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MEASURES AND FORKING

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Shelah's theory of forking (or stability theory) is generalized in a way which deals with measures instead of complete types. This allows us to extend the method of forking from the class of stable theories to the larger class of theories which do not have the independence property. When restricted to the special case of stable theories, this paper reduces to a reformulation of the classical approach. However, it goes beyond the classical approach in the case of unstable theories. Methods from ordinary forking theory and the Loeb measure construction from nonstandard analysis are used.

Introduction

In this paper we propose an extension of Shelah's theory of forking which deals with measures instead of complete types. The classical theory of forking has been highly successful in classifying types over models of stable theories. However, since stable theories are quite rare, one would like to apply the method to a larger class of theories. The principal aim of this paper is to apply the method of forking to the larger class of theories which do not have the independence property. In analysis one can often simplify a problem by considering behavior almost everywhere with respect to a measure instead of everywhere. Measures fulfill a similar purpose here. Roughly speaking, the results concern behavior everywhere on the stable part of a model and almost everywhere on the unstable part. A key property of forking for types in a stable theory is that a complete type over a small subset of a model has only a small number of nonforking extensions. With our notion of a nonforking extension of a measure, the analogous property holds for certain measures (called smooth measures) in a theory without the independence property. That is, each smooth measure over a small subset of the model has only a small number of nonforking extensions.

We shall make heavy use of methods from ordinary forking theory and light use of the Loeb measure construction from nonstandard analysis.

While the main focus of this paper is on the case of unstable theories, the approach may still be of interest in the special case of stable theories. In particular, the result in Section 5 that every complete type in a stable theory has a unique definable extension measure appears to have no analogue in the classical theory of forking.

As in the classical theory of forking, we work in a large saturated model M, and consider subsets of M of smaller cardinality. We restrict our consideration to models of countable languages. A complete type p over a subset A of M may be regarded as a finitely additive probability measure on the algebra $F(A)$ of subsets definable over A in which each set has measure zero or one. More generally, we consider arbitrary finitely additive problability measures α over $F(A)$. Such a finitely additive measure on formulas is extended to a countably additive probability measure as follows. We identify a type (set of formulas, possibly uncountable) t over A with the set of elements of M which satisfy t, and define the measure $\alpha(t)$ as the infimum of the measures of the finite conjunctions of formulas in t. Using a construction of Loeb [6], α then has a unique extension to a countably additive probability measure on the σ-algebra generated by the types over $F(A)$. In this extension, various subsets of M which arise in the classical theory of forking are measurable. Among these are the stable part of A, which is the union of all complete stable types over A, and the forking part of A over a subset C, which is the union of the set of all complete types over A which fork over C. The central notions of our theory are the notions of a pure extension of a measure, of a nonforking extension of a measure, and of a smooth measure. Given subsets $A \subset B$ of M, an extension of a measure α over A to a measure β over B is said to be pure if the stable part of α has the same measure as the stable part of β, and nonforking if the forking part of B over A has β-measure zero. An important fact in classical stability theory is that a stable complete type has only a small number of nonforking extensions. In order to have an analogous property for measures in an unstable theory, we need measures which decide everything on the unstable part of A. The smooth measures fill this need. A measure α over A is said to be smooth if for every $B \supset A$, all extensions β of α to B agree on the unstable part of A.

In Section 2 we show that the nonforking relation on the class of smooth measures in an arbitrary theory has the main properties enjoyed by the nonforking relation on the class of complete types in a stable theory. In particular, every smooth measure α over A is a nonforking extension of a smooth measure δ over a countable subset $D \subset A$, and has at least one and at most continuum many smooth nonforking extensions to each set $B \supset A$. In fact, the nonforking relation on smooth measures is characterized by a set of axioms analogous to the axioms of Lascar characterizing the nonforking relation on complete types in a stable theory.

The question of existence of smooth measures is dealt with in Section 3. It is shown that if the theory of M does not have the independence property, then smooth measures exist in abundance. In fact, for a theory without the independence property, every measure α has a smooth extension β over each set $B \supset A$ such that β is pure over α and β is nonforking on the stable part of α. Such an extension β is called a faithful extension of α. Conversely, the only theories such that every measure has a smooth faithful extension are the theories which do not have the independence property. If a measure α is already smooth, then the faithful extensions of α are exactly the nonforking extensions of α.

Section 4 deals with stationary measures, that is, smooth measures which have a unique nonforking extension to every larger set. These measures correspond to the stationary types in classical stability theory.

In Section 5 we show that every smooth measure over A induces in a natural way a unique extension to each set $B \supset A$, called the eventually definable extension. This unique extension β is nonforking and can be characterized in two ways. First, for any $C \supset B$, β has a further extension over C which is preserved under automorphisms of M which fix each element of A. Second, for every $C \supset B$, β has an extension over C which is 'definable' over A in the sense that the probability of each formula is an $F(A)$-measurable function of the parameters occurring in the formula. By contrast, in the classical theory only the stationary complete types have natural unique extensions to sets $B \supset A$. Given a complete type p in a stable theory, it may be useful to consider the unique eventually definable extension of p. In the case that p is not stationary, this extension will be a measure which takes values other than zero and one, rather than a complete type.

In Section 6 we prove the analogue of the symmetry theorem for stationary measures. In order to do this we first extend the results in the preceding sections to smooth measures over the whole model M rather than over small subsets of M. One important fact is that every smooth measure over M is definable over some countable subset of M. Measures definable over countable subsets of M play a key role in this section. For such measures α with special variable x and β with special variable y we define a new measure $[\alpha \times \beta]$ over M with special variables (x, y), called the nonforking product. The $[\alpha \times \beta]$-measure of a formula in two variables is computed as a double integral. $[\alpha \times \beta]$ is again definable over a countable fragment, so the nonforking product operation can be iterated, allowing us to pass from measures with single special variables to measures with tuples of special variables. The nonforking product is associative. When one factor is a smooth measure it is also commutative, that is, the double integral does not depend on the order of integration. This result is an analogue of the Fubini theorem for products of measures, but the formula $\phi(x, y)$ is in general measurable only in an extension of the usual product measure. In general, the nonforking product of two smooth measures over M is not smooth. However, the nonforking product of two faithful extensions of measures over a set C is again faithful over C. By combining results about nonforking products of measures with the symmetry theorem for ordinary complete stable types, we obtain a symmetry theorem for measures.

Other articles in the literature have considered the possibilities of extending the forking relation to unstable theories. The book [10] contains many results on unstable theories. The independence property is introduced there, and it is shown that a countable theory has the independence property if and only if for every infinite cardinal κ, there is the maximum possible number Z^κ of complete types over every model of power κ. This result suggests that theories without the independence property may be more tractable than arbitrary theories. The papers [4] and [8] pay particular attention to the extent to which the theory of forking for complete types applies to unstable theories. This paper is in a similar spirit, but while the previous treatments are noncommital about the unstable part of a model, the measures allow one to study both the stable and unstable parts. The paper [11] has results concerning the classification of the set of unstable theories which do not have the tree property, (this set is disjoint from the set of unstable theories which do not have the independence property). Pillay and Steinhorn [9] obtain results concerning definability and prime models in a special class of unstable theories without the independence property, the 0-minimal theories.

The reader is assumed to have some familiarity with the classical forking theory on complete types, as developed in Shelah [10], Harnik and Harrington [4], or Pillay [7]. We shall make heavy use of results in that classical theory. The basic definitions and notation we need are developed in Section 1. Following [4], we consider types and measures over fragments in M instead of subsets A of M. The usual examples of a fragment are the set of all formulas with parameters in a subset A, and the set of all formulas almost over A. The use of fragments rather than subsets of M is not necessary for the present exposition, but gives more flexibility for possible future applications and eliminates the step from the set A to the algebra of formulas $F(A)$. For an introductory treatment of model theory see Chang and Keisler [1]. For a treatment of standard measure theory see Halmos [3], and for the Loeb construction and nonstandard analysis see Loeb [6], Stroyan and Bayod [12], or Cutland [2].

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1. Preliminaries

We assume throughout that M is a saturated model of a theory T in a countable language L, and $\kappa = |M|$ is an uncountable inaccessible cardinal. A *small* set is a set of power less than κ. Throughout this paper we shall work with a specified tuple $\bar{x}$ of variables called *special variables*. To simplify notation, we work with a single special variable x unless we state otherwise. $F(M)$ is the algebra of formulas of M with at most x free and with parameters in M.

1.1. Definition. Given a small subset G of $F(M)$, the subsets of M defined by finite Boolean combinations of elements of G are called *basic* sets over G. The union of a set of basic sets is said to be *open* over G, and the complement of an open set is said to be *closed* over G. The sets in the σ-algebra σG generated by the open (or closed) sets over G are said to be *Borel* over G.

It follows from saturation that a subset of M is basic over G if and only if it is both open and closed over G.

We shall usually identify formulas with basic sets, and types (sets of formulas) with closed sets. Inclusion relations between types will refer to the corresponding subsets of M rather than sets of formulas. Thus if t and u are types and t is a subset of u as a set of formulas, then $t \supset u$ as subsets of M.

The following result defines a canonical extension of a finitely additive probability measure on G to a countably additive probability measure on σG. The measures obtained in this way are the primary object of study in this paper.

1.2. Theorem. *Any finitely additive probability measure α on a small Boolean closed subset G of $F(M)$ has a unique countably additive extension to the σ-algebra σG such that the measure of an open set over G is the supremum of the measures of its basic subsets. We shall call this extension the measure* generated by *α.*

Proof. Expand M to a κ-saturated structure M' with three new sorts, a sort for subclasses of M satisfying comprehension for formulas of L, a sort for the hyperreal unit interval $*[0, 1]$, and a sort for finitely additive mappings from the classes into $*[0, 1]$. We shall call the elements of M' *internal.* By saturation, α can be extended to an internal finitely additive mapping μ. Using the ω_1-saturation of M', let β be the Loeb measure generated by μ. β is a complete countably additive probability measure such that for each internal class X in M', $\beta(X)$ is the standard part of $\mu(X)$. For each subset Γ of G, we see by saturation that there is an internal class containing the union of Γ whose μ-measure is infinitely close to the supremum of the α-measures of the finite subsets of Γ. Therefore the union of Γ is β-measurable with the required measure. Hence every set in σG is β-measurable and the restriction of β to σG has the required properties. □

As in Harnik and Harrington [4], a *fragment* is a small (that is, of power less than κ) set of formulas of the diagram language of M containing all formulas of L and closed under connectives, quantifiers, and free substitutions of variables. Unless we explicitly mention other variables, we restrict our attention to formulas in F with only the special variable x free. Thus given a fragment F, σF is the set of Borel sets over F, that is, the σ-algebra of subsets of M generated by the types over F. By a *measure over* a fragment F we mean a measure α on σF which is generated (in the sense of Theorem 1.2) by a finitely additive probability measure on the basic sets over F. By a *measure on M* we mean a measure α over some fragment $F(\alpha)$ in M. (Note the use of the phrases 'over F' and 'on M'). A measure α on M is *countably generated* if its fragment $F(\alpha)$ is countable. The *measure algebra* of α is the quotient of α modulo the sets of $\sigma F(\alpha)$ of α-measure zero. By a *complete type* (in the special variable x) over a fragment F we mean a measure over F which only takes on the values 0 and 1. The formulas of measure

1 in a complete type form a complete type in the classical sense, i.e., a maximal consistent set of formulas in the variable x over F. By the *F-interior* of a set $X \subset M$ we mean the union of all open sets over F which are included in X.

A measure β is an *extension* of a measure α, and α is a *submeasure* of β, in symbols $\alpha \subset \beta$ and $\beta \supset \alpha$, if α and β are measures on M, $F(\alpha) \subset F(\beta)$, and α is the restriction $\beta \mid F(\alpha)$ of β to $\sigma F(\alpha)$. The *reduct* of a measure α on M to a sublanguage L' of L is the restriction of α to the part of the fragment $F(\alpha)$ in L'.

The *α-inner measure* of a set $X \subset M$ is defined as the supremum of the α-measures of all α-measurable subsets of X. The *α-outer measure* is defined analogously. If $X, Y \in F(\alpha)$ and $X \subset Y$, we say that X has *full measure* in Y (with respect to α) if $\alpha(X) = \alpha(Y)$. Equivalently, the set $Y - X$ has α-measure zero.

In the next three lemmas we present some basic facts about measures on M which will be needed later.

1.3. Lemma. *Let α be a measure on M and let G be a fragment containing $F(\alpha)$.*

(i) *The α-inner measure of a set $X \subset M$ is the surpremum of the measures of all closed sets C over $F(\alpha)$ such that $C \subset X$. The α-outer measure is the infimum of the measures of all open sets U over $F(\alpha)$ such that $U \supset X$.*

(ii) *For every set $X \in \sigma G$ and every real r between the α-inner and α-outer measures of X, α can be extended to a countably additive probability measure β on σG such that X has β-measure r.* (Note: Here β is not necessarily a measure over G in our sense, for the β-measure of an open set may be greater than the supremum of the β-measures of its basic subsets.)

(iii) *A formula ϕ in $F(M)$ has α-inner measure $>r$ if and only if there is a fomula $\theta \in F(\alpha)$ such that $\theta \subset \phi$ and $\alpha(\theta) > r$. Similarly for outer measure $<r$.*

(iv) *For any formula $\phi \in G$ and any real r between the α-inner and α-outer measure of ϕ, α can be extended to a measure β over G such that $\beta(\phi) = r$.*

(v) *The α-inner measure of an open set X over G is equal to the α-measure of the $F(\alpha)$-interior of X.*

(vi) *For every open set Y over $F(\alpha)$ there is a countable subfragment H of $F(\alpha)$ such that the H-interior of Y has full measure in Y.*

Proof. For (i) it suffices to prove that for every set $X \in \sigma F(\alpha)$, and every real $r > 0$, there is a closed set Y over $F(\alpha)$ such that $Y \subset X$ and $\alpha(Y)$ is within r of $\alpha(X)$. From results in [3], there is a set Z in the algebra generated by the open sets over $F(\alpha)$ such that $Z \subset X$ and $\alpha(Z)$ is within $r/2$ of $\alpha(X)$. Putting Z in disjunctive normal form, we may represent Z as a finite union of sets of the form $A \cap B$ where A is open and B is closed over $F(\alpha)$. By replacing each A by a basic set within $r/2n$ of A, we obtain a closed subset $Y \subset Z$ over $F(\alpha)$ such that $\alpha(Y)$ is within r of $\alpha(X)$.

(ii) follows from classical results in [3].

(iii) follows from (i) and saturation.

For (iv), first use (ii) to extend α to a probability measure δ on σG with

$\delta(\phi)=r$, then restrict δ to the basic sets in G and apply Theorem 1.2 to obtain the required measure β over G.

(v) follows from (i) and saturation.

(vi) By the definition of a measure on M, there is a countable union Z of $F(\alpha)$-basic subsets of Y which has full measure in Y. Let H be a countable subfragment of $F(\alpha)$ which contains each of the formulas used in Z. Then Z is contained in the H-interior of Y, so H has the required property. □

1.4. Definition. A *measurability pattern* is a set of finite positive Boolean combinations of statements asserting that the measure of a formula over M belongs to a closed subinterval of $[0, 1]$. A measurability pattern is *satisfiable* if there is a finitely additive probability measure on the formulas over M which agrees with the pattern.

Notice that a (non-strict) inequality between the measures of two formulas over M can be expressed by a countable measurability pattern. The next lemma is a compactness theorem for measurability patterns.

1.5. Lemma. *If every finite subset of a measurability pattern is satisfiable, then the whole pattern is satisfiable.*

Proof. Expand M by adding a sort for the ordered field of real numbers, a constant symbol for each real, and, for each formula $\phi(x, \bar{y})$ of L, a function symbol $f_\phi(\bar{a})$ for the measure of $\phi(x, \bar{a})$. Let T be the theory consisting of the diagrams of M and the real numbers, and axioms for a finitely additive probability measure on formulas $\phi(x, \bar{a})$. By hypothesis, T is consistent with every finite subset of the given measurability pattern P. By compactness, T has a model M' in which the whole pattern P holds. M' is an elementary extension of M with a finitely additive probability measure from formulas to an elementary extension of the reals. Taking standard parts and restricting to M, we obtain a finitely additive probability measure on the formulas over M which agrees with the pattern P. □

We now apply measurability patterns to prove a useful lemma on extending measures on M. Measurability patterns will be used again in Sections 3 and 5.

1.6. Lemma. *Let α be a measure on M and let G be a fragment containing $F(\alpha)$. Let X_i, $i \in I$, be a chain of open sets over G indexed by a linearly ordered set I. Then α has an extension β over G such that the β-measure of each X_i is equal to its α-inner measure. Moreover, for each set $V \in \sigma F(\alpha)$ and each i, the β-measure of $X_i \cap V$ is equal to its α-inner measure.*

Proof. For each i, let r_i be the α-inner measure of X_i. Let P be the measurability pattern which assigns $\alpha(\phi)$ to each $\phi \in F(\alpha)$ and states that each $\theta \in G$ which is included in X_i has measure $\leq r_i$. We must show that each finite subset of P is satisfiable. By taking unions of basic subsets of X_i, we may restrict our consideration to finite subsets of P which mention at most one basic subset λ_i of X_i for finitely many $i \in I$ and are such that λ_i increases as i increases. Let k be the largest of these i's. By classical measure theory, M can be extended to a probability measure for which λ_k has measure at most r_k and the inner measures of the sets λ_i for $i < k$ are unchanged. Repeating this process finitely many times, we see that the finite subset of P is satisfiable. By Lemma 1.5, P is satisfiable. By Theorem 1.2 there is a measure β over G which agrees with P and therefore is an extension of α such that each X_i has β-measure r_i. For each $V \in \sigma F(\alpha)$ and each i, the sum of the α-inner measures of $X_i \cap V$ and $X_i - V$ is r_i, and it follows that the β-measure of $X_i \cap V$ must equal its α-inner measure. □

For the remainder of this section we shall review some notions from classical forking theory, introduce corresponding notions for measures on M, and prove some basic lemmas which will be used later on. We shall now introduce the stable part of a fragment F. We always let Δ, or $\Delta(x, \bar{y})$, denote a finite set of formulas $\phi(x, \bar{y})$ of $L(M)$, with special variable x and parameter variables $\bar{y}$. A *Δ-type* in the variable x is a type consisting of formulas $\phi(x, \bar{b})$ where ϕ or its negation belongs to Δ. A *complete* Δ-type over a fragment F is a maximal consistent Δ-type of formulas over F. If p is a complete type over F, $p \mid \Delta$ denotes the complete Δ-type obtained by restricting p to formulas $\phi(x, \bar{b})$ where ϕ or its negation belongs to Δ. Let t be a type in x over F. The (Morley) *Δ-rank* of t is defined inductively as follows. t has Δ-rank ≥ 0 if and only if t is consistent. t has Δ-rank $\geq n+1$ if and only if for every finite subset u of t (as a set of formulas, hence $u \supset t$ as a subset of M), and every $m \in N$, there are at least m formally disjoint Δ-types q over some $G \supset F$ such that $u \cap q$ has Δ-rank $\geq m$. (Two types are formally disjoint if for some formula ϕ, ϕ belongs to one type and $\neg\phi$ to the other.) t is Δ-stable if the Δ-rank of t is finite. t is said to be *stable* if t is Δ-stable for every finite $\Delta \subset L$. If t is stable, then t is Δ-stable even for every finite $\Delta \subset L(M)$. Similar notation is used for an n-tuple $\bar{x}$ in place of the variable x. A formula $\phi(x, \bar{y})$ of $L(M)$ is *stable* if every type t is $\{\phi\}$-stable.

The *Δ-stable part* of a fragment F, denoted by $\mathrm{sbl}(\Delta, F)$, is the union of all Δ-stable basic types (or formulas) over F. For each n and Δ, $\mathrm{sbl}(n, \Delta, F)$ will denote the union of all basic types over F of Δ-rank at most n. Thus $\mathrm{sbl}(\Delta, F)$ is the union of $\mathrm{sbl}(n, \Delta, F)$ over n. Moreover, $\mathrm{sbl}(\Delta, F)$ and each $\mathrm{sbl}(n, \Delta, F)$ is open over F. The *stable part* of F, $\mathrm{sbl}(F)$, is the intersection of the sets $\mathrm{sbl}(\Delta, F)$ for all finite $\Delta \subset L$. Then $\mathrm{sbl}(F) \subset \mathrm{sbl}(\Delta, F)$ even for $\Delta \subset L(M)$. The complement of the stable part of F is the *unstable part* $\mathrm{usbl}(F)$. Similarly, the complement of $\mathrm{sbl}(\Delta, F)$ is denoted by $\mathrm{usbl}(\Delta, F)$. For a measure α on M, we put $\mathrm{sbl}(\alpha) = \mathrm{sbl}(F(\alpha))$, etc.

1.7. Lemma. *Let α be a measure on M, let $\Delta \subset F(\alpha)$, and let ϕ be a Δ-stable* *basic type over $F(\alpha)$. Then the union of the sets $\phi \cap p$, where p is a complete Δ-type over $F(\alpha)$ and $\phi \cap p$ has positive measure, has full measure in ϕ.*

Proof. We argue by induction on the Δ-rank of ϕ. Suppose the result holds for Δ-rank less than n, and ϕ has Δ-rank n. Then ϕ is the union of the open set $\phi \cap \mathrm{sbl}(n-1, \Delta, F(\alpha))$ and finitely many sets of Δ-rank n of the form $\phi \cap p$ where p is a complete Δ-type over $F(\alpha)$. The inductive hypothesis applies to each basic subset of $\phi \cap \mathrm{sbl}(n-1, \Delta, F(\alpha))$, and by the definition of measure on M, there is a union of countably many basic subsets which has full measure in $\phi \cap \mathrm{sbl}(n-1, \Delta, F(\alpha))$. It follows that the union of the sets $\phi \cap p$ of positive measure where p is a complete Δ-type over $F(\alpha)$ has full measure in ϕ, as required. □

1.8. Corollary. *Let α be a measure on M and let $\Delta \subset F(\alpha)$. Then the union of the sets* $\mathrm{sbl}(\Delta, \alpha) \cap p$ *of positive measure, where p is a complete Δ-type over $F(\alpha)$, has full measure in* $\mathrm{sbl}(\Delta, \alpha)$.

Proof. $\mathrm{sbl}(\Delta, \alpha)$ is an open set over $F(\alpha)$, so there is a countable union of Δ-stable formulas over $F(\alpha)$ which has full measure in $\mathrm{sbl}(\Delta, \alpha)$. A complete Δ-type p over $F(\alpha)$ has positive measure in $\mathrm{sbl}(\Delta, \alpha)$ if and only if it has positive measure in one of the sets of the countable union. The result now follows from Lemma 1.7. □

1.9. Lemma. (i) *If t and u are types and $t \subset u$, then for each finite Δ the Δ-rank of t is $\leqslant$ the Δ-rank of u.*

(ii) *If t is a type over F, then for each $\Delta \subset F$ there is a formula ϕ over F such that $t \subset \phi$ and the Δ-rank of ϕ equals the Δ-rank of t.*

(iii) *The sets* $\mathrm{sbl}(n, \Delta, F)$ *and the Δ-stable part of a fragment F are open over F and the stable part of F is Borel over F.*

(iv) *If G is an extension of F, then for each n and $\Delta \subset F$,* $\mathrm{sbl}(n, \Delta, F)$ *is the F-interior of* $\mathrm{sbl}(n, \Delta, G)$, $\mathrm{sbl}(\Delta, F)$ *is the F-interior of* $\mathrm{sbl}(\Delta, G)$, *and* $\mathrm{sbl}(F)$ *is a subset of* $\mathrm{sbl}(G)$.

(v) *A type t over F is Δ-stable if and only if $t \subset \mathrm{sbl}(\Delta, F)$, and has Δ-rank at most n if and only if $t \subset \mathrm{sbl}(n, \Delta, F)$.*

These results are implicit in [10].

We shall now introduce two kinds of well behaved extensions of measures on M, the pure extensions, which are well behaved on the unstable part of α, and the nonforking extensions, which are well behaved on the stable part of α. We begin with the pure extensions, which are extensions which do not increase the measure of the stable part. At the same time we introduce the stronger notion of a locally pure extension, which is an extension which does not increase the measure of any Δ-stable part.

1.10. Definition. A measure β is a *pure* extension of α if β is an extension of α and the stable part of α has full measure in the stable part of β. That is, the set $\mathrm{sbl}(\beta) - \mathrm{sbl}(\alpha)$ has β-measure zero. We say that β is pure over a fragment F if β is pure over its restriction to F.

β is a *locally pure* extension of α if β is an extension of α such that for each $\Delta \subset L$, the Δ-stable part of α has full measure in the Δ-stable part of β. That is, the union over $\Delta \subset L$ of the sets $\mathrm{sbl}(\Delta, \beta) - \mathrm{sbl}(\Delta, \alpha)$ has β-measure zero.

1.11. Lemma. (i) *Every locally pure extension of a measure α on M is a pure extension of α.*

(ii) *Every measure β on M is a locally pure extension of some countably generated submeasure.*

(iii) (Transitivity Properties). *If $\alpha \subset \mu \subset \beta$, then β is pure over α if and only if μ is pure over α and β is pure over μ. Similarly for locally pure.*

(iv) *β is a pure extension of α if and only if there is a restriction α_0 of α to a countable fragment such that every countable extension of α_0 which is included in β is a pure extension of α_0. Similarly for locally pure.*

(v) *The union of a chain of locally pure extensions of a measure α on M is a locally pure extension of α.*

(vi) *If L' is a sublanguage of L and β is a locally pure extension of α, then the L'-reduct of β is a locally pure extension of the L'-reduct of α.*

Proof. (i) If β is locally pure over α, then for each finite $\Delta \subset L$ the sets $\mathrm{sbl}(\Delta, \beta)$ and $\mathrm{sbl}(\Delta, \alpha)$ differ by a set of β-measure zero, and therefore the respective intersections over Δ differ by a set of β-measure zero, whence β is pure over α.

(ii) By 1.3(vi) and the fact that the Δ-stable part of β is open, for each finite Δ there is a countable subfragment G_Δ of $F(\beta)$ such that the Δ-stable part of G_Δ has full measure in the Δ-stable part of $F(\beta)$. Let G be the union of the fragments G_Δ. Then for each Δ, the Δ-stable part of $\beta \mid G$ has full measure in the Δ-stable part of β. Therefore β is a locally pure extension of $\beta \mid G$.

(iii) This is straightforward.

(iv) First let β be a pure extension of α. by (ii) there is a countably generated submeasure α_0 of α such that α is a pure extension of α_0. It follows from (iii) that every measure between α_0 and β is a pure extension of α_0. Conversely, suppose α_0 is a countably generated submeasure of α such that every countably generated measure between α_0 and β is a pure extension. By (i) there is a countably generated submeasure β_0 of β such that β is a pure extension of β_0. Then the union of α_0 and β_0 is a pure extension of α_0 and a pure submeasure of β, and by (ii), β is a pure extension of α.

(v) follows from the fact that if G_n is an increasing chain of fragments containing F, and G is the union of the chain, then for each Δ, $\mathrm{sbl}(\Delta, G)$ is the union of the sets $\mathrm{sbl}(\Delta, G_n)$.

(vi) follows from the corresponding result for complete types. □

The following example shows that in the preceding theorem, parts (v) and (vi) on unions of chains and reducts do not hold for pure extensions. Let T be the theory in a language with countably many binary relation symbols R_n, $n \in N$, such that each R_n is a dense pre-linear ordering without endpoints, and for any elements $a_1, \ldots, a_k$ and distinct $n_1, \ldots, n_k \in N$, the intersection of the equivalence classes $\{x : xR_{ni}a_i \,\&\, a_iR_{ni}x\}$ is dense in all the other relations R_n, $n \neq n_1, \ldots, n_k$. Let t be a complete type over an elementary submodel of M which has no equivalence class for any R_n. Let t_n be an increasing chain of complete types extending t such that t_n has an R_n equivalence class, but does not have an R_{n+1} equivalence class. Then each t_n is in the stable part of R_n but in the unstable part of R_{n+1}. It follows that each t_n is a pure extension of t, but the union of the t_n's is not pure over t. Moreover, the reduct of t_n to R_n is not pure over the reduct of t to R_n.

We are now ready to introduce the notion of forking. The following definition of a complete type q over G forking over F is from [4, Theorem 8.10]. For the stable case it is equivalent to the original definition in [10].

1.12. Definition. Let p be a complete type over a fragment F and let G be a fragment containing F. We shall say that an extension q of p to a complete type over G is *nonforking* over F if for each stable formula $\delta(x, \bar{y}) \in F$ and each element $b \in M$, if $q \models \delta(x, \bar{b})$ then the δ-rank of $p \cap \delta(x, \bar{b})$ is equal to the δ-rank of p. If q is not nonforking over F we say that q *forks* over F.

By the *forking part* of G over F, denoted by $\mathrm{fk}(G, F)$, we mean the union of all complete types q over G such that q forks over F. Given measures $\alpha \subset \beta$, we shall write $\mathrm{fk}(\beta, \alpha)$ for $\mathrm{fk}(F(\beta), F(\alpha))$, $\mathrm{fk}(G, \alpha)$ for $\mathrm{fk}(G, F(\alpha))$, and so forth.

Given a finite set Δ of formulas in F, we denote by $\mathrm{fk}(\Delta, G, F)$ the union of all complete types q over G such that the Δ-rank of q is less than the Δ-rank of $q \restriction F$.

1.13. Lemma. *Let F and G be fragments with $F \subset G$.*

(i) $\mathrm{fk}(G, F)$ *is open over* G.

(ii) $\mathrm{fk}(G, F) \cap \mathrm{sbl}(F) = \bigcup \{\mathrm{fk}(\Delta, G, F) : \Delta \subset L\} \cap \mathrm{sbl}(F)$

$$= \bigcup \{\mathrm{fk}(\Delta, G, F) : \Delta \subset F\} \cap \mathrm{sbl}(F).$$

(iii) *Every complete type p over F has a nonforking extension to a complete type q over G.*

(iv) $\mathrm{fk}(G, F) \subset \bigcup \{\mathrm{fk}(\Delta, G, F) : \Delta \subset F$ *and* Δ *is stable*$\}$.

These results (i) and (ii) are from [10], and (iii) is from [4, Proposition 8.9]. (iv) follows easily from the definitions. We now introduce the notion of a nonforking extension of a measure on M.

1.14. Definition. Let β be a measure on M. β is a *nonforking extension* of a measure α on M if the forking part of $F(\beta)$ over $F(\alpha)$ has β-measure zero,

$$\beta[\text{fk}(\beta, \alpha)] = 0.$$

1.15. Lemma. (i) *Every measure β on M is a nonforking extension of some countably generated submeasure of β.*

(ii) (Transitivity). *Let $\alpha \subset \mu \subset \beta$. If β is nonforking over μ and μ is nonforking over α, then β is nonforking over α. If β is nonforking over α, then μ is nonforking over α.*

(iii) *β is a nonforking extension of α if and only if there is a countably generated submeasure α_0 of α such that every countably generated submeasure β_0 of β which contains α_0 is nonforking over α_0.*

(iv) *If β is nonforking over its restriction to F and μ is a measure over $F(\beta)$ which is absolutely continuous with respect to β (that is, every set of β-measure zero has μ-measure zero), then μ is nonforking over its restriction to F.*

Proof. (i) By 1.3(vi) there is a countable subfragment H of $F(\beta)$ such that the H-interior of sbl(n, Δ, β) has full measure in sbl(n, Δ, β) for each n and $\Delta \subset H$. By Lemma 1.9(iv), sbl(n, Δ, H) is the H-interior of sbl(n, Δ, β). It follows that for each $\Delta \subset H$, the set fk(Δ, β, H) has β-measure zero. Therefore by 1.13(iv), β is a nonforking extension of $\beta \mid H$.

(ii) follows from the corresponding result for complete types.

The proof of (iii) is similar to the proof of 1.11(iv).

(iv) follows because if fk(G, α) has β-measure zero and μ is absolutely continuous with respect to β, then fk(G, α) has μ-measure zero. □

1.16. Lemma. *Let α be a measure on M.*

(i) *The union of a chain of nonforking extensions of α is nonforking over α.*

(ii) *Let L' be a sublanguage of L. If β is a nonforking extension of α, then the L'-reduct of β is a nonforking extension of the L'-reduct of α.*

Proof. (i) It is easy to check that if G_n is an increasing chain of fragments containing F, then

$$\bigcup \text{fk}(G_n, F) = \text{fk}(\bigcup G_n, F).$$

(ii) follows from the corresponding result for complete types. □

1.17. Lemma. *For every measure α on M and every fragment G containing $F(\alpha)$, the forking part of G over $F(\alpha)$ has α-inner measure zero.*

Proof. Let Y be closed over $F(\alpha)$ and $Y \subset \text{fk}(G, \alpha)$. Then Y is a union of complete types over $F(\alpha)$. By Lemma 1.13(iii), every complete type over F has a nonforking extension to G, and thus is not contained in fk(G, α). Therefore Y is empty, and fk(G, α) has α-inner measure zero. □

1.18. Theorem. *For every measure α on M and fragment $G \supset F(\alpha)$, α has a nonforking extension to G.*

Proof. By lemmas 1.6, 1.13(i), and 1.17, α has an extension β to G such that $\mathrm{fk}(G, \alpha)$ has β-measure zero. □

1.19. Definition. A fragment G is a *countable base* for a complete type t over a fragment F if G is countable, $G \subset F$, and for every $\Delta \subset G$, the restriction of t to G has the same Δ-rank and Δ-multiplicity as t. G is a *countable base* for a measure α on M if G is countable, $G \subset F(\alpha)$, and the set of $x \in M$ such that G is a countable base for the complete type of x over $F(\alpha)$ has α-measure one.

If G is a countable base for α, then for each $\Delta \subset G$ the set $\mathrm{fk}(\Delta, \alpha, G)$ has α-measure zero and hence α is a locally pure nonforking extension of $\alpha \mid G$. It follows from the transitivity property that the union of any countable chain of countable bases for α is also a countable base for α.

1.20. Proposition. *Every measure α on M has a countable base.*

Proof. By Lemma 1.11 and the proof of Lemma 1.15(iv), $F(\alpha)$ has a countable subfragment G such that for each $\Delta \subset G$, the set $\mathrm{fk}(\Delta, \alpha, G)$ has α-measure zero. There is a countable subfragment H of $F(\alpha)$ such that $G \subset H$ and for each finite $\Delta \subset G$ and each formula ϕ over H of finite Δ-rank n, there are the same number of complete Δ-types t with $t \cap \phi$ of rank n over H as over $F(\alpha)$. Repeating this construction countably many times and taking the union, we obtain a countable base for α. □

We shall say that two measures β and μ over G agree on a Borel set X over G if for every $\phi \in G$, $\beta(\phi \cap X) = \mu(\phi \cap X)$.

1.21. Theorem. *Let α be a measure on M and let G be a fragment containing $F(\alpha)$. There are at most continuum many nonforking extensions of α to G which disagree on the stable part of $F(\alpha)$.*

Proof. By the preceding results, we may assume without loss of generality that $F(\alpha)$ is countable and G is generated by a small elementary submodel N of M. Suppose β and μ are nonforking extensions of α over G. Let Δ be a finite subset of $F(\alpha)$. Suppose that for each Δ-stable formula $\phi \in F(\alpha)$ and each complete Δ-type p over G such that $\phi \cap p$ has the same Δ-rank as ϕ, $\beta(\phi \cap p) = \mu(\phi \cap p)$. Using Lemma 1.7, we shall show by induction on the Δ-rank of ϕ that for each Δ-stable formula $\phi \in F(\alpha)$ and each $\delta(x, \bar{b}) \in G$ with $\delta(x, \bar{y}) \in \Delta$,

$$\beta(\phi(x) \mathbin{\&} \delta(x, \bar{b})) = \mu(\phi(x) \mathbin{\&} \delta(x, \bar{b})). \tag{1}$$

Assume this holds for formulas of Δ-rank less than n, and let ϕ have Δ-rank n. By the inductive hypothesis,

$$\beta(\phi(x) \,\&\, \delta(x, \bar{b}) \cap \mathrm{sbl}(n-1, \alpha)) = \mu(\phi(x) \,\&\, \delta(x, \bar{b}) \cap \mathrm{sbl}(n-1, \alpha)).$$

There are only finitely many complete Δ-types $p_1, \ldots, p_k$ over G such that $\phi \cap p_i$ has the same Δ-rank as ϕ. Since $\phi(x)$ is Δ-stable, the formula $\phi(x) \,\&\, \delta(x, \bar{y})$ is stable for each $\delta \in \Delta$. It follows from the definition of $\mathrm{fk}(G, \alpha)$ that

$$[\phi \cap \mathrm{usbl}(n-1, \Delta, \alpha)] - [p_1 \cup \cdots \cup p_k] \subset \mathrm{fk}(G, \alpha),$$

so the above set has β-measure zero and μ-measure zero. By our hypothesis on β and μ, $\beta(\phi \cap p_i) = \mu(\phi \cap p_i)$ for each $i \leqslant k$. This shows that formula (1) holds for all $\delta(x, \bar{b})$. Our induction is complete.

It follows that for each $\delta(x, \bar{b}) \in G$ with $\delta(x, \bar{y}) \in \Delta$,

$$\beta(\mathrm{sbl}(\Delta, \alpha) \cap \delta(x, \bar{b})) = \mu(\mathrm{sbl}(\Delta, \alpha) \cap \delta(x, \bar{b}).$$

If the assumption on β and μ holds for all Δ, then β and μ agree on $\mathrm{sbl}(\alpha)$. Moreover, there are only countably many formulas in $F(\alpha)$, and for each $\Delta \subset F(\alpha)$ and Δ-stable formula $\phi \in F(\alpha)$ there are only finitely many complete Δ-types p over G such that $\phi \cap p$ has the same Δ-rank as ϕ. This shows that there are at most continuum many nonforking extensions of α to G which disagree on $\mathrm{sbl}(\alpha)$. □

2. Smooth measures

In this section we introduce the notion of a smooth measure and study the nonforking relation on the class of smooth measures on M. We shall see that this relation enjoys the same properties as the nonforking relation on the complete types of a stable theory, and can be characterized by a set of axioms in a manner analogous to Lascar's characterization (see [4]) of the nonforking relation on complete stable types. In addition, we show that a smooth measure on M has a countably generated measure algebra.

2.1. Definition. A measure α on M is *smooth* if for every fragment $G \supset F(\alpha)$, any two extensions of α to measures over G agree on the unstable part of α.

For example, any measure α on M whose unstable part has measure zero is smooth. In particular, in a stable theory every measure on M is smooth. At the other extreme among theories without the independence property is the theory DLO of dense linear order without endpoints, where the stable part of a measure α over a fragment F consists only of the elements of M which are definable over F. It can be seen using elimination of quantifiers that a measure α over a fragment F in DLO is smooth if and only if every complete type over F which is an infinite subset of M has α-measure zero. For example, the usual length measure on the rational subintervals of $[0, 1]$ is smooth. Moreover, if we identify $[0, 1]$ with a subset of the saturated model M, then any countably additive probability measure on the Borel subsets of $[0, 1]$ induces a smooth measure over the fragment $F([0, 1])$ generated by $[0, 1]$. However, the measure over $F([0, 1])$ which assigns measure one to the set of positive infinitesimals is not smooth.

We now give some examples in which both the stable and unstable parts of a fragment F are nontrivial. Let T be the theory of models M with a unary predicate U and a binary relation R such that R is a dense linear order on U and an equivalence relation with infinitely many infinite classes on the complement of U. Then the stable part of a fragment F consists of the elements of U which are definable in F and the complement of U. A measure over F will be smooth if and only if every complete type over F which is an infinite subset of U in M has measure zero. In particular, given any smooth measures α and β over F such that $\alpha(U) = 1$ and $\beta(U) = 0$, any linear combination $r \cdot \alpha + (1 - r) \cdot \beta$, $0 \leqslant r \leqslant 1$, will be a smooth measure over F in which U has measure r.

Let T be the theory with two binary relations E and R such that E is an equivalence relation with infinitely many equivalence classes, R is a dense linear ordering of the universe without endpoints, and each equivalence class of E is dense in R. Then an extension of a measure α over a fragment F is nonforking if and only if there are no new equivalence classes or elements of positive measure. A measure α over F is smooth if and only if the R-reduct of α is smooth and the union of the equivalence classes definable in F with positive α-measure has α-measure one.

We shall use the notation $X \triangle Y$ for the symmetric difference of the sets X and Y.

2.2. Lemma. *Let α be a measure on M. The following are equivalent.*

(i) *α is smooth.*

(ii) *For every formula ϕ in $F(M)$, the α-inner and α-outer measures of $\phi \cap \mathrm{usbl}(\alpha)$ are equal.*

(iii) *For every formula ϕ in $F(M)$ and each $r > 0$ and finite Δ, there are formulas θ, π, and δ in $F(\alpha)$ such that $\alpha(\pi) < r$, δ is Δ-stable, and $\theta \triangle \phi \subset \pi \cup \delta$.*

(iv) *For every formula $\phi(x, \bar{y})$ of L and each $r > 0$ and finite Δ there are a Δ-stable formula δ in $F(\alpha)$ and finite sequences of formulas θ_m, π_m, in $F(\alpha)$, $m = 1, \ldots, k$, such that $\alpha(\pi_m) < r$, and for each $\bar{b}$ in M there exists $m \leqslant k$ such that*

$$\theta_m \triangle \phi(x, \bar{b}) \subset \pi_m \cup \delta.$$

Proof. By Lemma 1.6, for each fragment $G \supset F(\alpha)$ and each formula $\phi \in G$, α

has an extension β over G such that $\beta(\phi \cap \mathrm{usbl}(\alpha))$ is equal to the α-inner measure of $\phi \cap \mathrm{usbl}(\alpha)$, and another extension β' such that $\beta'(\phi \cap \mathrm{usbl}(\alpha))$ is equal to the α-outer measure of $\phi \cap \mathrm{usbl}(\alpha)$. If α is smooth, then α and β' must agree on $\phi \cap \mathrm{usbl}(\alpha)$, so (i) implies (ii). The remaining implications follow from saturation, the fact that a finite disjunction of Δ-stable formulas is Δ-stable, and Lemma 1.3. □

2.3. Lemma. (i) *Every extension of a smooth measure on M is smooth.*

(ii) *A complete type over a fragment F is smooth if and only if it is stable.*

Proof. (i) follows from the fact that if $\alpha \subset \beta$, then $\mathrm{usbl}(\alpha) \supset \mathrm{usbl}(\beta)$. Since an unstable complete type has more than one extension to a complete type over some fragment which properly contains F, (ii) holds. □

2.4. Lemma. *Let α be a smooth measure on M and let β be an extension of α.*

(i) *β is a locally pure extension of α.*

(ii) *For each $\Delta \subset L$, the set* $\mathrm{fk}(\Delta, \beta, \alpha)$ *has β-measure zero in the unstable part of α.*

(iii) *If β is nonforking over α if and only if for each $\Delta \subset F(\alpha)$ the set* $\mathrm{fk}(\Delta, \beta, \alpha)$ *has β-measure zero, and also if and only if for each $\Delta \subset L$ the set* $\mathrm{fk}(\Delta, \beta, \alpha) \cap \mathrm{sbl}(\alpha)$ *has β-measure zero.*

Proof. Suppose (i) fails. Then there is a $\Delta \subset L$ and a Δ-stable formula δ over β of positive measure in the Δ-unstable part of α. By Lemma 2.2(ii), the intersection of δ with the Δ-unstable part of α has the same α-inner and α-outer measures. Thus by Lemma 1.3(i) and saturation there is a formula $\theta \in F(\alpha)$ which has positive measure in the Δ-unstable part of α and implies δ. Since δ is Δ-stable, θ is Δ-stable, contradicting the fact that θ has positive measure in the Δ-unstable part of α. This proves (i).

(ii) Suppose that the set $\mathrm{fk}(\Delta, \beta, \alpha)$ has positive β-measure in $\mathrm{usbl}(\alpha)$. Then for some n, the set

$$Y = (\mathrm{sbl}(n, \Delta, \beta) - \mathrm{usbl}(n, \Delta, \alpha)) \cap \mathrm{usbl}(\alpha)$$

has positive measure r. By Lemma 2.2(ii) the set Y has α-inner measure r, so there is a closed set $C \subset Y$ over $F(\alpha)$ of measure $>r/2$. Since $Y \subset \mathrm{sbl}(n, \Delta, \beta)$ and $\mathrm{sbl}(n, \Delta, \beta)$ is open over β, there is a basic set θ over $F(\alpha)$ with $C \subset \theta \subset \mathrm{sbl}(n, \Delta, \beta)$. But since $\theta \in F(\alpha)$, $\theta \subset \mathrm{sbl}(n, \Delta, \alpha)$, and thus C is empty, contradicting the fact that C has β-measure $>r/2$.

(iii) now follows by Lemma 1.13. □

By a *smooth countable base* for a measure β on M we mean a countable base G for β such that the restriction of β to G is smooth.

2.5. Proposition. *Every smooth measure β on M has a countable smooth base.*

Proof. Let β be smooth. We first show that β has a smooth countably generated submeasure. Let G be a countable subfragment of $F(\beta)$ which contains all the formulas θ_m, π_m, δ_m of Lemma 2.2 for each rational $r > 0$, each n, and each $\phi(x, \bar{y})$. Let $\mu = \beta \mid G$. By Lemma 2.2, μ is smooth and countably generated. By Lemma 1.20, β has a countable base H. Let J be a countable fragment such that $G \cup H \subset F(\beta)$. By Lemma 2.3(i), $\beta \mid J$ is smooth and J is a countable base for β. □

2.6. Proposition. *Every smooth measure has a countably generated measure algebra.*

Proof. Suppose the measure algebra of α is not countably generated. Then there is a sequence ϕ_i, $i < \omega_1$, of formulas over $F(\alpha)$ such that no ϕ_i is within a null set of the σ-algebra A_i generated by the preceding sets. Since the language L is countable, we may asume that all the ϕ_i are of the form $\phi(x, \bar{a}_i)$, that is, definable from the same formula ϕ. We may then find a positive real s and an infinite set of i's such that the measures of the symmetric differences of the $\phi(x, \bar{a}_i)$ are at least s.

Let r be a positive real number. Call a formula $\delta \in F(\alpha)$ *r-large* if there is an infinite set of i's such that the measures of the symmetric differences of the $\phi(x, \bar{a}_i)$ intersected with δ are at least r. By Ramsey's theorem, for each δ either δ or $\neg\delta$ is $s/2$-large. We claim that if δ is r-large for some positive r, then δ is not ϕ-stable. The proof is by induction on the ϕ-rank of δ. Suppose δ is an r-large ϕ-stable basic set over $F(\alpha)$ of ϕ-rank n. Then δ contains only finitely many complete ϕ-types of ϕ-rank n, so infinitely many of the $\phi(x, \bar{a}_i)$ must contain the same complete ϕ-types in δ of ϕ-rank n. There is a formula π of ϕ-rank less than n which is within $r/2$ of the open set $\mathrm{sbl}(n-1, \phi, \alpha) \cap \delta$. Whenever $i \neq j$ and $\phi(x, \bar{a}_i)$ and $\phi(x, \bar{a}_j)$ agree on complete ϕ-types in δ of rank n, the set $(\phi(x, \bar{a}_i) \triangle \phi(x, \bar{a}_j)) \cap \pi$ will have measure at least $r/2$. Then π is $r/2$-large and of ϕ-rank less than n. This proves the claim.

We conclude that for every ϕ-stable basic set δ over $F(\alpha)$, $\neg\delta$ is $s/2$-large. By taking δ of measure within $s/4$ of the ϕ-stable part of $F(\alpha)$, we obtain an infinite set of $\bar{a}_i$ such that the measures of the symmetric differences of the $\phi(x, \bar{a}_i)$ are bounded away from zero in the ϕ-unstable part of $F(\alpha)$.

By saturation, for any cardinal $\lambda < \kappa$, α has an extension β with λ formulas of the form $\phi(x, \bar{c})$ whose symmetric differences have β-measures bounded away from zero in the ϕ-unstable part of $F(\alpha)$. For λ large enough, two of these formulas must have the same Borel subsets and supersets in $\sigma F(\alpha)$. Therefore within the unstable part of α, the α-inner measure of $\phi(x, \bar{c})$ is less than the α-outer measure of $\phi(x, \bar{c})$. Hence the unstable part of α has more than one extension to $F(\beta)$. It follows that α is not smooth. □

2.7. Theorem. *Let α be a smooth measure on M and let G be a fragment containing $F(\alpha)$. Then α has at most continuum many nonforking extensions to G.*

Proof. This follows from Theorem 1.21, since by definition any two nonforking extensions of α agree on the unstable part of α. □

At this point let us recall the notions of elementary maps and conjugate formulas and types. If $F \subset G$ are fragments, an *F-elementary map* on G is an automorphism f of M such that the induced mapping on formulas maps G onto G and is the identity on F. Two formulas $\phi(x, \bar{b})$ and $\phi(x, \bar{c})$ in G are said to be *F-conjugates* on G if they are images of each other under F-elementary maps on G. Types over G which are F-conjugate are defined analogously. It is easily seen that if p is a complete Δ-type over G, then any F-conjugate of p on G is either equal to p or disjoint from p.

The following lemma is related to various results in the literature, for example in [10, Section III.2], and [4, p. 266]. We give a proof here because we need the result locally for $\Delta \subset L$ in a possibly unstable theory. We refer to [1, Chapter 5] for a treatment of special models.

Definition. By a *special fragment* over F we mean a fragment $G \supset F$ which is generated by a special elementary submodel N of M whose power has cofinality λ, where $\text{card}(F) < \lambda < \kappa$.

2.8. Lemma. *Let Δ be a finite set of formulas of L, let F be a small fragment, and let G be a special fragment over F.*

(i) *If $a \in \text{sbl}(\Delta, G)$ and the complete Δ-type of a over G has infinitely many F-conjugates on G, then $a \in \text{fk}(\Delta, G, F)$.*

(ii) *If $a \in \text{fk}(\Delta, G, F)$, then the complete Δ-type of a over G has at least λ F-conjugates on G.*

Proof. (i) Suppose the complete Δ-type p of a over G has infinitely many F-conjugates on G. Let r be the complete type of a over F. Then for any F-conjugate p' of p on G, $r \cap p$ and $r \cap p'$ have the same Δ-rank. If r has infinite Δ-rank, then $a \in \text{usbl}(\Delta, F) \cap \text{sbl}(\Delta, G)$, so $x \in \text{fk}(\Delta, G, F)$. If r has finite Δ-rank, then there are only finitely many complete Δ-types q over G such that $r \cap q$ has the same Δ-rank as r. It follows that $r \cap p$ has smaller Δ-rank than r, and hence

$$a \in r \cap p \subset \text{fk}(\Delta, G, F).$$

(ii) Now suppose that $a \in \text{fk}(\Delta, G, F)$. We may assume without loss of generality that F is generated by an elementary submodel N_0 of N. Let p be the complete type of a over F. Then for some finite Boolean combination $\delta(x, y)$ of formulas in Δ and some $\bar{c}$ in N, $\delta(a, \bar{c})$ holds and $p \cap \delta(x, \bar{c})$ has smaller Δ-rank than p. By transfinite recursion we may construct sequences

$$f_i,\ M_i,\quad i<\lambda,$$

such that f_0 is the identity map on M, $M_0 = N_0$, and:

(1) The M_i form an increasing chain of elementary submodels of N.

(2) For each $i<\lambda$, f_i is an F-elementary map on G.

(3) The tuple $f_i(\bar{c})$ belongs to M_{i+1}.

(4) The complete type of $f_i(a)$ over the fragment generated by M_i has the same Δ-rank as the complete type of $f_i(a)$ over F.

Then for each $j<\lambda$, we have $\delta(f_j(a), f_j(\bar{c}))$ but $\neg\delta(f_j(a), f_i(\bar{c}))$ for all $i<j$. Therefore the complete Δ-type of a over G has λ distinct F-conjugates on G. □

We shall now show that the nonforking relation has two more properties, convexity and closedness in a certain topology, and then give an axiomatic characterization of the nonforking relation.

2.9. Definition. A *neighborhood* of a measure μ over F is a set $N(r, X_1, \ldots, X_n)$, where $r>0$ and each $X_k \in \sigma F$, consisting of all measures β over F such that $|\beta(X_k) - \mu(X_k)| < r$ for $k = 1, \ldots, n$. This induces a topology on the set of all measures over F. A set of measures over F is *closed* if it is closed in this topology. A set of measures over F is *convex* if it is closed under sums $a\mu + b\beta$ where $0 \leqslant a$, $0 \leqslant b$, and $a+b=1$. If usbl(α) has positive measure, the *unstable piece* of α is the restriction of α to usbl(α) divided by the measure of usbl(α).

2.10. Lemma. *α is smooth if and only if either* usbl(α) *has measure zero or the unstable piece of α is smooth. The set of smooth measures over F is convex. The set of smooth measures over F is closed.*

Proof. We show that the set of smooth measures over F is closed. Suppose α is a measure over F which is not smooth. Then there is a formula $\phi(x, \bar{b})$ in $F(M)$ and an $r>0$ such that the α-inner and α-outer measures of $\phi(x, \bar{b}) \cap \text{usbl}(F)$ differ by at least r. It follows that there is a finite Δ and a Δ-stable formula $\delta \in F$ such that the α-outer measure of $\phi(x, \bar{b}) \cap \text{usbl}(\Delta, F)$ is at least the α-inner measure of $\phi(x, \bar{b}) \cap \neg\delta$ plus $r/2$. Let X be the minimal closed set over F containing $\phi(x, \bar{b}) \cap \text{usbl}(\Delta, F)$ and let Y be the F-interior of $\phi(x, \bar{b}) \cap \neg\delta$. By Lemma 1.3(v), for any measure μ over F, $\mu(X)$ is the μ-outer measure of the closed set $\phi(x, \bar{b}) \cap \text{usbl}(\Delta, F)$ and $\mu(Y)$ is the μ-inner measure of the basic set $\phi(x, \bar{b}) \cap \neg\delta$. Then $\alpha(X) - \alpha(Y) \geqslant r/2$. Also, for any measure μ over F such that $\mu(X) - \mu(Y) \geqslant r/2$, the μ-outer measure of $\phi(x, \bar{b}) \cap \text{usbl}(\Delta, F)$ must exceed the μ-inner measure by at least $r/2$, whence μ is not smooth. Therefore α has a neighborhood which contains no smooth measures over F. The proofs of the other parts of the lemma are routine. □

2.11. Lemma. *Every neighborhood of a measure α over F contains a finite convex combination of the unstable piece of α and finitely many stable complete types. (In fact, the coefficients of the convex combination may be taken to be rational.)*

Proof. Given finitely many Borel sets over F, form all finite Boolean combinations, pick a stable complete type in each portion which meets $\mathrm{sbl}(\alpha)$, and form the appropriate convex combination with the unstable piece of α. □

2.12. Lemma. *Let F and G be fragments with $F \subset G$. The set of pairs of measures α over F and β over G such that β is a nonforking extension of α is convex and closed in the set of all pairs of measures over F and G with the product topology.*

Proof. Convexity is easy. The relation $\alpha \subset \beta$ is obviously closed. The set of nonforking extensions is closed because β forks over α if and only if the Borel set $\mathrm{fk}(G, F)$ has β-measure greater than zero. □

2.13. Theorem. *The nonforking relation is the unique relation $\leqslant$ on smooth measures such that Axioms* 0–5 *below hold.* (Axioms 0–4 are similar to the Lascar axioms for complete types, as stated in [4, p. 246].)

0. *The relation $\alpha \leqslant \beta$ implies $\alpha \subset \beta$ and is preserved under elementary maps.*
1. *If $\alpha \subset \beta \subset \delta$, then $\alpha \leqslant \delta$ if and only if $\alpha \leqslant \beta$ and $\beta \leqslant \delta$.*
2. *On every fragment $G \supset F(\alpha)$ there exists $\beta \geqslant \alpha$. If α is a complete type there exists a complete type $\beta \geqslant \alpha$ over G.*
3. *For every α there is a countably generated β with $\beta \leqslant \alpha$.*
4. *On each fragment $G \supset F(\alpha)$, there are at most continuum many $\beta \geqslant \alpha$.*
5. *For each pair of fragments $F \subset G$, the set of pairs α over F and β over G such that $\alpha \leqslant \beta$ is closed and convex.*

Proof. The preceding results show that the nonforking relation satisfies Axioms 0–5. (The transitivity Axiom 1 follows from Lemmas 1.15 and 2.4.)

To prove uniqueness of the relation $\leqslant$, assume that a relation $\leqslant$ on the smooth measures satisfies Axioms 0–5. Suppose $\alpha \leqslant \beta$. We show that β is a nonforking extension of α. By Axiom 0, $\alpha \subset \beta$. Let G be a special fragment over $F(\beta)$. By Axiom 2 there is a $\delta \geqslant \beta$ over G. Let X be the set of all x in $\mathrm{sbl}(G)$ such that for each finite Δ the complete Δ-type of x over G has positive δ-measure. By Corollary 1.8, X has full δ-measure in $\mathrm{sbl}(G)$. Suppose δ forks over α. By Lemma 2.4, $\mathrm{fk}(G, \alpha) \cap \mathrm{sbl}(\alpha)$ has positive δ-measure. Then $X \cap \mathrm{fk}(G, \alpha) \cap \mathrm{sbl}(\alpha)$ has positive δ-measure and thus has an element x. By Lemma 2.8, for some finite $\Delta \subset L$ there are more than continuum many mutually disjoint $F(\alpha)$-conjugates of the complete Δ-type of x over G. Since $x \in X$, the complete Δ-type of x over G has positive δ-measure, and its $F(\alpha)$-conjugates have positive measure in the corresponding $F(\alpha)$-conjugate of δ. It follows that there are more than continuum many mutually distinct $F(\alpha)$-conjugates of δ. By Axiom 0, there are more than continuum many $\mu \geqslant \alpha$ over G, contradicting Axiom 4. Therefore δ is a nonforking extension of α, and hence β is a nonforking extension of α.

Now suppose that p and q are complete stable types and q is a nonforking extension of p. Using Axiom 2 and the result of the preceding paragraph, the argument in [4, p. 266], shows that $p \leqslant q$.

Finally, suppose β is a nonforking extension of α. Let $N(r, \bar{X})$ be a neighborhood of α and $N(s, \bar{Y})$ a neighborhood of β. We may assume that $r = s$, and that $\bar{X}$ is a subsequence of $\bar{Y}$. By Lemma 2.11, $N(s, \bar{Y})$ contains a finite convex sum β' of complete types q_i over $F(\beta)$ and the unstable piece of β. Moreover, from the proof of 2.11, since $\beta(\text{fk}(\beta, \alpha)) = 0$, each q_i may be taken outside of $\text{fk}(\beta, \alpha)$, so that q_i is nonforking over $F(\alpha)$. Let p_i be the restriction of q_i to $F(\alpha)$, and let α' be the convex sum of the p_i and the unstable piece of α with the same coefficients as were used for β'. By the preceding paragraph, each $q_i \geqslant p_i$. The unstable piece of β is $\geqslant$ the unstable piece of α because it is the unique extension. It then follows by Axiom 5 that $\alpha' \leqslant \beta'$ since the relation $\leqslant$ is convex, and that $\alpha \leqslant \beta$ since $\leqslant$ is closed. □

3. Existence of smooth extensions

In this section we take up the question of when well behaved smooth extensions of a measure α exist. In general, a measure on M does not necessarily have a smooth nonforking extension. However, we shall see that for a theory without the independence property, every measure has a smooth extension which is pure and is nonforking on the stable part. We shall call such extensions faithful. We shall also obtain the converse result that if every measure on M has a smooth extension, then the theory does not have the independence property. Another necessary and sufficient condition for a theory to not have the independence property is that every measure on M has a countably generated measure algebra.

At the end of this section we list several open questions. We also discuss a natural alternative to the notion of a smooth measure. We call a measure α weakly smooth if over every fragment containing $F(\alpha)$, all nonforking extensions agree on the unstable part of α. The notion of weakly smooth behaves better than smooth in at least one respect, that every measure on M has a weakly smooth nonforking extension. However, we have not been able to determine whether the weakly smooth measures have all the properties needed to have a good theory of forking, and we include some of these properties in the list of open problems.

Before going into the details let us see what goes wrong with a theory which does have the independence property. For example, consider the complete theory T of the model $\mathfrak{A} = \langle A, N^A, S^A, \in \rangle$ where N^A is a countable set and S^A is the set of all finite subsets of N^A. On the saturated elementary extension M of $\mathfrak{A}$, let α be a measure over N^A such that the basic sets of the form $a \in x$ for a in N^A are independent of each other and have measure $\frac{1}{2}$. Then no extension β of α can be smooth, because if c is an element of N^M such that the formula $c \in x$ is not in the fragment $F(\beta)$, then the subset of M defined by $c \in x$ has inner measure zero and outer measure one with respect to β.

For an example in a theory which does not have the independence property of a measure which has no smooth nonforking extension, we return to an example from Section 2. Let T be the theory with two binary relations R and E such that R is a dense linear order without endpoints and E is an equivalence relations with infinitely many classes such that each class in dense in R. T does not have the independence property. Now let α be a measure on M such that the R-reduct of α is smooth but each equivalence class definable over $F(\alpha)$ has α-measure zero. Then α has no smooth nonforking extension, because in any nonforking extension β, each equivalence class definable over $F(\beta)$ must have β-measure zero. However, β is not smooth because each 'new' equivalence class has β-inner measure zero and β-outer measure one.

3.1. Definition. An extension β of a measure α over M is said to be *faithful* if for each $\Delta \subset L$, the set $\mathrm{sbl}(\beta) \cap \mathrm{fk}(\Delta, \beta, \alpha)$ has β-measure zero.

In general, neither one of the notions of a faithful extension or a nonforking extension implies the other.

3.2. Lemma. (i) *Every pure nonforking extension is faithful.*

(ii) *An extension of a measure α on M is faithful if and only if it is pure and is nonforking on the stable part of α, that is, the sets* $\mathrm{sbl}(\beta) - \mathrm{sbl}(\alpha)$ *and* $\mathrm{fk}(\beta, \alpha) \cap \mathrm{sbl}(\alpha)$ *have β-measure zero.*

(iii) *An extension of a smooth measure is faithful if and only if it is nonforking.*

(iv) *If $\alpha \subset \beta \subset \mu$, then μ is a faithful extension of α if and only if β is a faithful extension of α and μ is a faithful extension of β.*

Proof. (i), (ii), and (iv) are immediate, and (iii) follows from Lemma 2.4. □

Our main task in this section is to show that if T does not have the independence property, then every measure α *on* M has a smooth faithful extension. To do this we must add enough information to the unstable part of α to determine the measure of every formula over M, and at the same time make sure that the extension is pure (i.e. the stable part is not enlarged) and is nonforking on the stable part of α. Our first step will be to obtain a faithful extension β of α with the property that every extension of β is pure over β. This is done by introducing a technical notion called a dense measure which implies that every extension is pure, and showing in Proposition 3.12 that every α has a dense faithful extension. The second step is to show that if T does not have the independence property, then every dense measure β has a smooth faithful extension μ. This is done in Theorem 3.16.

In order to obtain pure extensions of measures, we introduce the technical notion of a $\langle \Delta_n \rangle$-pure extension, which is like a locally pure extension except that it applies only to a particular increasing chain of finite sets of formulas Δ_n instead of to all finite sets $\Delta \subset L$.

3.3. Definition. Fix a countable enumeration of the formulas $\phi(x, \bar{y})$ of L and let Δ_n be the set consisting of the first n formulas. An extension β of α is a $\langle \Delta_n \rangle$*-pure* extension of α if for each n, the Δ_n-stable part of α has full measure in the Δ_n-stable part of β.

3.4. Lemma. (i) *Every $\langle \Delta_n \rangle$-pure extension of α is a pure extension.*

(ii) *The union of a chain of $\langle \Delta_n \rangle$-pure extensions of α is a $\langle \Delta_n \rangle$-pure extension of α.*

Proof. The stable part of β is the intersection of the Δ_n-stable parts of β and the Δ_n-stable parts of β form a decreasing chain. Given an increasing chain of measures β_k, the Δ_n-stable part of β is the union of the Δ_n-stable parts of the β_k. Thus

$$\mathrm{sbl}(\beta) = \bigcap_n \mathrm{sbl}(\Delta_n, \beta), \qquad \mathrm{sbl}(\Delta_n, \beta) = \bigcup_k \mathrm{sbl}(\Delta_n, \beta_k). \quad \square$$

3.5. Lemma. *Let α be a measure on M and let G be a fragment containing $F(\alpha)$. Then α has a $\langle \Delta_n \rangle$-pure faithful extension to a measure β over G.*

Proof. The sequence $\mathrm{sbl}(\Delta_n, G)$ is a decreasing chain of open sets over G. It follows from Lemma 1.9(v), Lemma 1.3(v), and Lemma 1.6 that α has an extension μ to G such that each $\mathrm{sbl}(\Delta_n, \alpha)$ has full μ-measure in $\mathrm{sbl}(\Delta_n, G)$, and hence μ is a $\langle \Delta_n \rangle$-pure extension of α. By Theorem 1.18, α has a nonforking extension δ to G. Let β be the measure over G which agrees with μ on the unstable part of α and with δ on the stable part of α. Then β is a $\langle \Delta_n \rangle$-pure faithful extension of α to G. $\square$

We shall now consider measurability patterns of the sort discussed in Section 1 but with an n-tuple of additional parameters $\bar{y}$. Such a pattern $p(\bar{y})$ consists of a set of finite positive Boolean combinations of inequalities between a real number and the probability in x of a formula $\phi(x, \bar{y})$ in the diagram language of M.

3.6. Lemma. *Let $P(\bar{y})$ be a measurability pattern involving formulas over a fragment F and a finite sequence of additional parameters $\bar{y}$. The set of $\bar{b}$ in M such that the pattern $P(\bar{b})$ is satisfiable is closed over F. Moreover, if $P(\bar{y})$ is finite, then the set of $\bar{b}$ in M such that $P(\bar{b})$ is satisfiable is basic over F.*

Proof. Since the pattern $P(\bar{b})$ involves a set of fewer than κ formulas, it suffices by Lemma 1.5 to prove the result for the case that $P(\bar{y})$ is finite. Let $P(\bar{y})$ be a finite pattern involving formulas $\phi_1(x, \bar{y}), \ldots, \phi_n(x, \bar{y})$ over F. Let us call $\bar{b}$ and $\bar{c}$ *equivalent* if the sequences $\phi_1(x, \bar{b}), \ldots, \phi_n(x, \bar{b})$ and $\phi_1(x, \bar{c}), \ldots, \phi_n(x, \bar{c})$ have exactly the same nonempty Boolean combinations. If $\bar{b}$ and $\bar{c}$ are equivalent, then the pattern $P(\bar{b})$ is satisfiable if and only if $P(\bar{c})$ is satisfiable. It follows that the set of $\bar{b}$ in M such that $p(\bar{b})$ is satisfiable is basic over F. □

3.7. Definition. A Δ-tree in F is a tree of basic Δ-types over F with free variable x whose branches are pairwise inconsistent. The nodes of the tree are ordered by inclusion and the union of the branches is M.

3.8. Lemma. *Let α be a measure on M and let $\Delta \subset L$. Suppose θ is a formula of $F(\alpha)$ with measure $r > 0$ in the Δ-unstable part of α. Then α has a faithful $\langle \Delta_n \rangle$-pure extension β such that $F(\beta)$ has a finite Δ-tree for which each branch has β-measure $\leq 0.9\, r$ in θ intersected with the Δ-unstable part of α.*

Proof. Suppose not. Let G be an extension of $F(\alpha)$. Then for any Δ-formula $\delta(x, \bar{b})$ over G and every faithful $\langle \Delta_n \rangle$-pure extension β of α to G, either

(1) $\beta(\delta(x, \bar{b}) \cap \theta \cap \mathrm{usbl}(\Delta, \alpha)) < 0.1\, \mathrm{r}$, or

(2) $\beta(\delta(x, \bar{b}) \cap \theta \cap \mathrm{usbl}(\Delta, \alpha) > 0.9\, \mathrm{r}$.

Since the average of two faithful $\langle \Delta_n \rangle$-pure extensions of α to G is again a $\langle \Delta_n \rangle$-pure faithful extension of α to G, either (1) holds for all β or (2) holds for all β. By extending G further, it follows that either (1) holds for all fragments G containing $\delta(x, \bar{b})$ and all β or (2) holds for all G and all β. Call an n-tuple $\bar{b}$ in M of *type* 1 if condition (1) holds for all fragments G containing $\delta(x, \bar{b})$ and all faithful $\langle \Delta_n \rangle$-pure extensions β of α to G, and call $\bar{b}$ of *type* 2 if it is not of type 1. We shall show that the set of n-tuples $\bar{b}$ in M of type 1 is basic over $F(\alpha)$.

Let G be a fragment determined by a special elementary submodel A of M of power greater than the power of $F(\alpha)$ which contains $F(\alpha)$. There are measurability patterns $P(\bar{b}, A)$ and $Q(\bar{b}, A)$ over G such that an n-tuple $\bar{b}$ in A is of type 1 if and only if $P(\bar{b}, A)$ is satisfiable, and of type 2 if and only if $Q(\bar{b}, A)$ is satisfiable. By Lemma 3.6, the sets of $\bar{b}$ in A of types 1 and 2 are both closed over G, and hence are basic over G. It follows that there is a formula $\delta'(\bar{y}, A)$ over G such that $\bar{b}$ is of type 1 if and only if $\delta'(\bar{b}, A)$ holds. Since A is special, every $\bar{b}$ in M belongs to an elementary submodel A' of M which is isomorphic to A over $F(\alpha)$. Thus an n-tuple $\bar{b}$ in M is of type 1 if and only if the diagram of A over $F(\alpha)$ is consistent with $\delta'(\bar{b}, A)$, and is of type 2 if and only if the diagram of A over $F(\alpha)$ is consistent with $\neg\delta'(\bar{b}, A)$. By compactness and replacing the constants in A by new existentially quantified variables, there is a formula $\delta''(\bar{y})$ over $F(\alpha)$ such that an n-tuple $\bar{b}$ in M is of type 1 if and only if $\delta''(\bar{b})$. This shows that the set of n-tuples $\bar{b}$ in M of type 1 is basic over $F(\alpha)$.

Let $k(x)$ be the conjunction of the formulas

$$(\forall\bar{y})[\delta(x, \bar{y}) \text{ iff } \delta''(\bar{y})]$$

in $F(\alpha)$ for $\sigma \in \Delta$. Then $k(x)$ is a branch of a finite Δ-tree such that every branch except $k(x)$ has measure less than 0.1 r in θ intersected with the Δ-unstable part of α. It follows from the hypotheses that $k(x)$ & θ has measure greater than 0.9 r in the Δ-unstable part of α. However, $k(x)$ implies a complete Δ-type and is therefore Δ-stable, a contradiction. This completes the proof. □

3.9. Definition. A measure α on M is *dense* if for every real $r > 0$, every finite sequence $\bar{b}$ of elements of M and $\phi_i(x, \bar{y})$ of formulas in $F_{n+1}(\alpha)$, and every extension $\beta \supset \alpha$ such that each $\phi_i(x, \bar{b})$ is in $F(\beta)$, there exists a finite sequence $\bar{c}$ such that each formula $\phi_i(x, \bar{c})$ is in $F(\alpha)$ and for each i, $\alpha[\phi_i(x, \bar{c}) \cap \text{usbl}(\alpha)]$ is within r of $\beta[\phi_i(x, \bar{b}) \cap \text{usbl}(\alpha)]$.

3.10. Lemma. *Every extension β of a dense measure α on M is locally pure.*

Proof. Suppose α is dense and β is an extension of α which is not locally pure. Then for some $\Delta \subset L$, $\text{sbl}(\Delta, \beta) - \text{sbl}(\Delta, \alpha)$ has positive measure $r > 0$ in β. Choose a Δ-stable formula $\delta(x, \bar{b})$ in $F(\beta)$ whose β-measure is within 0.1 r of $\text{sbl}(\Delta, \beta)$, and a Δ-stable formula θ in $F(\alpha)$ whose α-measure is within 0.1 r of $\text{sbl}(\Delta, \alpha)$. Then $\delta(x, \bar{b}) \cap \neg\theta$ has β-measure at least 0.8 r. Since $\neg\theta$ is within 0.1 r of $\text{usbl}(\Delta, \alpha)$ and $\text{usbl}(\Delta, \alpha)$ is a subset of $\text{usbl}(\alpha)$, $\delta(x, \bar{b}) \cap \neg\theta \cap \text{usbl}(\alpha)$ has β-measure at least 0.7 r. The formula $\delta(x, \bar{y})$ may be chosen so that for every $\bar{c}$, $\delta(x, \bar{c})$ is Δ-stable. Since α is dense, there is a $\bar{c}$ such that $\delta(x, \bar{c})$ is a Δ-stable formula in $F(\alpha)$ and $\delta(x, \bar{c}) \cap \neg\theta \cap \text{usbl}(\alpha)$ has measure at least 0.6 r. But then $\text{sbl}(\Delta, \alpha) \cap \neg\theta$ has measure at least 0.6 r, contrary to the hypothesis that θ is within 0.1 r of $\text{sbl}(\Delta, \alpha)$. □

3.11. Lemma. *The union of a chain of dense measures is dense.*

The proof is straightforward.

3.12. Proposition. *Let α be a measure on M. Then α has a dense faithful extension β.*

Proof. Let Δ be a finite set of formulas of L. By iterating Lemma 3.8, obtain a $\langle \Delta_n \rangle$-pure faithful extension μ of α with a countable set of constants C such that every complete Δ-type over C has measure zero in $\text{usbl}(\Delta, \alpha)$. Repeat this process for each $\Delta \subset L$ and take the union of the chain. The resulting measure δ is a pure extension of α with a countable set C of constants such that for every $\Delta \subset L$, every complete Δ-type over C has measure zero in $\text{usbl}(\Delta, \delta)$. Let δ_0 be a countably generated submeasure of δ containing all formulas with constants from C. Then in any countably generated fragment G containing $F(\delta_0)$, every stable complete Δ-type in G contained in the Δ-unstable part of δ_0 has outer measure zero. Since there are only countably many stable complete Δ-types in G, the Δ-stable part of G has outer measure zero in the Δ-unstable part of δ_0. Then every countably generated extension of δ_0 is pure over δ_0. By Lemma 1.11, every extension of δ_0 is pure over δ_0, and hence every extension of δ is pure over δ. Now extend δ to a dense measure β by forming a countable chain of extensions. β is a pure extension of δ. By Theorem 1.18, we can adjust the measure β outside usbl(δ) to obtain a dense faithful extension. □

The remainder of this section is devoted to proving that if the theory T does not have the independence property, then every measure on M has a smooth faithful extension.

3.13. Definition (Shelah [10]). A theory T has the *independence property* if there is a formula $\phi(x, \bar{y})$ such that for each n there exist $\bar{b}_1, \ldots, \bar{b}_n$ in M such that each nontrivial Boolean combination of $\phi(x, \bar{b}_1), \ldots, \phi(x, \bar{b}_n)$ is satisfiable in M.

3.14. Theorem. *The following are equivalent.*

(i) *T does not have the independence property.*

(ii) *For every measure α on M and formula $\phi(x, \bar{y})$, every set of $\phi(x, \bar{a}) \in F(\alpha)$ such that the measures of the symmetric differences of the $\phi(x, \bar{a})$ are bounded away from zero is finite.*

(ii′) *Same as* (ii) *but with the word* 'finite' *replaced by* 'countable'.

(iii) *Every measure on M has a countably generated measure algebra.*

Proof. (iii) implies (i) is easy. Assume (iii) fails for a measure α. Then there is a sequence ϕ_i, $i < \omega_1$, of definable sets in $F(\alpha)$ such that no ϕ_i is within a null set of the σ-algebra A_i generated by the preceding sets. Since L is countable, we may assume that all the ϕ_i are of the form $\phi(x, \bar{a}_i)$, that is, definable from the same formula ϕ. We may then find an uncountable set of i's such that the measures of the symmetric differences of the sets $\phi(x, \bar{a}_i)$ are bounded away from zero, so (ii)′ fails. (ii) trivially implies (ii)′. Now assume (ii) fails and (i) holds. Let Γ be a finite set of formulas and let $n \in \omega$. We may find countably many $\bar{a}_k$, $k < \omega$, which are Γ-n-indiscernible, and such that the measure of the symmetric differences of the basic sets $\phi(x, \bar{a}_k)$ are bounded away from zero. By a result of Shelah [10], Γ and n may be chosen to guarantee that for every $x \in M$ the truth value of $\phi(x, \bar{a}_k)$ is constant for all sufficiently large k. Therefore the basic sets ϕ_k converge pointwise to a set S. Then the measures of $\phi_k \triangle S$ approach zero, contradicting the fact that the measures of $\phi_k \triangle \phi_1$ are bounded away from zero. □

3.15. Lemma. *If α is a measure on M which is dense and not smooth, then α has a faithful extension μ with a basic set ϕ such that the set $\{\mu(\phi \triangle \theta) : \theta \in F(\alpha)\}$ is bounded away from zero.*

Proof. Let λ and β be two extensions of α to the same fragment G which are distinct on usbl(α). For some positive real r and some basic set ϕ over G, $|\lambda(B) - \beta(B)| > r$ where $B = \phi \cap \text{usbl}(\alpha)$. By Lemma 3.11, λ and β are pure extensions of α, and by Lemma 1.14 they may be taken to be faithful extensions of α. Let μ be the average of λ and β. Then μ is also a faithful extension of α. For any α-measurable set A, either $\lambda(A \triangle \phi) > r/2$ or $\beta(A \triangle \phi) > r/2$, and hence $\mu(A \triangle \phi) > r/4$. Then μ and ϕ have the required property. □

3.16. Theorem. *The following are equivalent.*

(i) *T does not have the independence property.*
(ii) *Every measure α on M has a smooth extension.*
(iii) *Every measure α on M has a smooth faithful extension.*

Proof. It is trivial that (iii) implies (ii). Assume that (i) fails, that is, T has the independence property. By Theorem 3.14, there is a measure α on M whose measure algebra is not countably generated. The same applies to any extension of α. Then by Proposition 2.6, α does not have a smooth extension, so (ii) fails. Therefore (ii) implies (i).

We now assume (i) and prove (iii). Using Proposition 3.12, let β be a dense faithful extension of α. Iterating results 3.12 and 3.14 ω_1 times, we either obtain a smooth faithful extension of β or an uncountable sequence of formulas such that the measures of the symmetric differences of each formula with the previous ones is bounded away from zero. If the second alternative occurs, then there is a formula $\phi(x, \bar{y})$ which has uncountably many instances in the sequence of formulas, and hence there are infinitely many instances of ϕ such that the measures of the symmetric differences are bounded away from zero. This is impossible by Theorem 3.14, so α has a smooth faithful extension. □

Here is one more extension result for theories without the independence property whose proof uses methods similar to the preceding results.

3.17. Theorem. *Suppose T does not have the independence property. Then every measure α on M has a nonforking extension β such that β has a unique nonforking extension over each fragment $G \supset F(\beta)$.*

Proof. Recall from Theorem 1.18 that every measure β on M has a nonforking extension to each fragment $G \supset F(\beta)$. Suppose that for any nonforking extension β of α there is a fragment $G \supset F(\beta)$ over which β has two distinct nonforking extensions. Since the average of two nonforking extensions is again nonforking, we see as in the proof of Lemma 3.15 that β has a nonforking extension μ such that for some basic set ϕ over G, the μ-measures of the symmetric differences of ϕ with sets in $F(\beta)$ are bounded away from zero. Since the union of a chain of nonforking extensions is again nonforking over α, there is a nonforking extension β of α which has an uncountable sequence of basic sets such that for each set ϕ in the sequence, the β-measures of the symmetric differences of ϕ with the previous sets are bounded away from zero. As in the proof of Theorem 3.16, it follows from Theorem 3.14 that T has the independence property. □

We conclude this section with some open questions which arise in connection with the central result that in a theory without the independence property, every measure on M has a smooth faithful extension.

3.18. Open Problems. (i) To what extent can the results of this paper be extended to the case that the language L is uncountable? The main place where the countability of L was used in an essential way is in the construction of a smooth faithful extension by means of $\langle \Delta_n \rangle$-pure extensions.

(ii) In a theory without the independence property, does every measure α on M have a locally pure smooth extension? A locally pure faithful smooth extension?

(iii) In a theory T without the independence property, is there a 'canonical' way to choose a smooth faithful extension of an arbitrary measure α on M?

(iv) Does every measure α over M have a locally pure extension to each fragment $G \supset F(\alpha)$? Equivalently, does the union of the sets $\mathrm{sbl}(\Delta, G) - \mathrm{sbl}(\Delta, \alpha)$ over all $\Delta \subset L$ always have α-inner measure zero?

(v) Does every measure α on M have a locally pure nonforking extension to each fragment $G \supset F(\alpha)$? Equivalently, does the union of the sets $\mathrm{fk}(\Delta, G, \alpha)$ over all $\Delta \subset L$ always have α-inner measure zero?

For the following problems we consider a weakening of the notion of a smooth measure. Let us call a measure α on M *weakly smooth* if any two nonforking extensions of α agree on the unstable part of α. One reason to consider weakly smooth measures is that, while it is not true that every measure on M has a smooth nonforking extension, it follows from Theorem 3.17 above that every measure on M has a weakly smooth nonforking extension. It follows from the transitivity of nonforking that every nonforking extension of a weakly smooth measure is weakly smooth. Moreover, by the proof to Theorem 2.7, for every weakly smooth measure α on M, α has at most continuum many nonforking extensions over each fragment $G \supset F(\alpha)$. In fact the nonforking relation between weakly smooth measures on M satisfies all the Axioms 0 through 5 of Theorem 2.13 except possibly for Axiom 3, which is listed as an open question below. (The proof of the transitivity Axiom 1 uses Lemma 1.13.) There is hope that a well behaved theory can be obtained with weakly smooth in place of smooth and locally pure nonforking extensions in place of faithful extensions. However, the outcome of such a program would depend on the answers to some of the open questions which follow.

For an example of a measure which is weakly smooth but not smooth, let T be the theory with a dense linear order R and an equivalence relation E with infinitely many classes all dense in R; then any measure α on M with a smooth R-reduct and such that each equivalence class definable over $F(\alpha)$ has α-measure zero is weakly smooth but not smooth.

We now resume our list of open questions.

(vi) Does every measure α on M have a locally pure weakly smooth extension? A locally pure smooth extension? A locally pure weakly smooth nonforking extension?

(vii) Does every weakly smooth measure on M have a weakly smooth countable base?

(viii) Does every weakly smooth measure on M have a countable measure algebra?

(ix) Is every nonforking extension of a weakly smooth measure on M a locally pure extension?

4. Stationary measures

In this section we shall study measure which have unique faithful extensions to any larger fragment. We call such measures stationary. Stationary complete stable *types* are defined in Shelah [10] and several equivalent characterizations are given in the literature. Harnik and Harrington [4] introduced a definition of a stationary complete type which is weaker than the notion we shall use in the case that p is unstable.

4.1. Definition. A measure α on M is *stationary* if α has a unique faithful extension to every fragment containing $F(\alpha)$.

4.2. Lemma. *Every stationary measure α on M is smooth.*

Proof. Suppose that α is not smooth. By Lemma 2.2, there is a formula ϕ in $F(M)$ such that the α-inner measure of $\phi \cap \text{usbl}(\alpha)$ is less then the α-outer measure of $\phi \cap \text{usbl}(\alpha)$. Then there is a $\Delta \subset L$ such that the α-inner measure of $\phi \cap \text{usbl}(\Delta, \alpha)$ is less than the α-outer measure of $\phi \cap \text{usbl}(\Delta, \alpha)$. Let G be a fragment containing $F(\alpha)$ and the formula ϕ. It follows from the proof of Lemma 3.5 that α has a faithful extension β to G such that $\beta[\neg\phi \cup \text{sbl}(\Delta, G)]$ equals the α-inner measure of $\neg\phi \cup \text{sbl}(\Delta, G)$. It follows that $\beta[\neg\phi \cup \text{sbl}(\Delta, \alpha)]$ equals the α-inner measure of $\neg\phi \cup \text{sbl}(\Delta, \alpha)$, and hence that $\beta[\phi \cap \text{usbl}(\Delta, \alpha)]$ equals the α-outer measure of $\phi \cap \text{usbl}(\Delta, \alpha)$. Similarly, α has a faithful extension μ to G such that $\mu[\neg\phi \cap \text{usbl}(\Delta, \alpha)]$ equals the α-outer measure of $\neg\phi \cap \text{usbl}(\Delta, \alpha)$.

Then $\mu[\phi \cap \text{usbl}(\Delta, \alpha)]$ equals the α-inner measure of $\phi \cap \text{usbl}(\Delta, \alpha)$. Therefore β and μ differ on $\phi \cap \text{usbl}(\Delta, \alpha)$, and α is not stationary. □

Recall that a complete type on M is smooth if and only if it is stable. Therefore every stationary complete type on M is stable. For stable complete types, our notion of a stationary type coincides with the notions from the literature.

4.3. Lemma. *If α is a measure on M, then the union of the stationary complete types over α is Borel over $F(\alpha)$, and in fact is a countable intersection of open sets over $F(\alpha)$.*

Proof. It is shown in [10, p. 107] that a complete stable type p over F is stationary if and only if for every finite Δ, p is Δ-stable and the Δ-multiplicity of p is one, that is, over each $G \supset F$ there is only one complete Δ-type of maximal Δ-rank consistent with p. Moreover, for each n and finite Δ, the set $\text{sbl}(n, \Delta, F)$ intersected with the set of all x such that the complete type of x over F has Δ-multiplicity one is open over F. The result follows. □

4.4. Lemma. *If α is a stationary measure on M, then every nonforking extension of α is stationary, and the restriction of α to any smooth base for α is stationary.*

Proof. It follows from the transitivity property of nonforking that every nonforking extension of α is stationary. If G is a smooth base for α, then by Theorem 1.18 every nonforking extension of $\alpha \mid G$ has a nonforking extension to a fragment containing $F(\alpha)$. Since α is stationary, any two nonforking extensions of $\alpha \mid G$ to a fragment H must be equal, so $\alpha \mid G$ is stationary. □

4.5. Theorem. *Let α be a smooth measure on M. The following are equivalent.*

(i) *α is stationary.*

(ii) *The union of the nonstationary complete stable types over $F(\alpha)$ has measure zero.*

(iii) *For every $r > 0$ and formula $\phi(x)$ in $F(M)$ there are formulas θ and π over $F(\alpha)$ and a fragment $G \supset F(\alpha)$ such that $\alpha(\pi) < r$ and*

$$\theta \triangle \phi \subset \pi \cup \text{fk}(G, \alpha).$$

Proof. We first prove that (ii) implies (i). Assume (ii). Let G be a fragment containing $F(\alpha)$. On the unstable part of G, an extension β of α to G is uniquely determined because α is smooth. On the stable part of G, the faithful extension β of α to G is also uniquely determined as follows. Let X be the union of the stationary complete types over $F(\alpha)$, and let X' be the union of the unique nonforking extensions of the complete stationary types over $F(\alpha)$ to G. By (ii), X has full α-measure in $\text{sbl}(\alpha)$. For any Borel subset Y' of X' over G, we have

$Y \subset Y' \subset Y \cup \mathrm{fk}(G, \alpha)$ where Y is the union of the complete types of elements of Y' over $F(\alpha)$. Also, Y is Borel over $F(\alpha)$. Therefore for any faithful extension β of α to G, the β-measure of Y' must equal the α-measure of Y, and the β-measure of X' must equal the α-measure of $\mathrm{sbl}(\alpha)$. It follows that α is stationary.

We now prove that (i) implies (ii). Assume that (ii) fails, so the union of the nonstationary complete stable types over $F(\alpha)$ has positive measure. Let $\mathrm{mlt}(1, \Delta, \alpha)$ be the union of all complete Δ-stable types over $F(\alpha)$ of Δ-multiplicity one. Then $\mathrm{mlt}(1, \Delta, \alpha)$ is open over $F(\alpha)$. It follows from the proof of Lemma 2.12 that for some finite Δ, the set $\mathrm{sbl}(\alpha) - \mathrm{mlt}(1, \Delta, \alpha)$ has positive α-measure. Then for some n, the set $\mathrm{sbl}(\alpha) \cap \mathrm{sbl}(n, \Delta, \alpha) - \mathrm{mlt}(1, \Delta, \alpha)$ has positive α-measure. Take the smallest such n. By Lemma 1.7, there is a complete Δ-type p over α such that the intersection of p with $\mathrm{sbl}(\alpha) \cap \mathrm{sbl}(n, \Delta, \alpha) - \mathrm{mlt}(1, \Delta, \alpha)$ has positive α-measure. Then for some formula $\theta \subset \mathrm{sbl}(n, \Delta, \alpha)$ over $F(\alpha)$, the closed set $\theta \cap p - \mathrm{mlt}(1, \Delta, \alpha)$ has positive α-measure in $\mathrm{sbl}(\alpha)$. Let $G \supset F(\alpha)$ be a fragment generated by an elementary submodel of M. Then there exist $k > 1$ and complete Δ-types $q_1, \ldots, q_k$ over G such that $q_i \subset p$, each $\theta \cap q_i$ has Δ-rank n, and $\theta \cap p$ is contained in the union of the $\theta \cap q_i$ and $\mathrm{fk}(\Delta, G, \alpha)$. Let π_i be formulas over G such that $\theta \cap q_i \subset \pi_i$ and the π_i are pairwise disjoint. For each i, the open set $\pi_i \cup \mathrm{fk}(G, \alpha)$ intersected with $\theta \cap p \cap \mathrm{sbl}(\alpha) - \mathrm{mlt}(1, \Delta, \alpha)$ has α-inner measure zero, because any complete type

$$t \subset \theta \cap p \cap \mathrm{sbl}(\alpha) - \mathrm{mlt}(1, \Delta, \alpha)$$

over $F(\alpha)$ contains points outside of $\mathrm{fk}(G, \alpha)$ and points outside of π_i. By Lemma 1.6, α has an extension μ_i to G such that

$$X_i = [\pi_i \cup \mathrm{fk}(G, \alpha)] \cap \theta \cap p \cap \mathrm{sbl}(\alpha) - \mathrm{mlt}(1, \Delta, \alpha)$$

has μ_i-measure zero. By Theorem 2.4, μ_i is a pure extension of α, and by Theorem 1.18 the μ_i may be taken to be nonforking extensions of α. The union of the sets X_i contains $\theta \cap p \cap \mathrm{sbl}(\alpha) - \mathrm{mlt}(1, \Delta, \alpha)$ and hence has positive μ_i-measure. It follows that α has two nonforking extensions to G which differ on the sets X_i. Therefore α is nonstationary, and (i) fails.

If (iii) holds, then for any formula $\phi \in F(M)$, any two nonforking extensions of α to the same fragment G must assign the same measure to ϕ, so (i) holds.

Finally, we prove that (i) implies (iii). Assume that α is stationary. Let $\phi \in F(M)$ and let G be a fragment containing $F(\alpha)$ with $\phi \in G$. Let β be the unique nonforking extension of α to G. It follows from results in [10] (or from the open mapping theorem, cf. [4]) that there is an open set X' over $F(\alpha)$ such that a stationary type p over $F(\alpha)$ is contained in X' if and only if ϕ belongs to the unique nonforking extension of p to G. Let Z be the union of all stationary complete types over $F(\alpha)$. Then

$$X' \cap (Z - \mathrm{fk}(G, \alpha)) = \phi \cap (Z - \mathrm{fk}(G, \alpha)),$$

so

$$X' \triangle \phi \subset (M - Z) \cup \mathrm{fk}(G, \alpha).$$

Since α is smooth, there are Borel sets X'', Y'' over $F(\alpha)$ such that $\alpha(Y'') = 0$ and

$$(X'' \triangle \phi) \cap \mathrm{usbl}(\alpha) \subset Y''.$$

Since α is stationary, the set $Y' = (M - Z) \cap \mathrm{sbl}(\alpha)$ has α-measure zero. Therefore putting

$$X = (X' \cap \mathrm{sbl}(\alpha)) \cup (X'' \cap \mathrm{usbl}(\alpha)), \qquad Y = Y' \cup Y'',$$

we have

$$\alpha(Y) = 0 \quad \text{and} \quad X \triangle \phi \subset Y \cup \mathrm{fk}(G, \alpha).$$

Now let $r > 0$. Choose a closed set V and an open set W over $F(\alpha)$ such that

$$V \subset X - Y, \qquad X \cup Y \subset W, \qquad \alpha(W - V) < r/2.$$

By saturation there are formulas θ, π in $F(\alpha)$ such that $V \subset \theta \subset W$, $W - V \subset \pi$, and $\alpha(\pi) < r$. Then

$$\theta \triangle \phi \subset \pi \cup \mathrm{fk}(G, \alpha)$$

as required. □

4.6. Corollary. *Every smooth measure α on M such that $F(\alpha)$ is algebraically closed is stationary. In particular, every smooth measure α on M such that $F(\alpha)$ is generated by an elementary submodel of M is stationary.*

Proof. For the definition of an algebraically closed fragment see [14]. Every stable complete type over an algebraically closed fragment $F(\alpha)$ is stationary in our sense (see [4, Corollary 5.4]). The result now follows from Theorem 4.5(ii). □

5. Definable and flat extensions

In this section we show that every smooth measure α on M induces a unique extension β to every fragment $G \supset F(\alpha)$. We shall call β the flat extension of α, or using an equivalent characterization, the eventually definable extension of α. This extension is necessarily nonforking over α. Thus in the case that α is a stationary measure, the unique nonforking extension of α to G is the flat extension. The flat extension of a smooth measure α to G may be thought of as the 'average' of all nonforking extensions of α to G. In the case that α is a complete stable type over $F(\alpha)$, the flat extensions of α will be complete types if α is stationary, but will be measures with some values strictly between 0 and 1 if α is not stationary and G is large enough. In the next section we shall need the notions of flat and eventually definable extensions of measures which are not necessarily smooth. In general, we show that every eventually definable extension is flat, and every flat extension is faithful.

5.1. Definition. An extension μ of a measure α on M is said to be a *flat* extension if for every fragment $G \supset F(\mu)$, μ has an extension β over G which is invariant under $F(\alpha)$-elementary maps on G, that is, for each $F(\alpha)$-elementary map g on G and each formula $\phi \in G$, $\beta(\phi) = \beta(g\phi)$.

5.2. Lemma. *Let α be a measure on M such that for each fragment $G \supset F(\alpha)$, α has a unique nonforking extension to G. Then every nonforking extension of α is flat. In particular, if α is stationary, then every nonforking extension of α is flat.*

Proof. This follows from the fact that if β is a nonforking extension of α over G and f is an $F(\alpha)$-elementary map on G, then $f(\beta)$ is nonforking over α. □

5.3. Theorem. *Every flat extension of a measure on M is a faithful extension.*

Proof. We shall use Lemma 2.8. Let α be a measure on M and let β be a flat extension of α. Let G be a special fragment over $F(\beta)$. Let μ be an extension of β to G such that μ is invariant under $F(\alpha)$-elementary maps on G. We shall show that μ is a faithful extension of α. Suppose not. Then for some finite $\Sigma \subset L$, the set $\text{sbl}(G) \cap \text{fk}(\Sigma, G, \alpha)$ has positive μ-measure. Let X be the set of all $x \in \text{sbl}(G)$ such that for each $\Delta \subset L$ the complete Δ-type of x over G has positive μ-measure. By Corollary 1.8, X has full measure in $\text{sbl}(G)$, and hence the set $\text{sbl}(G) \cap \text{fk}(\Sigma, G, \alpha) \cap X$ has positive μ-measure and contains an element x. By Lemma 2.8(ii), for some finite Δ the complete Δ-type of x over G has infinitely many distinct $F(\alpha)$-conjugates on G. But these $F(\alpha)$-conjugates are pairwise disjoint and must all have the same positive μ-measure, which is impossible. Therefore μ is a faithful extension of α, and by transitivity, β is a faithful extension of α. □

Remark. The proof of the above theorem also shows that if G is a special fragment over $F(\alpha)$ and μ is an extension of α to G which is invariant under $F(\alpha)$-elementary maps on G, then μ is a faithful extension of α.

5.4. Lemma. *Let α be a smooth measure on M and μ an extension of α. Let α_0 and μ_0 be the restrictions of α and μ to countable smooth bases such that μ_0 is an extension of α_0. Then μ is flat over α if and only if μ_0 if flat over α_0.*

The proof is routine.

The following lemma is a consequence of the open mapping theorem (see [4, Theorem 10.6]).

5.5. Lemma. *Let F be a small fragment and let G be a special fragment over F. Let ϕ be a basic set over G and let X be the union of all the F-conjugates of ϕ on G. Then there is an open set Y over F such that*

$$X \cap (\mathrm{sbl}(F) - \mathrm{fk}(G, F)) \subset Y \subset X \cup \mathrm{usbl}(F) \cup \mathrm{fk}(G, F).$$

Proof. The set X is open over G. By the open mapping theorem the set

$$Y' = \bigcup \{\mathrm{tp}(x, F) : x \in X \cap (\mathrm{sbl}(F) - \mathrm{fk}(G, F)\}$$

is open in F relative to $\mathrm{sbl}(F)$, that is, $Y' = Y \cap \mathrm{sbl}(F)$ for some open set Y over F. Here $\mathrm{tp}(x, F)$ is the complete type of x over F. Then

$$X \cap (\mathrm{sbl}(F) - \mathrm{fk}(G, F)) \subset Y' \subset Y.$$

Moreover, for any x in $\mathrm{sbl}(F)$, all nonforking extensions of $\mathrm{tp}(x, F)$ over G are F-conjugates of each other [4, Theorem 5.5]. Therefore

$$Y' \subset X \cup \mathrm{fk}(G, F),$$

and hence

$$Y \subset X \cup \mathrm{usbl}(F) \cup \mathrm{fk}(G, F). \qquad \square$$

5.6. Theorem. *A smooth measure α on M has a unique flat extension to each fragment $G \supset F(\alpha)$.*

Proof. Let $G \supset F(\alpha)$. Assume first that G is a special fragment over $F(\alpha)$. Let P be the measurability pattern which contains the diagram of α and states that for each formula $\delta(x, \bar{c})$ over G, all the $F(\alpha)$-conjugates of $\delta(x, \bar{c})$ on G have the same measure. We claim that every finite subset P' of P is satisfiable. To prove the claim, first observe that the claim holds if $\mathrm{usbl}(\alpha)$ has α-measure one since α is smooth. Next show that the claim holds if $\Delta \subset L$, all the formulas $\delta(x, \bar{y})$ involved in P' belong to Δ, $\phi(x)$ is a formula over $F(\alpha)$ of Δ-rank n, and the set

$$X = \phi \cap \mathrm{sbl}(\alpha) \cap \mathrm{usbl}(n-1, \Delta, \alpha)$$

has α-measure one. This is because there are only finitely many complete Δ-types t over G such that $t \cap \phi$ has Δ-rank n, and the union of these types contains $X - \mathrm{fk}(\Delta, G, \alpha)$ and thus has α-outer measure one. Finally, the claim in general follows by approximating α by a convex sum of measures of the above kinds. Then by Lemma 1.5, P is satisfiable by a measure β over G. β is preserved under $F(\alpha)$-elementary maps on G.

We now prove that there is a unique extension of α to G which is preserved under $F(\alpha)$-elementary maps on G. Let β' be any extension of α over G which is preserved under $F(\alpha)$-elementary maps on G. Since α is smooth, β and β' agree on $\mathrm{usbl}(\alpha)$. By Theorem 5.3, both β and β' are nonforking over α. By Lemma 5.5, for any basic set ϕ over G, β and β' assign the same measure to the union of the $F(\alpha)$-conjugates of ϕ on G. It follows that for each closed set t over G, β and β' assign the same measure to the union of the $F(\alpha)$-conjugates of t on G. For every $\Delta \subset L$ and every complete Δ-type t over G, the $F(\alpha)$-conjugates of t on G are either equal or disjoint. Since β and β' are flat, they both assign measure r/k to t where r is the measure of the union of the $F(\alpha)$-conjugates of t and k is the number of conjugates. It follows that $\beta' = \beta$.

Finally, consider an arbitrary fragment H containing $F(\alpha)$. Let G be a special fragment over H. It follows from the preceding paragraphs that the restriction of β to H is the unique flat extension of α to H. □

5.7. Corollary. *Let α be a smooth measure on M.*

(i) *If $\alpha \subset \beta \subset \mu$, then μ is a flat extension of α if and only if β is a flat extension of α and μ is a flat extension of β.*

(ii) *The union of a chain of flat extensions of α is a flat extension of α.*

We now turn to the definable extensions of a measure.

5.8. Definition. Let β be a measure over a fragment G and let F be a subfragment of G, and let $\phi(x, \bar{y})$ be a formula of L with special variable x and an n-tuple of variables $\bar{y}$. Let $F(\bar{y})$ be the algebra of formulas of F in the variables $\bar{y}$. A real-valued function $f(\bar{y})$ is said to be *Borel* over F if f is measurable with respect to the σ-algebra $\sigma F(\bar{y})$ generated by the open (or closed) sets over $F(\bar{y})$. β is *ϕ-definable over F* if there is a Borel function $f(\bar{y})$ on F such that whenever $\phi(x, \bar{b}) \in G$, $\beta(\phi(x, \bar{b})) = f(\bar{b})$. β is *definable over F* if β is ϕ-definable over F for every formula ϕ of L. β is *eventually definable over F* if for every fragment $G \supset F(\beta)$, β has an extension δ over G which is definable over F. (Thus eventual definability implies definability.)

5.9. Proposition. *Every eventually definable extension of a measure α on M is a flat extension* (*and hence a faithful extension*).

Proof. An extension of α over G which is definable over $F(\alpha)$ is preserved by $F(\alpha)$-elementary maps on G, and hence any eventually definable extension is flat. Flat extensions are faithful by Lemma 5.3. □

5.10. Proposition. *If a measure α on M is definable over a fragment H and if $H \subset G \subset F(\alpha)$, then α is definable over G and $\alpha \mid G$ is definable over H. Similarly for eventual definability.*

The proof is straightforward.

5.11. Proposition. *Suppose a measure α on M is eventually definable over a fragment H. Then there is a mapping f_ϕ from formulas $\phi(x, \bar{y})$ of L to real-valued*

Borel functions $f_\phi(\bar{y})$ over H such that for every fragment $G \supset F$, α has an extension β over G which is ϕ-definable over H by f_ϕ. (In other words, the ϕ-definition depends only on ϕ and not on the extension G of F.)

Proof. Since κ is inaccessible and there are fewer than κ mappings from formulas to Borel functions over H, there must be one mapping f_ϕ which gives a ϕ-definition for extensions of α over arbitrarily large fragments containing $F(\alpha)$. $\square$

5.12. Theorem. *Let α be a smooth measure on M. Then every flat extension of α is eventually definable over α.*

Proof. Let G be a special fragment over $F(\alpha)$, and let β be the unique flat extension of α to G. Let

$$f(\bar{b}) = \beta[\phi(x, \bar{b}) \cap \text{usbl}(\phi, \alpha)].$$

We first show that for each $r \geq 0$, the set of $\bar{b}$ such that $f(\bar{b}) > r$ is open over $F(\alpha)$. We have $f(\bar{b}) > r$ if and only if there is a basic set θ over α such that $\alpha[\theta \cap \text{usbl}(\phi, \alpha)] > r$ and $\theta \cap \text{usbl}(\phi, \alpha) \subset \phi(x, \bar{b})$. Since $\text{usbl}(\phi, \alpha)$ is closed over $F(\alpha)$, the last inclusion holds if and only if there is a ϕ-stable basic set π over $F(\alpha)$ such that

$$(\forall x)[\theta \mathbin{\&} \neg\pi \rightarrow \phi(x, \bar{b})].$$

Therefore the set of $\bar{b}$ such that $f(\bar{b}) > r$ is open over $F(\alpha)$.

We next show by induction on n that if θ is a ϕ-stable basic set over F of ϕ-rank n, then the function

$$g_\theta(\bar{b}) = \beta[\phi(x, \bar{b}) \cap \theta]$$

is Borel over $F(\alpha)$. Assume the inductive hypothesis for all $m < n$. The argument which follows uses the finite equivalence relation theorem and related results from [10]. Let θ be a ϕ-stable basic set over $F(\alpha)$ of ϕ-rank n. Then the function

$$\beta[\phi(x, \bar{b}) \cap \theta \cap \text{sbl}(n-1, \phi, \alpha)] \tag{1}$$

is Borel over $F(\alpha)$, because it is the limit of the Borel functions

$$\beta[\phi(x, \bar{b}) \cap \theta \cap \pi]$$

where $\pi(x)$ is a basic set over $F(\alpha)$ of ϕ-rank less than n.

Let K be the set of all types of the form $\theta \cap t$ where t is a complete ϕ-type over G and $\theta \cap t$ has ϕ-rank n. K is finite and nonempty. By the finite equivalence relation theorem, there is a finite equivalence relation $E(x, z)$ over $F(\alpha)$ such that if x and z have different types in K then $\neg E(x, z)$. For each $k \in K$, let k' be the union of all types $h \in K$ which are $F(\alpha)$-conjugates of k. Let $k \in K$. Since K is finite, k' is closed over G. Let $\delta(x)$ be the union of all E-classes which meet k'.

Since E has only finitely many classes, δ is almost over $F(\alpha)$. But δ is its only $F(\alpha)$-conjugate on G, so δ is basic over $F(\alpha)$. Moreover,

$$\delta \cap \theta \cap \mathrm{usbl}(n-1, \phi, G) = k' \cap \mathrm{usbl}(n-1, \phi, G).$$

Since β is flat over α, β is faithful over α, and therefore $\mathrm{usbl}(n-1, \phi, G)$ has full β-measure in $\mathrm{usbl}(n-1, \phi, \alpha)$. Then

$$\beta[k' \cap \mathrm{usbl}(n-1, \phi, \alpha)] = \alpha[\delta \cap \theta \cap \mathrm{usbl}(n-1, \phi, \alpha)] = j.$$

Define the natural numbers a, b, c, d by
a = number of $F(\alpha)$-conjugates of k- on G which meet $\phi(x, \bar{b})$;
b = number of $F(\alpha)$-conjugates of k on G;
c = number of E-classes which meet $\phi(x, \bar{b})$ & $\delta(x)$;
d = number of E-classes which meet $\delta(x)$.
Suppose $\phi(x, \bar{b}) \in h$ for some $F(\alpha)$-conjugate h of k on G, so that $a > 0$. Let e be the number of E-classes which meet h. Then $e > 0$. Since any $F(\alpha)$-elementary map sends E-classes to E-classes, the integer e does not depend on h. Therefore

$$c/a = d/b = e,$$

and hence

$$a/b = c/d.$$

The number d does not depend on $\bar{b}$. For each c, the set of $\bar{b}$ with the value c is basic over $F(\alpha)$, since c can be defined from the formulas $E(x, y)$, $\delta(x)$, and $\phi(x, \bar{b})$. Since β is preserved under $F(\alpha)$-maps on G, all $F(\alpha)$-conjugates on G of a type $k \in K$ must have the same β-measure. Thus $\beta[k' \cap \phi(x, b) \cap \mathrm{usbl}(n-1, \phi, \alpha)] = (a/b) \cdot j$. Then

$$\beta[\theta(x) \cap \phi(x, b) \cap \mathrm{usbl}(n-1, \phi, \alpha)] \tag{2}$$

is a finite sum of terms of the form $(a/b) \cdot j$, one term for each F-conjugacy class of types in the finite set K. It follows that the function (2) is Borel over $F(\alpha)$. Then g_θ, which is the sum of the functions (1) and (2), is Borel over $F(\alpha)$. This completes our induction.

The function

$$g(b) = \beta[\phi(x, \bar{b}) \cap \mathrm{sbl}(\phi, \alpha)]$$

is the limit of the Borel functions g_θ where θ is a ϕ-stable formula over $F(\alpha)$, and is therefore Borel over $F(\alpha)$. We conclude that the function

$$\beta[\phi(x, \bar{b})] = f(b) + g(b)$$

is Borel over $F(\alpha)$. Therefore β is definable over $F(\alpha)$.

Since every flat extension of α can be extended to a flat extension β over a special fragment G over $F(\alpha)$, every flat extension of α is eventually definable over $F(\alpha)$. □

5.13. Corollary. (i) *An extension of a stationary measure α on M is nonforking if and only if it is eventually definable over $F(\alpha)$.*

(ii) *If α is a smooth measure on M and G is a smooth base of α, then α is eventually definable over G.*

Proof. In each case, there is a unique nonforking extension, which must be flat by Lemma 5.2 and Theorem 5.6 and eventually definable by Theorem 5.12. □

5.14. Theorem. *Let α be a smooth measure and let μ be the flat extension of α to a fragment $G \supset F(\alpha)$. An extension β of α to G is nonforking over α if and only if every basic set over G of μ-measure zero has β-measure zero.*

Proof. μ is nonforking over α by Theorem 5.3. Suppose that every basic set over G of μ-measure zero has β-measure zero. $\text{fk}(G, \alpha)$ is open over G and has μ-measure zero, because μ is nonforking over α. Therefore every basic subset of $\text{fk}(G, \alpha)$ has μ-measure zero, and hence has β-measure zero. Then $\text{fk}(G, \alpha)$ has β-measure zero, so β is nonforking over α.

Now suppose β is nonforking over α. By the transitivity properties we may assume that G is a special fragment over $F(\alpha)$. Let ϕ be a basic set over G with positive β-measure. Let H be the algebraic closure of $F(\alpha)$. Then $F(\alpha) \subset H \subset G$, $\text{sbl}(H) = \text{sbl}(\alpha)$, and $\text{fk}(G, H) = \text{fk}(G, \alpha)$. Also, all H-conjugates of ϕ on G agree on the nonforking part of G over H. By Lemma 5.5. there is an open set Y over H such that

$$\phi \cap (\text{sbl}(H) - \text{fk}(G, H)) \subset Y \subset \phi \cup \text{usbl}(H) \cup \text{fk}(G, H).$$

Since α is smooth, β and μ agree on $\text{usbl}(H)$ and since they are nonforking they assign measure zero to $\text{fk}(G, H)$. If $\beta(\phi \cap \text{usbl}(H))$ is positive, then $\mu(\phi)$ is positive, so we may assume that $\beta(\phi \cap \text{usbl}(H)) = 0$. Therefore $\beta(Y) \geq \beta(\phi)$ and $\mu(Y) \geq \mu(\phi)$. Thus $\beta(Y) > 0$, and there is a basic set θ over H such that $\theta \subset Y$ and $\beta(\theta) > 0$. Since H is the algebraic closure of $F(\alpha)$, θ is almost over $F(\alpha)$, that is, θ has finitely many $F(\alpha)$-conjugates on G. Using Lemma 5.5 again, we see that μ and β assign the same measure to the union X of the $F(\alpha)$-conjugates of θ on G. We have $0 < \beta(\theta) < \beta(X) = \mu(X)$. Since μ is flat over G, all the $F(\alpha)$-conjugates of θ on G have the same μ-measure. But there are only finitely many conjugates, so $\mu(\theta) > 0$. Then $\mu(\phi) > 0$ as required. □

5.15. Corollary. *Let p be a stable complete type over a fragment F and let μ be the flat extension of p to a fragment $G \supset F$. A complete type $q \supset p$ over G is nonforking over p if and only if every $\phi \in q$ has positive μ-measure.*

5.16. Theorem. *The flat extension relation is the unique relation $\leq'$ on smooth measures such that:*

0. *$\alpha \leq' \beta$ implies $\alpha \subseteq \beta$.*

1. *If $\alpha \subset \beta \subset \delta$, then $\alpha \leqslant' \delta$ if and only if $\alpha \leqslant' \beta$ and $\beta \leqslant' \delta$.*

2. *For every α and every fragment $G \supset F(\alpha)$ there is a unique β over G such that $\alpha \leqslant' \beta$.*

3. *If $\alpha \leqslant' \beta$ and f is an $F(\alpha)$-elementary map on $F(\beta)$, then $\beta = f\beta$.*

Proof. The flat extension relation satisfies Axioms 0–3 by Theorem 5.6 and Corollary 5.5.

To prove uniqueness let $\leqslant'$ be a relation satisfying Axioms 0–3. Let α and β be smooth measures on M such that $\alpha \leqslant' \beta$. Let G be a fragment containing $F(\beta)$. By Axiom 2 there is a measure δ over G such that $\beta \leqslant' \delta$. By Axiom 1, $\alpha \leqslant' \delta$. By Axiom 3, δ is preserved under $F(\alpha)$-elementary maps. Then β is a flat extension of α. On the other hand, if β is the flat extension of α to G, then $\alpha \leqslant' \beta$ by Axiom 2 and the uniqueness of the flat extension. □

6. The nonforking product

In this section we introduce the nonforking product $[\alpha \times \beta]$ of two measures α and β, which will allow us to pass from measures in the special variables x and y separately to measures in the pair of special variables (x, y), and to tuples of special variables. It will be convenient to work with measures over M itself as well as measures over small fragments in M. We shall see that each smooth measure over M is definable over a countable fragment. When α and β are measures over M which are definable over countable fragments, the $[\alpha \times \beta]$-measure of a formula $\phi(x, y)$ over M is computed by integrating the α-measure of $\phi(x, y)$ as a function of y with respect to β. This product is an extension of the ordinary product measure, and will usually be a proper extension because the formula $\phi(x, y)$ is in general not product measurable. The nonforking product of two measures which are definable over a countable fragment is again definable over a countable fragment, so the operation can be iterated. The nonforking product operation is associative, and if one of the measures α and β is smooth, then it is also commutative. The commutativity result is an analogue of the Fubini theorem for ordinary products of measures.

The symmetry theorem for complete types (from Shelah [10, p. 112]) states that if T is stable a and b are elements of M and C is a small subset of M, then the complete type of a over $C \cup b$ is nonforking over C if and only if the complete type of b over $C \cup a$ is nonforking over C. Using the symmetry theorem for complete types and the commutativity of the nonforking product, we obtain a symmetry theorem for measures.

Since this section deals with products of measures, we shall state our results for finite sequences of special variables instead of just one special variable. All the results in the preceding sections which were stated for a single special variable x also hold for a finite sequence of special variables $\bar{x}$.

We begin with a series of definitions and results concerning measures over M.

6.1. Definition. The family of *Borel sets over M* (in the special variables $\bar{x}$) is the σ-algebra σM generated by the set of unions of fewer than κ basic sets (in the special variables $\bar{x}$) over M. A *measure over M* is a countably additive probability measure α defined on σM such that the measure of the union of a set S of fewer than κ basic sets over M is the supremum of the measures of the unions of the finite subsets of S. A measure α over M is *smooth* if there is a (small) fragment F in M such that the restriction of α to F is smooth.

Warning. Notice our use of the words 'over' and 'on' to distinguish measures *over M*, which are defined on the family of all Borel sets over M, from measures *on M*, which are only defined on the Borel sets over some small fragment F.

A set is Borel over M if and only if it is Borel over some fragment F in M, because the family of sets which are Borel over some fragment is a σ-algebra containing all unions of fewer than κ basic sets over M. Similarly, a real-valued function is Borel over M if and only if it is Borel over some fragment.

It follows from saturation that every complete type over M is a measure over M. A complete type over M is a smooth measure if and only if it is stable.

6.2. Lemma. *Any finitely additive probability measure α on the algebra $F_n(M)$ of basic subsets of M in the variables $\bar{x}$ has a unique extension to a measure over M.*

Proof. By Theorem 1.2, for each fragment F in the variables $\bar{x}$, the restriction of $\alpha \mid F$ generates a unique measure over F. Let us denote this measure by $[\alpha \mid F]$. For each fragment $G \subset F$, the restriction $[\alpha \mid F] \mid G$ of $[\alpha \mid F]$ to G is a measure over G which extends $\alpha \mid G$, and by uniqueness we must have $[\alpha \mid F] \mid G = [\alpha \mid G]$. Thus if $F \supset G$, then $[\alpha \mid F]$ is an extension of $[\alpha \mid G]$. Since κ is inaccessible, it follows that the union of all the measures $[\alpha \mid F]$ is the unique measure over M which extends α. □

Measures over M behave somewhat differently than measures over fragments. For instance, we do not introduce the notion of the stable part of M because every element b of M satisfies the trivial stable formula $x = b$ of rank zero. To describe the properties of a measure α over M, we consider the properties of the restrictions of α to (small) fragments in M.

6.3. Definition. Let α be a measure over M and let F be a fragment in M. The restriction of α to F is denoted by $\alpha \mid F$. α is *nonforking* over F if for every fragment $G \supset F$, $\alpha \mid G$ is nonforking over F. α is *faithful* over F if for every fragment $G \supset F$, $\alpha \mid G$ is faithful over F. α is *stationary* if there is a fragment F such that $\alpha \mid F$ is stationary and α is nonforking over F. α is *flat* over F if for every fragment $G \supset F$, $\alpha \mid G$ is flat over F. α is *definable* over F if for every fragment $G \supset F$, $\alpha \mid G$ is definable over F. F is a *base* for α if for every fragment $G \supset F$, F

is a base for $\alpha \mid G$. α is *pure* over F if for every fragment $G \supset F$, $\alpha \mid G$ is pure over F.

Note that if a measure α over M is faithful over F, flat over F, definable over F, or is pure over F, then the corresponding property also holds for any fragment $G \supset F$.

6.4. Proposition. (i) *Every measure on M has a nonforking extension to a measure over M.*

(ii) *T does not have the independence property if and only if every measure on M has a faithful extension to a smooth measure over M.*

(iii) *Every smooth measure α on M has a unique extension to a measure over M which is definable over $F(\alpha)$.*

(iv) *Every stationary measure α on M has a unique nonforking extension to a measure over M, and this measure is definable over $F(\alpha)$.*

(v) *If α is a measure over M and α is definable over F, then $\alpha \mid G$ is eventually definable over F for every fragment $G \supset F$.*

Proof. (i) Let α be a measure on M. By Theorem 1.18 and Lemma 1.16, there is an increasing chain of nonforking extensions of α, whose union is a nonforking extension of α over M.

(ii) This follows from Theorem 3.16 and (i).

(iii) Let α be a smooth measure over a fragment F. Let μ be the union of the unique eventually definable extensions of α to G for all fragments $G \supset F$. Then μ is a definable extension of α to a measure over M. μ is clearly the only definable extension of α to a measure over M.

(iv) Let μ be the union of the unique nonforking extensions of α to G for all fragments $G \supset F$. These extensions agree where they are both defined because of the transitivity properties of nonforking, and μ is the unique nonforking extension to α to M. This argument only uses the fact that α has a unique nonforking extension to each $G \supset F$. Since α is stationary, α is smooth, so μ is definable over G by (iii).

(v) Let $G \supset F$. For each fragment $H \supset G$, $\alpha \mid H$ is an extension of $\alpha \mid G$ which is definable over F, so $\alpha \mid G$ is eventually definable over F. □

6.5. Proposition. *If α is a smooth measure over M, then α has a smooth countable base H, and $\alpha \mid H$ is stationary.*

Proof. Since α is smooth, there is a fragment F such that $\alpha \mid F$ is smooth. Let G be a countable smooth base for $\alpha \mid F$. Then $\alpha \mid G$ is smooth. For each $\Delta \subset L$ and each complete Δ-type t over M, let $\alpha'(t)$ be the infimum of the measures of all finite conjunctions of formulas in t. There are only countably many pairs (Δ, t) such that t is a complete Δ-type over M and $\alpha'(t) > 0$. It follows that there is a countable fragment $H \supset G$ such that for each $\Delta \subset L$ and each complete Δ-type u over H, there is a complete Δ-type t extending u to M such that $\alpha(u) = \alpha'(t)$. Then H is a base for $\alpha \mid J$ whenever $J \supset H$, so H is a base for α. $\alpha \mid H$ must be stationary, because otherwise $\alpha \mid H$ would have two distinct nonforking extensions to some fragment $J \supset H$, which could in turn be extended to distinct nonforking extensions of $\alpha \mid H$ over M. □

6.6. Proposition. *Let α be a smooth measure over M and let F be a smooth base for α. Then α is definable over F.*

Proof. This follows from Corollary 5.13. □

6.7. Corollary. *Every smooth measure over M is definable over a countable fragment.*

Proof. By Proposition 6.5, α has a countable smooth base F, and by Proposition 6.6, α is definable over F. □

6.8. Proposition. *A theory T does not have the independence property if and only if every measure over the saturated model M has a countably generated measure algebra.*

Proof. This follows from Theorem 3.12 and the fact that a measure α over M has a countably generated measure algebra if and only if $\alpha \mid F$ has a countably generated measure algebra for every fragment F in M. □

6.9. Lemma. *A measure α over M is definable over F if and only if for every formula $\phi(\bar{x}, \bar{y})$ of L (or of F), the real-valued function $f(\bar{y}) = \alpha\{\bar{x} : \phi(\bar{x}, \bar{y})\}$ is Borel over F.*

Proof. We prove only the nontrivial direction. Suppose α is definable over the fragment F. Then for each formula $\phi(\bar{x}, \bar{y})$ of L and fragment $G \supset F$ there is a Borel function g over F such that $g(\bar{b}) = f(\bar{b})$ whenever $\phi(\bar{x}, \bar{b}) \in G$. Since the cardinal κ of M is inaccessible, there is a Borel function h over F such that $g = h$ for arbitrarily large fragments G. Then $f = h$, so f is Borel over F. When $\phi(\bar{x}, \bar{y}, \bar{c})$ is a formula of F, the corresponding function f is a section of the function for the formula $\phi(\bar{x}, \bar{y}, \bar{z})$ of L, so f is still Borel over F. □

In order to introduce a product of measures, we need to deal with integrals with respect to measures over M. The next lemma shows that integrals of Borel functions with respect to measures over M which are definable over a countable fragment are themselves Borel over the fragment.

6.10. Lemma. *Let F be a countable fragment. A measure α over M is definable over F if and only if for every bounded real-valued function $f(\bar{x}, \bar{y})$ which is Borel over F, the integral*

$$g(\bar{y}) = \int f(\bar{x}, \bar{y})\, \mathrm{d}\alpha(\bar{x})$$

is Borel over F.

Proof. The integrability condition implies that α is definable over F because the integral of the characteristic function of a formula $\phi(\bar{x}, \bar{y})$ with respect to $\alpha(\bar{x})$ is the α-measure of the formula in $\bar{x}$. For the converse, assume that α is definable over F. Since F is countable, a relation $R(\bar{x}, \bar{y})$ is Borel over F if and only if it belongs to the σ-algebra generated by the basic sets over F. By the monotone class theorem (see [3]), the α-measure of any Borel relation over F is Borel over F as a function of $\bar{y}$. Then the integral of any Borel simple function over F is Borel over F, and hence the integral of any Borel function over F is Borel over F. $\square$

Remark. When the fragment F is not assumed to be countable, the above lemma holds provided that for each real r and each formula $\phi(\bar{x}, \bar{y})$ over F, the set of $\bar{y}$ such that $\alpha[\phi(\bar{x}, \bar{y})] > r$ is open over F. The proof must be modified by first proving the result for f the characteristic function of an open set over F, then the characteristic function of a finite Boolean combination of open sets over F, and then applying the monotone class theorem as before.

We are now ready to introduce the nonforking product of two definable measures over M.

6.11. Definition. Let α and β be measures over M in the special variables $\bar{x}$ and $\bar{y}$ respectively, and let α be definable over some fragment. The *nonforking product* $[\alpha \times \beta]$ of α and β is the measure over M such that for each formula $\phi(\bar{x}, \bar{y})$ in $F(M)$,

$$[\alpha \times \beta]\phi(\bar{x}, \bar{y}) = \int \alpha\{\bar{x} : \phi(\bar{x}, \bar{y})\}\, \mathrm{d}\beta(\bar{y}).$$

It follows from Lemma 6.9 that the function $f(\bar{y}) = \alpha\{\bar{x} : \phi(\bar{x}, \bar{y})\}$ is Borel over M, and since its values are in the interval $[0, 1]$, the integral exists and $[\alpha \times \beta]$ is defined and is a measure over M. It is clear that for any pair of formulas $\pi(\bar{x})$ and $\theta(\bar{y})$ in $F(M)$, $[\alpha \times \beta](\pi \,\&\, \theta) = \alpha(\pi) \cdot \beta(\theta)$. The same holds for pairs of Borel sets over F in the variables $\bar{x}$ and $\bar{y}$. Thus the nonforking product $[\alpha \times \beta]$ is an extension of the ordinary product measure of α and β. In general, $[\alpha \times \beta]$ will be a proper extension of the ordinary product measure, and in fact a formula $\phi(\bar{x}, \bar{y})$

will ordinarily not be measurable even with respect to the completion of the ordinary product of α and β.

6.12. Lemma. *Let α and β be measures over M which are both flat over a fragment F. Then $[\alpha \times \beta]$ is flat over F.*

Proof. α and β are preserved under F-elementary maps, and thus the integral which gives $[\alpha \times \beta]$ is preserved under F-elementary maps. □

The next lemma shows that nonforking products of measures which are definable over countable fragments are again definable over a countable fragment, so that the nonforking product operation can be iterated.

6.13. Lemma. *Let α and β be measures over M in the special variables $\bar{x}$ and $\bar{y}$ which are both definable over a countable fragment F. Then the nonforking product $[\alpha \times \beta]$ is definable over F. Moreover, for every bounded function $f(\bar{x}, \bar{y}, \bar{z})$ which is Borel over F,*

$$\int f(\bar{x}, \bar{y}, \bar{z})\, \mathrm{d}[\alpha \times \beta] = \int\int f(\bar{x}, \bar{y}, \bar{z})\, \mathrm{d}\alpha\, \mathrm{d}\beta \tag{1}$$

for all $\bar{z}$ in M, and the function of $\bar{z}$ given by either side of (1) *is Borel over F.*

Proof. We first prove that the right side of equation (1) is Borel over F. Let $h(\bar{y}, \bar{z})$ be the integral of f with respect to α. Since α is definable over F, the function h is Borel over F by Lemma 6.10. Since β, is definable over F, the integral of $h(\bar{y}, \bar{z})$ with respect to β is Borel over F. This integral is the right side of equation (1). We next prove equation (1). By definition of $[\alpha \times \beta]$, equation (1) holds for all $\bar{z}$ whenever f is the characteristic function of a basic set over F. Since F is a countable fragment, σF is the σ-algebra generated by the basic sets over F in the variables $\bar{x}$, $\bar{y}$, and $\bar{z}$. It follows that equation (1) holds for all simple functions over F, and therefore holds for all bounded Borel functions over F. Then the left side of (1) is Borel over F, and by Lemma 6.10, $[\alpha \times \beta]$ is definable over F. □

6.14. Corollary (Associative law). *If α, β, and μ are measures over M and are definable over a countable fragment F, then*

$$\alpha \times [\beta \times \mu] = [\alpha \times \beta] \times \mu.$$

Proof. A simple computation using Lemma 6.13 shows that for any bounded Borel function $f(\bar{x}, \bar{y}, \bar{z})$ over F, the integral of f with respect to either $\alpha \times [\beta \times \mu]$ or $[\alpha \times \beta] \times \mu$ is equal to the triple iterated integral with respect to α, β, and μ. □

We now prove a commutativity law for nonforking products, which is a Fubini type theorem for pairs of measures over M.

6.15. Theorem. *Let α and β be measures over M in the special variables $\bar{x}$ and $\bar{y}$ respectively, such that α is smooth and β is definable over a countable fragment. Then for every formula $\phi(x, y)$ in $F(M)$,*

$$\int \alpha\{\bar{x} : \phi(\bar{x}, \bar{y})\}\, \mathrm{d}\beta(\bar{y}) = \int \beta\{\bar{y} : \phi(\bar{x}, \bar{y})\}\, \mathrm{d}\alpha(\bar{x}). \qquad (2)$$

Remark. The theorem shows that $[\alpha \times \beta]$ in the special variables $\langle \bar{x}, \bar{y} \rangle$ is equal to $[\beta \times \alpha]$ in the special variables $\langle \bar{y}, \bar{x} \rangle$.

Proof. By Proposition 6.5, α has a smooth base F. By Proposition 6.6, α is definable over F. F may be chosen so that β is also definable over F, whence both integrals exist. To simplify notation, we shall give the proof for the case that α and β have single special variables x and y.

We first prove the formula (2) in the case that the formula $\phi(x, y)$ of $L(M)$ is stable, that is, every type is ϕ-stable. From [10], $\phi(x, y)$ is stable with special variable x and parameter variable y if and only if it is stable with special variable y and parameter variable x. Let G be a special fragment over F which is a base for α and β. It follows from [4, Corollary 2.8], that every complete ϕ-type $t(x)$ over G in the variable x has a $\phi(x, y)$-definition $\delta t(y)$ over G which is a positive Boolean combination of formulas in G of the form $\phi(c, y)$. Also, for each fragment $H \supset G$, each complete ϕ-type $t(x)$ over G has a unique extension to a complete ϕ-type $t'(x)$ over H with the same ϕ-rank, and $t'(x)$ has the same $\phi(x, y)$-definition over G. Similarly for complete types u over G in y. Let J be the set of all the complete ϕ-types in x over G which have positive α-measure, and let K be the set of all complete ϕ-types in y over G which have positive β-measure. By Lemma 1.7, $\bigcup J$ has full α-measure in M, and $\bigcup K$ has full β-measure in M. The sets J and K are countable. Consider types $t(x) \in J$ and $u(y) \in K$, with ϕ-definitions $\delta t(y)$ and $\delta u(x)$ over G.

We claim that $\delta t(y) \supset u(y)$ if and only if $\delta u(x) \supset t(x)$. The proof of this claim is like the proof that every coheir is an heir (see [7]). Choose sequences a_n, b_n of elements such that for each n, A_n is an elementary extension of B_{n-1} (with $G = B_0$) containing a_n, B_n is an elementary extension of A_n containing b_n, the ϕ-type of a_n over B_{n-1} is the extension of $t(x)$ given by $\delta t(y)$, and the ϕ-type of b_n over A_n is the extension of $u(y)$ given by $\delta u(x)$. If the claim is not true for $t(x)$ and $u(y)$, say $\delta u(x) \supset t(x)$ but not $\delta t(y) \supset u(y)$, then $\phi(a_m, b_n)$ if and only if $m \leqslant n$. Thus ϕ has the order property (see [10, p. 30]), which contradicts the hypothesis that ϕ is stable.

Let I be the set of all pairs $\langle t, u \rangle \in J \times K$ such that $\delta u(x) \supset t(x)$. If a has $\phi(x, y)$-type $t \in J$, b has $\phi(y, x)$-type $u \in K$, and the ϕ-type of b over $G \cup a$ is given by $\delta u(x)$, then $\phi(a, b)$ if and only if $\langle t, u\rangle \in I$. The ϕ-type of b over $G \cup a$ is given by $\delta u(x)$ if and only if b is not an element of $\mathrm{fk}(\phi, G \cup a, G)$. Since G is a base for β, $\mathrm{fk}(\phi, G \cup a, G)$ has β-measure zero. Therefore $\beta(\phi, (a, y))$ is the sum of $\beta(u(y))$ over all u such that $\langle t, u\rangle \in I$. Similarly, for any b of $\phi(y, x)$-type $u \in K$, $\alpha(\phi(x, b))$ is the sum of $\alpha(t(x))$ over all t such that $\langle t, u\rangle \in I$. It follows that both integrals in (2) are equal to the sum of $\alpha(t) \cdot \beta(u)$ over all $\langle t, u\rangle \in I$. This completes the proof of (2) in the case that $\phi(x, y)$ is stable.

We now take up the general case where ϕ is not assumed to be stable. Let $r > 0$. Let $\Gamma(x)$ be a $\phi(x, y)$-stable formula over G such that $\mathrm{sbl}(\phi(x, y), G) - \Gamma(x)$ has α-measure less than r. Then $\Gamma(x)\ \&\ \phi(x, y)$ is stable, and it follows that equation (2) holds for the formula $\Gamma(x)\ \&\ \phi(x, y) = \phi'(x, y)$.

By Lemma 2.2(iv) there are finite sequences of formulas $\theta_m(x)$, $\pi_m(x)$, $\delta_m(x)$ in G, $m = 1, \ldots, k$, such that $\alpha(\pi_m) < r$, δ_m is ϕ-stable, and for every b in M there exists $m \leqslant k$ such that

$$\theta_m \,\triangle\, \phi'(x, b) \subset \pi_m \cup \delta_m. \tag{3}$$

(The fact that α is smooth is used here.) We may take the θ_m so that $\theta_m \subset \neg\Gamma$, and thus $\theta_m \triangle \phi'(x, b) \subset \neg\Gamma$. Since $\alpha(\delta_m \cap \neg\Gamma) < r$, the symmetric difference $\theta_m \triangle \phi'(x, b)$ has α-measure less than $2r$. Let $\lambda_m(y)$ be the set of $b \in M$ such that (3) holds for m and fails for all $m' < m$. The λ_m form a finite partition of M by formulas in y over G. Let $\sigma_1(x, y)$ be the disjunction of the formulas

$$\theta_m(x)\ \&\ \neg\pi_m(x)\ \&\ \neg\delta_m(x)\ \&\ \neg\Gamma(x)\ \&\ \lambda_m(y), \quad m = 1, \ldots, k,$$

and let $\sigma_2(x, y)$ be the disjunction of the formulas

$$[\theta_m(x) \vee \pi_m(x) \vee \delta_m(x)]\ \&\ \neg\Gamma(x)\ \&\ \lambda_m(y), \quad m = 1, \ldots, k.$$

Then $\sigma_1(x, y) \subset \phi'(x, y) \subset \sigma_2(x, y)$, σ_1 and σ_2 are finite unions of products of basic sets over G in the variables x and y, and $\sigma_2 - \sigma_1$ has measure less than $2r$ in the product measure $\alpha \times \beta$. It follows that the intersection of $\phi(x, y)$ with the ϕ-unstable part of G in the variable x is measurable in the completion of the product of α and β. By the Fubini theorem, equation (2) holds when $\phi(x, y)$ is replaced by $\phi(x, y) \cap \mathrm{usbl}(\phi, G)(x)$. Since (2) also holds when $\phi(x, y)$ is replaced by $\phi(x, y)\ \&\ \Gamma(x)$ for any ϕ-stable formula $\Gamma(x)$, it holds for the original formula $\phi(x, y)$ as required. □

6.16. Corollary. *Let α and β be measures over M such that α is smooth and β is definable over a countable fragment. Then for any bounded function $f(\bar{x}, \bar{y})$ which is Borel over some countable fragment,*

$$\int\!\!\int f(\bar{x}, \bar{y})\, \mathrm{d}\alpha(\bar{x})\, \mathrm{d}\beta(\bar{y}) = \int\!\!\int f(\bar{x}, \bar{y})\, \mathrm{d}\beta(\bar{y})\, \mathrm{d}\alpha(\bar{x}).$$

By Lemma 6.13, both integrals are also equal to the integral of $f(\bar{x}, \bar{y})$ with respect to $[\alpha \times \beta]$.

We do now know whether the above equation holds when f is only assumed to be Borel over some fragment, rather than Borel over a countable fragment.

The nonforking product $[\alpha \times \beta]$ of two smooth measures α and β over M is not necessarily smooth over M. For example, let T be the theory with a unary relation U and a binary relation R which is a dense linear order without endpoints on U and an equivalence relation with infinitely many infinite classes on the complement of U. Let α be a smooth measure over M such that $\alpha(U) = \frac{1}{2}$, each singleton in U has α-measure zero, and each equivalence class in the complement of U has α-measure zero. Then $[\alpha \times \alpha]$ is not smooth. To see this, consider any fragment G and an equivalence class $R(b, y)$ & $\neg U(y)$ which is not definable over G, where $\neg U(b)$. The set $U(x)$ & $R(b, y)$ & $\neg U(y)$ is included in the unstable part of $G(x, y)$ but has inner measure 0 and outer measure $\frac{1}{4}$ with respect to $[\alpha \times \alpha] \mid G$.

In the special case that the theory T is stable, every measure over M is smooth, and in particular the nonforking product $[\alpha \times \beta]$ is smooth.

Since only one measure is required to be smooth in Corollary 6.16, the result can be iterated to apply to finite sequences of smooth measures. We state this iterated result as a corollary.

6.17. Corollary. *Let $\alpha_1, \ldots, \alpha_n$ be smooth measures over M in the special variables $\bar{y}_1, \ldots, \bar{y}_n$. For any bounded function $f(\bar{y}_1, \ldots, \bar{y}_n)$ which is Borel over a countable fragment, all the iterated integrals of f with respect to the measures $\alpha_1, \ldots, \alpha_n$ are the same regardless of the order in which the integrals are taken.*

Proof. By Lemma 6.13 and Corollaries 6.14 and 6.16. □

We shall now give some applications of the above results on products and the symmetry theorem for complete stable types. For the remainder of this section we consider fragments generated by small subsets C of M and complete types of elements b of M over C. The complete type of b over C is denoted by $\text{tp}(b, C)$. Given a set C and an element b, we sometimes write $C \cup b$ for $C \cup \{b\}$. To simplify notation, we consider pairs of measures with single special variables x and y.

6.18. Lemma. *If α is a measure over M and $b \in M$, then $\alpha \mid C \cup b$ is nonforking over C if and only if for α-almost all $a \in M$,* $\text{tp}(a, C \cup b)$ *is nonforking over C.*

Proof. Immediate from the definition of nonforking extension. □

6.19. Lemma (see [10]). *For any small subset C of M, the set of pairs $\langle a, b \rangle$ in M^2 such that* $\text{tp}(a, C \cup b)$ *forks over C is open over C.*

6.20. Lemma (Symmetry Theorem for types, [4, Theorem 10.1]). *Let a and b be* *elements of M and C a subset of M such that* tp(a, C) *stable. Then* tp$(a, C \cup b)$ *does not fork over C if and only if* tp$(b, C \cup a)$ *does not fork over C.*

6.21. Theorem (Symmetry Theorem for measures). *Let C be a small subset of M and let α and β be measures over M such that $\alpha \mid C$ is smooth and β is definable over some countable fragment. The following are equivalent:*

(i) *For α-almost all $a \in$ sbl(C), $\beta \mid C \cup a$ does not fork over C.*
(ii) *For β-almost all $b \in M$, $\alpha \mid C \cup b$ does not fork over C.*

Proof. By the symmetry theorem for types, if $a \in$ sbl(C) then tp$(b, C \cup a)$ does not fork over C if and only if tp$(a, C \cup b)$ does not fork over C. By Lemma 6.19, the sets of pairs $\langle a, b\rangle$ such that tp$(b, C \cup a)$ does not fork over C, and such that tp$(a, C \cup b)$ does not fork over C, are closed over C. Let D be a countable subset of C which is a smooth base for $\alpha \mid C$ and a base for $\beta \mid C$. Then for any a, $\beta \mid C \cup a$ does not fork over C if and only if $\beta \mid C \cup a$ does not fork over D, and also if and only if for each countable E with $D \subset E \subset C$, $\beta \mid E \cup a$ does not fork over E. The analogous result also holds for α. Moreover, since $\alpha \mid D$ is smooth, if $D \subset E \subset C$ then $\alpha \mid E \cup b$ does not fork over E if and only for α-almost all $a \in$ sbl(C), tp$(a, E \cup b)$ does not fork over E. Therefore by Theorem 6.15 and Lemma 6.20, the following are equivalent:

For α-almost all $a \in$ sbl(C), $\beta \mid C \cup a$ does not fork over C.

For α-almost all $a \in$ sbl(C), for all countable $D \subset E \subset C$, $\beta \mid E \cup a$ does not fork over E.

For all countable $D \subset E \subset C$, for α-almost all $a \in$ sbl(C), $\beta \mid E \cup a$ does not fork over β (this step uses the fact that the intersection of a family of fewer than κ closed sets over M has full α-measure in the intersection of some countable subfamily).

For all countable $D \subset E \subset C$, for α-almost all $a \in$ sbl(C), for β-almost all $b \in M$, tp$(b, E \cup a)$ does ot fork over E.

For all countable $D \subset E \subset C$, for α-almost all $a \in$ sbl(C), for β-almost all $b \in M$, tp$(a, E \cup b)$ does not fork over E.

For all countable $D \subset E \subset C$, for β-almost all $b \in M$, for α-almost all $a \in$ sbl(C), tp$(a, E \cup b)$ does not fork over E.

For all countable $D \subset E \subset C$, for β-almost all $b \in M$, $\alpha \mid E \cup b$ does not fork over E (this step uses the fact that $\alpha \mid D$ is smooth).

For β-almost all $b \in M$, for all countable $D \subset E \subset C$, $\alpha \mid E \cup b$ does not fork over E.

For β-almost all $b \in M$, $\alpha \mid C \cup b$ does not fork over C. □

When both measures are smooth over C, we have the following form of the symmetry theorem.

6.22. Corollary. *Let C be a small subset of M and let α and β be measures over M such that both $\alpha \mid C$ and $\beta \mid C$ are smooth. Then the following are equivalent.*

(i) *For α-almost all $a \in M$, $\beta \mid C \cup a$ does not fork over C.*
(ii) *for α-almost all $a \in \mathrm{sbl}(C)$, $\beta \mid C \cup a$ does not fork over C.*
(iii) *For β-almost all $b \in M$, $\alpha \mid C \cup b$ does not fork over C.*
(iv) *For β-almost all $b \in \mathrm{sbl}(C)$, $\alpha \mid C \cup b$ does not fork over C.*

Proof. By Theorem 6.21, (i) is equivalent to (iv), and (ii) is equivalent to (iii). By a minor modification of the proof of 6.21, we also can show that (ii) is equivalent to (iv). □

We conclude with some additional results showing that various properties are preserved by the nonforking product of measures.

6.23. Lemma. *For any fragment F, the stable part of F in the variables (x, y) is the cartesian product of the stable part of F in the variable x and the stable part of F in the variable y. In symbols.*

$$\mathrm{sbl}(F)(x, y) = \mathrm{sbl}(F)(x) \times \mathrm{sbl}(F)(y).$$

This follows from the fact that a type p over a countable F is Δ-stable if and only if p has countably many extensions by complete Δ-types over each countable $G \supset F$ [10, Theorem 3.1].

6.24. Proposition. *Let α and β be measures over M such that α is definable over a countable fragment, and let C be a small subset of M. Then $[\alpha \times \beta]$ is pure over C if and only if both α and β are pure over C.*

Proof. Let C' be a countable base for $\alpha \mid C$, $\beta \mid C$, and $[\alpha \times \beta] \mid C$. Then α, β, or $[\alpha \times \beta]$ is pure over C if and only if it is pure over C'. For any countable set D with $C' \subset D \subset M$, it follows from Lemma 6.24 and 6.13 that $[\alpha \times \beta] \mid D$ is pure over C' if and only if $\alpha \mid D$ and $\beta \mid D$ are pure over C'. The result now follows by Lemma 1.11. □

We shall use the following consequence of the symmetry theorem for complete types.

5. Lemma [4, Theorem 10.7]. *Let C and D be small subsets of M with $C \subset D$, and let (a, b) be a pair of elements of M whose complete type over D is stable. Then* $\mathrm{tp}((a, b), D)$ *does not fork over C if and only if* $\mathrm{tp}(a, D)$ *does not fork over C and* $\mathrm{tp}(b, D \cup a)$ *does not fork over C.*

6.26. Theorem. *Let α and β be measures over M which are definable over some countable fragment. Let C be a small subset of M. Then $[\alpha \times \beta]$ is faithful over C if and only if both α and β are faithful over C.*

Proof. It suffices to prove the theorem for C countable. Let D be a countable subset of M such that $C \subset D$ and both α and β are definable over D. By Lemma 6.13, $[\alpha \times \beta]$ is definable over D. By Proposition 5.9, α, β, and $[\alpha \times \beta]$ are faithful over D. By Proposition 6.24, we may assume that α, β, and $[\alpha \times \beta]$ are pure over C. By Lemmas 6.23 and 6.25, the following are equivalent.

$[\alpha \times \beta]$ is faithful over C.

$[\alpha \times \beta] \mid D$ is faithful over C.

For $[\alpha \times \beta]$-almost all (a, b) in sbl(D), tp$((a, b), D)$ does not fork over C.

For $[\alpha \times \beta]$-almost all (a, b) in sbl(D), tp(a, D) and tp$(b, D \cup a)$ do not fork over C.

For α-almost all a in sbl(D), for β-almost all b in sbl(D), tp(a, D) and tp$(b, D \cup a)$ do not fork over C.

For α-almost all a in sbl(D), for β-almost all b in sbl(D), tp(a, D) and tp(b, D) do not fork over C (because $\beta \mid D \cup a$ is faithful over $\beta \mid D$).

$\alpha \mid D$ and $\beta \mid D$ are faithful over C.

α and β are faithful over C. □

For the following corollary, given a finite sequence $\langle a_1, \ldots, a_n \rangle$ and a set $J \subset \{1, \ldots, n\}$, let $a_J = \langle a_j : j \in J \rangle$, and let $a_{n-J} = \{a_i : i \in \{1, \ldots, n\} - J\}$.

6.27. Corollary. *Let $\alpha_1, \ldots, \alpha_n$ be smooth measures over M in the special variables $x_1, \ldots, x_n$. Let C be a small subset of M such that each α_k is nonforking over C. Then for $[\alpha_1 \times \cdots \times \alpha_n]$-almost all n-tuples $\langle a_1, \ldots, a_n \rangle$ of elements of M, and for each subset $J \subset \{1, \ldots, n\}$, if* tp$(a_J, C \cup a_{n-J})$ *is stable, then it is nonforking over C.*

Proof. By Theorem 6.26 and Corollary 6.17, the nonforking product α_J of the measures α_j for $j \in J$ is faithful over C. Therefore for α_J-almost all a_J, and all a_{n-J}, if tp$(a_J, C \cup a_{n-J})$ is stable then it is nonforking over C. □

The preceding results also hold for finite sequences $\bar{x}$ and $\bar{y}$ of special variables instead of single special variables x and y.

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