The probability theories for IVFSs and IVIFSs

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Since the concept of IFS-probability was proposed, the probability theory for IF-events has been constructed. In this paper the concepts of IVFS-probability and IVIFS-probability are proposed for the first time. And the representation theorem for IVFS-probability is given. Furthermore, the relationships between IFS-probability, IVFS-probability and IVIFS-probability are discussed.

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1. Introduction

Since the concept of fuzzy sets was introduced by Zadeh in 1965 [1], the theories of fuzzy sets and fuzzy systems have been developed rapidly. In 1986, the notion of an intuitionistic fuzzy set was introduced by Atanassov [2] as a generalization of a fuzzy set.

An intuitionistic fuzzy set [2] in \(X, X \neq \emptyset\), is an expression \(A\) given by

\[ A = \{(x, (\mu_A(x), \nu_A(x)))|x \in X\} \]

with \(\mu_A(x), \nu_A(x) : X \rightarrow [0, 1]\) satisfying the condition \(0 \leq \mu_A(x) + \nu_A(x) \leq 1\) for all \(x \in X\).

The numbers \(\mu_A(x)\) and \(\nu_A(x)\) denote, respectively, the degree of membership and the degree of nonmembership of the element \(x\) in the set \(A\). We will denote by IFSSs the set of all the intuitionistic fuzzy sets in \(X\). And for simplicity, we define \(A = (\mu_A, \nu_A)\).

Let \(D[0, 1]\) be the set of all closed subintervals of the interval \([0, 1]\).

\[ \forall[a, b], [c, d] \in D[0, 1], \text{ we define} \]

\[ [a, b] = [c, d] \iff a = c \text{ and } b = d, \]

\[ [a, b] + [c, d] = [a + c, b + d], \]

\[ [a, b] \leq [c, d] \iff a \leq c \text{ and } b \leq d. \]

Therefore, the interval \([1, 1]\) is the greatest element of \(D[0, 1]\) and the interval \([0, 0]\) is the least element of \(D[0, 1]\).

In [3], the notion of an interval-valued fuzzy set was introduced by Sambuc as a generalization of a fuzzy set. An interval-valued fuzzy set [3] in \(X, X \neq \emptyset\), is an expression \(A\) given by

\[ A = \{[x, [f_A(x), g_A(x)]]|x \in X\} \]

where \([f_A(x), g_A(x)] \in D[0, 1]\).
The closed interval \([a, b]\) denotes the degree of membership of an element \(x\) in the set \(A\). We will denote by IVFSs the set of all interval-valued fuzzy sets in \(X\). And for simplicity, we define \(A = [a_A, b_A]\).

**Theorem 1** ([4]). The maps \(f : \text{IVFSs} \to \text{IFSs}\), \(g : \text{IFSs} \to \text{IVFSs}\) are defined in the following:

(i) The map \(f\) assigns to every IVFSs \(A = \{(x, [a_A(x), b_A(x)])|x \in X\}\) an IFSs \(\{(x, \mu_B(x), \nu_B(x))|x \in X\}\) = \(B = f(A)\) where \(\mu_B(x) = f_A(x)\), \(1 - \nu_B(x) = g_A(x)\).

(ii) The map \(g\) assigns to every IFSs \(B = \{(x, \mu_B(x), \nu_B(x))|x \in X\}\) an IVFSs \(\{(x, [f_A(x), g_A(x)])|x \in X\}\) = \(A = g(B)\) where \(f_A(x) = \mu_B(x)\), \(1 - g_A(x) = \nu_B(x)\).

Then IFSs and IVFSs are equivalent, and are generalizations of the notion of FSs.

The notion of an interval-valued intuitionistic fuzzy set was introduced for the first time by Atanassov and Gargov [4] as a generalization of an intuitionistic fuzzy set.

**Definition 1** ([4]). An interval-valued intuitionistic fuzzy set in \(X, X \neq \emptyset\), is an expression \(A\) given by

\[A = \{(x, ([a_A(x), b_A(x)], [h_A(x), i_A(x)])|x \in X\}\}

where \([a_A(x), b_A(x)], [h_A(x), i_A(x)]\) \(\in D[0, 1]\\), and \(0 \leq i_A(x) + h_A(x) \leq 1, \forall x \in X\).

The intervals \([a_A(x), b_A(x)]\) and \([h_A(x), i_A(x)]\) denote, respectively, the degree of membership and the degree of nonmembership of the element \(x\) in the set \(A\). We will denote by IVFSs the set of all the interval-valued intuitionistic fuzzy sets in \(X\). And for simplicity, we define \(A = ([a_A], [b_A], [h_A], [i_A])\).

In [5,6], the definition of an IFS-probability was introduced. And in [7], the representation theorem for probability for IFS-events was proved.

In this paper, we introduce the definitions of IVFS-probability and IVIFS-probability for the first time. On the basis of the proposed new concepts, the relationships between IFS-probability, IVFS-probability and IVIFS-probability are investigated.

2. Preliminaries

Let \((\Omega, S, \mathbb{P})\) be a classical probability space. An IF-event is a pair \(A = (\mu_A, \nu_A)\) of \(S\)-measurable real functions \(\mu_A, \nu_A : \Omega \to [0, 1]\\) such that \(\mu_A + \nu_A \leq 1\).

In this paper we shall use the additivity based on Łukasiewicz connectives:

\[x \circ y = (x + y - 1) \vee 0, \quad x \oplus y = (x + y) \wedge 1.\]

for all \(x, y \in [0, 1]\).

\[\forall A, B = (\mu_B, \nu_B) \in \text{IFSs}, \text{we define}\]

\[A \circ B = (\mu_A \circ \mu_B, \nu_A \circ \nu_B), \quad A \oplus B = (\mu_A \oplus \mu_B, \nu_A \oplus \nu_B)\]

\[A \leq B \iff \mu_A \leq \mu_B, \nu_A \geq \nu_B.\]

With respect to the ordering,

\[A = (a, b), B = (c, d) \iff a \leq c, d \leq b, [a, b], [c, d] \in D[0, 1].\]

**Definition 2** ([5]). An IFS-probability is a function \(\mathcal{P} : \text{IFSs} \to D[0, 1]\) satisfying the following axioms:

(i) \(\mathcal{P}(\{1, 0\}) = [1, 1], \mathcal{P}((0, 1)) = [0, 0]\\);

(ii) if \(A \circ B = (0, 1)\), then \(\mathcal{P}(A \circ B) = \mathcal{P}(A) + \mathcal{P}(B)\\);

(iii) if \(A_n \not\supset A, \text{then } \mathcal{P}(A_n) \not\supset \mathcal{P}(A)\\).

In [6], Ciungu and Riečan gave a representation theorem for the IFS-probability.

**Theorem 2** ([7]). For every IFS-probability \(\mathcal{P} : \text{IFSs} \to \text{IVFSs}, \text{there exist constants } \beta, \gamma \text{ and probabilities } P, R_1, R_2 \text{ with } \beta R_1 \leq \gamma R_2 \text{ such that } \mathcal{P} \text{ is given by the following formula:}\)

\[\mathcal{P}(\mu_A, \nu_A) = \left[\int_{\Omega} \mu_A dP + \beta \int_{\Omega} [1 - (\mu_A + \nu_A)] dR_1, \int_{\Omega} \mu_A dP + \gamma \int_{\Omega} [1 - (\mu_A + \nu_A)] dR_2\right].\]

3. IVFS-probabilities

\[\forall A = [a_A, b_A], B = [a_B, b_B] \in \text{IVFSs}, \text{we define}\]

\([A \circ B, A \oplus B] = [a_A \circ b_A, a_A \oplus b_A], \quad [A, B] = [a_A \oplus b_A, a_A \oplus b_A].\]
**Definition 3.** An IVFS-probability is a function \( \mathcal{P} : \text{IVFS} \rightarrow D[0, 1] \) satisfying the following axioms:

(i) \( \mathcal{P}([1, 1]) = [1, 1], \mathcal{P}([0, 0]) = [0, 0] \);
(ii) if \( A \cap B = [0, 0] \), then \( \mathcal{P}(A \cup B) = \mathcal{P}(A) + \mathcal{P}(B) \);
(iii) if \( A_n \nsubseteq A \), then \( \mathcal{P}(A_n) \nsubseteq \mathcal{P}(A) \).

**Theorem 3.** Let \( \mathcal{P} \) be an IVFS-probability and \( g : \text{IFSs} \rightarrow \text{IVFSs} \) be the map defined in Theorem 1; then \( \mathcal{P} \circ g \) is an IFS-probability.

**Proof.** (i) \( \mathcal{P} \circ g(([1, 0]) = \mathcal{P}(g(([1, 0]))) = [1, 1], \mathcal{P} \circ g(([0, 1])) = \mathcal{P}(g([0, 1])) = [0, 0] \).

(ii) If \( (\mu_A, \upsilon_A) \cap (\mu_B, \upsilon_B) = [0, 0] \), then \( \mu_A \cap \mu_B = 0, \upsilon_A \cap \upsilon_B = 0 \).

Since \( g((\mu_A, \upsilon_A)) = [\mu_A, 1-\upsilon_A], g((\mu_B, \upsilon_B)) = [\mu_B, 1-\upsilon_B], [\mu_A, 1-\upsilon_A] \cap [\mu_B, 1-\upsilon_B] = [\mu_A \cap \mu_B, (1-\upsilon_A) \cap (1-\upsilon_B)] = [\mu_A \cap \mu_B, 1-(\upsilon_A + \upsilon_B)] = [0, 0] \), then \( g((\mu_A, \upsilon_A)) \cap g((\mu_B, \upsilon_B)) = [0, 0] \). Therefore \( \mathcal{P} \circ g((\mu_A, \upsilon_A) \cap (\mu_B, \upsilon_B)) = \mathcal{P}(g((\mu_A, \upsilon_A)) + \mathcal{P}(g((\mu_B, \upsilon_B))) = \mathcal{P} \circ g((\mu_A, \upsilon_A)) \).

(iii) If \( A_n = (\mu_{A_n}, \upsilon_{A_n}) \nsubseteq A = (\mu_A, \upsilon_A) \), i.e. \( \mu_{A_n} \nsubseteq \mu_A, \upsilon_{A_n} \nsubseteq \upsilon_A \), then \( \mu_{A_n} \cap \mu_A = 1-\upsilon_{A_n} \cap 1-\upsilon_A \), i.e. \( g(A_n) \nsubseteq g(A) \); therefore \( \mathcal{P} \circ g(A_n) = \mathcal{P}(g(A_n)) \nsubseteq \mathcal{P}(g(A)) = \mathcal{P} \circ g(A) \).

From Definition 2, we have that \( \mathcal{P} \circ g \) is an IFS-probability.

**Theorem 4.** Let \( \mathcal{P} \) be an IFS-probability and \( f : \text{IFSs} \rightarrow \text{IFSs} \) be as defined in Theorem 1; then \( \mathcal{P} \circ f \) is an IVFS-probability.

**Proof.** (i) \( \mathcal{P} \circ f(([1, 1]) = \mathcal{P}(f([1, 1])) = [1, 1], \mathcal{P} \circ f([0, 0]) = \mathcal{P}(f([0, 0])) = [0, 0] \).

(ii) If \( (f_A, g_A) \cap (f_B, g_B) = [0, 0] \), then \( f_A \cap f_B = [0, 0], g_A \cap g_B = [0, 0] \).

Since \( f((f_A, g_A)) = (f_A, 1-g_A), f((f_B, g_B)) = (f_B, 1-g_B), (f_A, 1-g_A) \cap (f_B, 1-g_B) = (f_A \cap f_B, 1-(g_A \cap g_B)) = (0, 0) \), then \( f((f_A, g_A)) \cap f((f_B, g_B)) = (0, 0) \). Therefore \( \mathcal{P} \circ f((f_A, g_A) \cap (f_B, g_B)) = \mathcal{P}(f((f_A, g_A) \cap (f_B, g_B))) = \mathcal{P}(f((f_A, g_A)) \cap f((f_B, g_B))) = \mathcal{P}(f((f_A, g_A)) \cap f((f_B, g_B))) \).

(iii) If \( A_n = (f_{A_n}, g_{A_n}), A = (f_A, g_A), \) i.e. \( f_{A_n} \nsubseteq f_A, g_{A_n} \nsubseteq g_A \), then \( f_{A_n} \nsubseteq f_A, 1-g_{A_n} \nsubseteq 1-g_A \), i.e. \( f(A_n) \nsubseteq f(A) \); therefore \( \mathcal{P} \circ f(A_n) = \mathcal{P}(f(A_n)) \nsubseteq \mathcal{P}(f(A)) = \mathcal{P} \circ f(A) \).

From Definition 3, we have that \( \mathcal{P} \circ f \) is an IVFS-probability.

The IVFS-probability enjoys the following representation theorem.

**Theorem 5.** For every IVFS-probability \( \mathcal{P} : \text{IVFSs} \rightarrow \text{IVFSs} \), there exist constants \( \beta, \gamma \) and probabilities \( P, R_1, R_2 \) with \( \beta R_1 \leq \gamma R_2 \) such that \( \mathcal{P} \) is given by the following formula:

\[ \mathcal{P}((f_A, g_A)) = \left[ \int_{\Omega} f_A dP + \beta \int_{\Omega} (g_A-f_A) dR_1, \int_{\Omega} f_A dP + \gamma \int_{\Omega} (g_A-f_A) dR_2 \right] \]

**Proof.** Since \( [f_A, g_A] = g((f_A, 1-g_A)), \) then \( \mathcal{P}((f_A, g_A)) = \mathcal{P}(g((f_A, 1-g_A))) = \mathcal{P} \circ g((f_A, 1-g_A)). \) From Theorems 2 and 3, we know that

\[ \mathcal{P} \circ g((f_A, 1-g_A)) = \left[ \int_{\Omega} f_A dP + \beta \int_{\Omega} (g_A-f_A) dR_1, \int_{\Omega} f_A dP + \gamma \int_{\Omega} (g_A-f_A) dR_2 \right] \]

and therefore

\[ \mathcal{P}((f_A, g_A)) = \left[ \int_{\Omega} f_A dP + \beta \int_{\Omega} (g_A-f_A) dR_1, \int_{\Omega} f_A dP + \gamma \int_{\Omega} (g_A-f_A) dR_2 \right]. \]

**4. IVIFS-probabilities**

\[ \forall A = ([f_A, g_A], [h_A, i_A]), B = ([f_B, g_B], [h_B, i_B]) \in \text{IVIFSs}, \] we define

\[ (f_A, g_A), [h_A, i_A]) \cap (f_B, g_B), [h_B, i_B]) = ([f_A, g_A] \cap [f_B, g_B], [h_A, i_A] \cap [h_B, i_B]) \]

\[ ([f_A, g_A], [h_A, i_A]) \cap (f_B, g_B), [h_B, i_B]) = ([f_A, g_A] \cap [f_B, g_B], [h_A, i_A] \cap [h_B, i_B]) \]

\[ ([f_A, g_A], [h_A, i_A]) \leq ([f_B, g_B], [h_B, i_B]) \} \iff [f_A, g_A] \leq [f_B, g_B], [h_A, i_A] \geq [h_B, i_B] \}

**Definition 4.** An IVIFS-probability is a function \( \mathcal{P} : \text{IVIFSs} \rightarrow D[0, 1] \) satisfying the following axioms:

(i) \( \mathcal{P}(([1, 1], [0, 0])) = [1, 1], \mathcal{P}(([0, 0], [1, 1])) = [0, 0] \);
(ii) if \( A \cap B = [0, 0], [1, 1] \), then \( \mathcal{P}(A \cup B) = \mathcal{P}(A) + \mathcal{P}(B) \);
(iii) if \( A_n \nsubseteq A \), then \( \mathcal{P}(A_n) \nsubseteq \mathcal{P}(A) \).
Definition 5. A map $h : IVIFSs \rightarrow IFSs$ is defined as follows:

$$\forall ([A, g_A], [h_A, i_A]) \in IVIFSs, \quad h(([A, g_A], [h_A, i_A])) = \left( \frac{f_A + g_A}{2}, \frac{h_A + i_A}{2} \right).$$

Proposition 1. Let $h$ be the map defined in Definition 5; then $h : (IVIFSs, \oplus, \ominus) \rightarrow (IFSs, \oplus, \ominus)$ is a homomorphism. That is, for all $([A, g_A], [h_A, i_A], B = ([B, g_B], [h_B, i_B]) \in IVIFSs$, we have

$$h(([A, g_A], [h_A, i_A]) \oplus ([B, g_B], [h_B, i_B])) = h(([A, g_A] \oplus [B, g_B], [h_A] \ominus [B, g_B]) \ominus ((h_B, i_B))).$$

Proof. (i) $h(([A, g_A], [h_A, i_A]) \oplus ([B, g_B], [h_B, i_B])) = h(([A, g_A] \oplus [B, g_B], [h_A] \ominus [B, g_B]) \ominus ((h_B, i_B))) = h(([A, g_A] \oplus [B, g_B], [h_A] \ominus [B, g_B]) \ominus ((h_B, i_B))).$

(ii) $h(([A, g_A], [h_A, i_A]) \ominus ([B, g_B], [h_B, i_B])) = h(([A, g_A] \ominus [B, g_B], [h_A] \ominus [B, g_B]) \ominus ((h_B, i_B))).$

Theorem 6. Let $\mathcal{P}$ be an IFS-probability and $h : IVIFSs \rightarrow IFSs$ be a map defined in Definition 5; then $\mathcal{P} \circ h$ is an IVIFS-probability.

Proof. (i) $\mathcal{P} \circ h([1, 0]) = \mathcal{P}(h([1, 0])) = [1, 0].$

(ii) If $([A, g_A], [h_A, i_A], [h_B, i_B], [B, g_B] \in IVIFSs, then $[A, g_A] \ominus [B, g_B], [h_A] \ominus [B, g_B], [h_B, i_B]).$

That is, $f_A \ominus f_B = g_A \ominus g_B = [h_A] \ominus [h_B].$ Finally, for $h([A, g_A], [h_A, i_A]) = \mathcal{P}(h([A, g_A], [h_A, i_A]))) = \mathcal{P}(h([A, g_A], [h_A, i_A])).$

Corollary 1. If $\mathcal{P}$ is an IVIFS-probability, then the map $g : IVIFSs \rightarrow IVIFSs$ is defined in Theorem 1, the map $h : IVIFSs \rightarrow IFSs$ is as defined in Definition 5, then $\mathcal{P} \circ g \circ h$ is an IVIFS-probability.

5. Conclusions

In this paper, the concepts of IVIFS-probability and IVIFS-probability were proposed for the first time. Then on the basis of the new concepts, the relationships between the IFS-probability, IVIFS-probability and IVIFS-probability were discussed. There are still lots of interesting topics along these research lines, for instance, probability theories for IVF-events and IVIF-events, etc. All of these will emerge in forthcoming papers.

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