every 2-arc-strong digraph has an out-branching $B^+_s$ such that $D - A(B^+_s)$ is connected for every choice of the root $s$. On the other hand, there exist strong digraphs $D$ with $UG(D)$ arbitrarily highly edge-connected and a vertex $s$ which can reach all other vertices by a directed path and yet $D$ has no out-branching $B^+_s$ s.t. $D - A(B^+_s)$ is connected. To see this take a directed path $P = u_1 u_2 u_3 \ldots u_t$ and add all arcs $u_i u_j$ where $j < i$ and both $i$ and $j$ are even or both odd. This has no good out-branching from $s$ because every $B^+_s$ will use all arcs on $P$.

**Theorem 3.3.** The following problems are all NP-complete

(i) Given a digraph $D$ and $s$, $t \in V(D)$; does $D$ have an $(s, t)$-path $P$ such that $D - A(P)$ is connected?

(ii) Given a digraph $D$ and $s$, $t \in V(D)$; does $UG(D)$ have an $(s, t)$-path $P$ such that $D - A(P)$ contains an out-branching rooted in $s$?

(iii) Given a strong digraph $D$; does $D$ contain a cycle $C$ such that $D - A(C)$ is connected?

(iv) Given a strong digraph $D$; does $D$ contain a cycle $C$ such that $D - A(C)$ is strongly connected?

(v) Given a strong digraph $D$; does $UG(D)$ contain a cycle $W$ such that $D - A(W)$ is strongly connected?

**Proof.** Let $F$ be an instance of 3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$, let $D^+_F$ be the digraph that we build as in the proof of Theorem 1.5, except that instead of using $H(j)$ as the clause gadget for $C_j$ we use a directed 6-cycle $a_j d_1^j d_2^j d_3^j d_4^j d_5^j a_j$ as clause gadget and where the vertices $a_j, d_1^j, d_2^j, a_3$ are identified with vertices of the variable gadgets as we did in the proof of Theorem 1.5. Define $D^*$ as we did in the proof of Theorem 1.5. To prove that problem (i) is NP-complete, it suffices to note that if $P$ is an $(s, t)$-path in $D^*_F$ such that $D^*_F - A(P)$ is connected, then $P$ does not use any arc from any of the clause gadgets. Now it is easy to see that $D^*_F$ contains such a path if and only if $D^*$ contains a path which avoids at least one vertex from $\{a_j, d_2^j, a_3\}$ for each $j \in [m]$ and we are done by Claim 1. If $Q$ is path between $s$ and $t$ in $UG(D^+_F)$ such that $D^+_F - A(Q)$ contains an out-branching, then $Q$ does not use any arc from any of the clause gadgets. Again this and Claim 1 easily implies that (ii) is NP-complete. To prove that (iii) is NP-complete we consider the digraph $D^*$ which we obtain from $D^+_F$ by adding the arc $ts$. Then the argument above for (i) shows that $D^*$ has a cycle $C$ such that $D - A(C)$ is connected if and only if $F$ is satisfiable so (iii) is NP-complete. To prove that (iv) and (v) are NP-complete we consider $D'^*$ which we obtain from $D^*$ by adding a new vertex $t'$ and the arcs $tt', ts'$. By choice of clause gadget, a cycle $C$ such that $D'^* - A(C)$ is strongly is a cycle formed by an $(s, t)$-path $P$ in $D'$ and the arc $ts$ with the property that $P$ avoids at least one vertex from each of the sets $\{a_j, d_2^j, a_3\}$ and now we can apply Claim 1 to see that (iv) is NP-complete. Finally, observe that $UG(D'^*)$ has a cycle $W$ such that $D - A(W)$ is strongly connected if and only if $W$ is formed by an $(s, t)$-path $P$ in $D'$ and the arc $ts$ with the property that $P$ avoids at least one vertex from each of the sets $\{a_j, d_2^j, a_3\}$ and again we can use Claim 1. □

It was shown in [3] that there is a polynomial algorithm to check whether the underlying digraph $UG(D)$ of a given strong digraph $D$ contains two vertex disjoint cycles $C, C'$ such that $C$ is also a cycle in $D$. On the other hand it was shown in [6] that the same problem becomes NP-complete if we do not require that $D$ is strong.

In [7] the authors posed the following conjecture and proved it for some digraphs following where $N = 3$ is necessary and sufficient (a digraph is semicomplete if it has no non-adjacent vertices). Recently [2] the conjecture was also confirmed for locally semicomplete and again $N = 3$ is best possible for this much larger class of digraphs. A digraph is locally semicomplete if and only if the in-neighbourhood and the out-neighbourhood of every vertex induces a semicomplete digraph.

**Conjecture 3.4 ([7]).** There exist a natural number $N$ such that every $N$-arc-strong $D$ contains two arc-disjoint spanning strong subdigraphs.

A (much) weaker version of this is the following.

**Conjecture 3.5 ([5]).** There exist a natural number $K$ such that every $K$-arc-strong digraph $D$ has a strongly spanning subdigraph $D'$ such that $UG(D - A(D'))$ is connected.

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**References**


