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The first-integral method applied to the Eckhaus equation

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ABSTRACT

The first-integral method is a direct algebraic method for obtaining exact solutions of some nonlinear partial differential equations. This method can be applied to nonintegrable equations as well as to integrable ones. This method is based on the theory of commutative algebra. In this work, we apply the first-integral method to study the exact solutions of the Eckhaus equation.

1. Introduction

It is well known that nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various fields of science, such as physics, biology, chemistry, etc. Therefore, seeking exact solutions of NPDEs is very important and significant in the nonlinear sciences. In the past few decades, a great effort has been made in this task and many powerful methods have been presented, such as the inverse scattering method [1], Hirota's direct method [2], the tanh method [3], the extended tanh function method [4], the Jacobian elliptic function expansion method [5], and so on.

The first-integral method was first proposed by Feng [6] in solving the Burgers–KdV equation which is based on the ring theory of commutative algebra. Recently, this useful method has been widely used by many researchers, such as in [7–11] and the references therein.

The Eckhaus equation is in the following form:

$$i\Psi_t + \Psi_{xx} + 2(|\Psi|^2)_x\Psi + |\Psi|^4\Psi = 0,$$

where $\Psi = \Psi(x, t)$ is a complex-valued function of two real variables x, t. This equation is of nonlinear Schrödinger type. The Eckhaus equation was found in [12] as an asymptotic multiscale reduction of certain classes of nonlinear partial differential equations. In [13], many of the properties of the Eckhaus equation were investigated. In [14], the Eckhaus equation was (exactly) linearized by a change of (dependent) variable. The aim of this work is to find exact solutions of the Eckhaus equation by the first-integral method.

2. The first-integral method

Raslan summarized the use of the first-integral method [8].

Step 1. Consider a general nonlinear PDE in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0. \qquad (1)$$

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Using a wave variable where $\xi = x - ct$, we can rewrite Eq. (1) as the following nonlinear ODE:

$$G(U, U', U'', \ldots) = 0, \tag{2}$$

where the prime denotes differentiation with respect to ξ.

Step 2. Suppose that the solution of ODE (2) can be written as follows:

$$u(x, t) = f(\xi). \tag{3}$$

Step 3. We introduce a new independent variable

$$X(\xi) = f(\xi), \qquad Y = \frac{\partial f(\xi)}{\partial \xi}, \tag{4}$$

which leads a system of nonlinear ordinary differential equations:

$$\begin{aligned} \frac{\partial X(\xi)}{\partial \xi} &= Y(\xi), \\ \frac{\partial Y(\xi)}{\partial \xi} &= F_1(X(\xi), Y(\xi)). \end{aligned} \tag{5}$$

Step 4. By the qualitative theory of ordinary differential equations [15], if we can find the integrals for Eq. (5) under the same conditions, then the general solutions to Eq. (5) can be obtained directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for us to tell what these first integrals are. We will apply the Division Theorem to obtain one first integral for Eq. (5) which reduces Eq. (2) to a first-order integrable ordinary differential equation. An exact solution to Eq. (1) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem. *Suppose that $P(w, z)$ and $Q(w, z)$ are polynomials in $C[w, z]$; and $P(w, z)$ is irreducible in $C[w, z]$. If $Q(w, z)$ vanishes at all zero points of $P(w, z)$, then there exists a polynomial $G(w, z)$ in $C[w, z]$ such that*

$$Q(w, z) = P(w, z)G(w, z).$$

3. The Eckhaus equation

Let us consider the Eckhaus equation:

$$i\Psi_t + \Psi_{xx} + 2(|\Psi|^2)_x\Psi + |\Psi|^4\Psi = 0. \tag{6}$$

We use the wave transformation

$$\Psi(x, t) = u(\xi)e^{i(\alpha x + \beta t)}, \qquad \xi = k(x - 2\alpha t), \tag{7}$$

where k, α and β are constants to be determined later.

Substituting (7) into (6), we obtain an ordinary differential equation:

$$k^2u'' - (\beta + \alpha^2)u + 4ku'u^2 + u^5 = 0. \tag{8}$$

Using (4) and (5), we get

$$\dot{X}(\xi) = Y(\xi), \tag{9}$$

$$\dot{Y}(\xi) = -\frac{4}{k}(X(\xi))^2Y(\xi) + \frac{(\beta + \alpha^2)}{k^2}X(\xi) - \frac{1}{k^2}(X(\xi))^5. \tag{10}$$

According to the first-integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (9) and (10), and

$$Q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0$$

is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$Q(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0, \tag{11}$$

where the $a_i(X)$ $(i = 0, 1, \dots, m)$ are polynomials in X and $a_m(X) \neq 0$. Eq. (11) is called the first integral for (9) and (10). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X, Y]$ such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX}\frac{dX}{d\xi} + \frac{dQ}{dY}\frac{dY}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i. \tag{12}$$

In this example, we take two different cases, assuming that $m = 1$ and $m = 2$ in (11).

Case A: Suppose that $m = 1$; by comparing the coefficients of the Y^i $(i = 2, 1, 0)$ on either side of (12), we have

$$\dot{a_1}(X) = h(X)a_1(X), \tag{13}$$

$$\dot{a_0}(X) = \left[\frac{4}{k}(X(\xi))^2 + g(X)\right]a_1(X) + h(X)a_0(X), \tag{14}$$

$$a_1(X)\left[\frac{(\beta + \alpha^2)}{k^2}X(\xi) - \frac{1}{k^2}(X(\xi))^5\right] = g(X)a_0(X). \tag{15}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from (13) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that deg $(g(X)) = 2$ only.

Suppose that $g(X) = A_0 + A_1X + A_2X^2$; then we find $a_0(X)$.

$$a_0(X) = B_0 + A_0X + \frac{A_1}{2}X^2 + \left(\frac{4}{3k} + \frac{A_2}{3}\right)X^3, \tag{16}$$

where B_0 is an arbitrary integration constant.

Substituting $a_0(X)$ and $g(X)$ into (15) and setting all the coefficients of powers of X to zero, we then obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$B_0 = 0, \qquad A_0 = -\frac{\sqrt{\beta + \alpha^2}}{k}, \qquad A_1 = 0, \qquad A_2 = -\frac{1}{k}, \tag{17}$$

$$B_0 = 0, \qquad A_0 = \frac{\sqrt{\beta + \alpha^2}}{k}, \qquad A_1 = 0, \qquad A_2 = -\frac{1}{k}, \tag{18}$$

where k, α and β are arbitrary constants.

Using the conditions (17) in (11), we obtain

$$Y(\xi) = \frac{\sqrt{\beta + \alpha^2}}{k}X(\xi) - \frac{1}{k}X^3(\xi). \tag{19}$$

Combining (19) with (9), we obtain the exact solution to Eq. (8) and the exact solution to the Eckhaus equation can be written as

$$\Psi(x, t) = \pm\sqrt{\sqrt{(\beta + \alpha^2)}}\left[\frac{e^{\frac{2}{k}\sqrt{(\beta+\alpha^2)}(k(x-2\alpha t)+\xi_0)}}{1 + e^{\frac{2}{k}\sqrt{(\beta+\alpha^2)}(k(x-2\alpha t)+\xi_0)}}\right]^{\frac{1}{2}} e^{i(\alpha x+\beta t)}, \tag{20}$$

where ξ_0 is an arbitrary constant.

If $\lambda = \frac{2}{k}\sqrt{(\beta + \alpha^2)}$, then

$$\begin{aligned}\Psi(x, t) &= \pm\sqrt{\frac{k\lambda}{2}}\left[\frac{e^{\lambda(k(x-2\alpha t)+\xi_0)}}{1 + e^{\lambda(k(x-2\alpha t)+\xi_0)}}\right]^{\frac{1}{2}} e^{i(\alpha x+\beta t)} \\ &= \pm\sqrt{\frac{k\lambda}{2}}\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\lambda}{2}(k(x - 2\alpha t) + \xi_0)\right)\right]^{\frac{1}{2}} e^{i(\alpha x+\beta t)}.\end{aligned}$$

Similarly, in the case of (18), from (11), we obtain

$$Y(\xi) = -\frac{\sqrt{\beta + \alpha^2}}{k}X(\xi) - \frac{1}{k}X^3(\xi), \tag{21}$$

and then the exact solution of the Eckhaus equation can be written as

$$\Psi(x, t) = \pm\sqrt{\sqrt{(\beta + \alpha^2)}}\left[\frac{e^{-\frac{2}{k}\sqrt{(\beta+\alpha^2)}(k(x-2\alpha t)+\xi_0)}}{1 - e^{-\frac{2}{k}\sqrt{(\beta+\alpha^2)}(k(x-2\alpha t)+\xi_0)}}\right]^{\frac{1}{2}} e^{i(\alpha x+\beta t)}, \tag{22}$$

where ξ_0 is an arbitrary constant.

If $\lambda = \frac{2}{k}\sqrt{(\beta+\alpha^2)}$, then

$$\Psi(x,t) = \pm\sqrt{\frac{k\lambda}{2}}\left[\frac{e^{-\lambda(k(x-2\alpha t)+\xi_0)}}{1-e^{-\lambda(k(x-2\alpha t)+\xi_0)}}\right]^{\frac{1}{2}} e^{i(\alpha x+\beta t)}$$
$$= \pm\sqrt{\frac{k\lambda}{2}}\left[-\frac{1}{2}+\frac{1}{2}\coth\left(\frac{\lambda}{2}(k(x-2\alpha t)+\xi_0)\right)\right]^{\frac{1}{2}} e^{i(\alpha x+\beta t)}.$$

Comparing our results with Zhang's results [16], it can be seen that the results are same.

Case B: Suppose that $m = 2$; by equating the coefficients of Y^i $(i = 3, 2, 1, 0)$ on either side of (12), we have

$$\dot{a_2}(X) = h(X)a_2(X), \tag{23}$$

$$\dot{a_1}(X) = \left(\frac{8}{k}X^2 + g(X)\right)a_2(X) + h(X)a_1(X), \tag{24}$$

$$\dot{a_0}(X) = -2a_2(X)\left[\frac{(\beta+\alpha^2)}{k^2}X - \frac{1}{k^2}X^5\right] + \left(\frac{4}{k}X^2 + g(X)\right)a_1(X) + h(X)a_0(X), \tag{25}$$

$$a_1(X)\left[\frac{(\beta+\alpha^2)}{k^2}X - \frac{1}{k^2}X^5\right] = g(X)a_0(X). \tag{26}$$

Since $a_i(X)$ $(i = 0, 1, 2)$ are polynomials, then from (23) we deduce that $a_2(X)$ is constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees of $g(X)$, $a_1(X)$ and $a_2(X)$, we conclude that $\deg(g(X)) = 2$ only. Suppose that $g(X) = A_0 + A_1X + A_2X^2$; then we find $a_1(X)$ and $a_0(X)$ as follows:

$$a_1(X) = B_0 + A_0X + \frac{A_1}{2}X^2 + \left(\frac{8}{3k} + \frac{A_2}{3}\right)X^3, \tag{27}$$

$$\begin{aligned} a_0(X) = {} & d + A_0B_0X + \frac{1}{2}\left(-\frac{2(\beta+\alpha^2)}{k^2} + A_0^2 + A_1B_0\right)X^2 \\ & + \frac{1}{3}\left(\frac{3}{2}A_0A_1 + \left(\frac{4}{k} + A_2\right)B_0\right)X^3 + \frac{1}{4}\left(A_0\left(\frac{8}{3k} + \frac{A_2}{3}\right) + \frac{A_1^2}{2} + A_0\left(A_2 + \frac{4}{k}\right)\right)X^4 \\ & + \frac{1}{5}\left(A_1\left(\frac{8}{3k} + \frac{A_2}{3}\right) + \frac{1}{2}\left(A_2 + \frac{4}{k}\right)A_1\right)X^5 + \frac{1}{6}\left(\frac{2}{k^2} + \left(A_2 + \frac{4}{k}\right)\left(\frac{8}{3k} + \frac{A_2}{3}\right)\right)X^6. \end{aligned} \tag{28}$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in the last equation in (26) and setting all the coefficients of powers of X to zero, we then obtain a system of nonlinear algebraic equations and by solving it with the aid of Maple, we obtain

$$d = 0, \quad B_0 = 0, \quad A_0 = 0, \quad A_1 = 0, \quad A_2 = -\frac{2}{k}, \tag{29}$$

where α and β are arbitrary constants.

$$d = 0, \quad B_0 = 0, \quad A_0 = 0, \quad A_1 = 0, \quad A_2 = -\frac{4}{k}, \quad \beta = -\alpha^2, \tag{30}$$

where α is an arbitrary constant.

Using the conditions (29) in (11), we get

$$Y(\xi) = \frac{(-X^2(\xi) \pm \sqrt{(\beta+\alpha^2)})X(\xi)}{k}. \tag{31}$$

Combining (31) with (9), we obtain the exact solution to Eq. (8) and then the exact solutions to the Eckhaus equation can be written as

$$\Psi(x,t) = \pm\sqrt{\frac{k\lambda}{2}}\left[\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{\lambda}{2}(k(x-2\alpha t)+\xi_0)\right)\right]^{\frac{1}{2}} e^{i(\alpha x+\beta t)}, \tag{32}$$
$$\Psi(x,t) = \pm\sqrt{\frac{k\lambda}{2}}\left[-\frac{1}{2}+\frac{1}{2}\coth\left(\frac{\lambda}{2}(k(x-2\alpha t)+\xi_0)\right)\right]^{\frac{1}{2}} e^{i(\alpha x+\beta t)},$$

where $\lambda = \frac{2}{k}\sqrt{(\beta+\alpha^2)}$ and ξ_0 is an arbitrary constant.

Similarly, in the case of (30), from (11), we obtain

$$Y(\xi) = -\frac{1}{k}X^3(\xi), \tag{33}$$

and then the exact solution of the Eckhaus equation can be written as

$$\Psi(x,t) = \pm\sqrt{\frac{k}{2}}\left[\frac{1}{(k(x-2\alpha t)+\xi_0)}\right]^{\frac{1}{2}} e^{i(\alpha x-\alpha^2 t)}, \tag{34}$$

where ξ_0 is an arbitrary constant.

4. Conclusion

In this work, we obtained exact solutions of the Eckhaus equation by using the first-integral method. The results show that this method is efficient.

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