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Common Eigenvalue Problem and Periodic Schrödinger Operators

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Let $\mathfrak{A}$ be a subset of the family of all self-adjoint extensions of a symmetric operator A_0 with equal deficiency indices in a Hilbert space. Assuming that A_0 has a purely residual spectrum we describe the set of eigenvalues common to all self-adjoint extensions from $\mathfrak{A}$. This abstract result is used to show that the one-dimensional periodic Schrödinger operator with local point interactions is absolutely continuous. © 1999 Academic Press

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1. INTRODUCTION

The aim of the paper is to propose a new point of view on the spectral theory of periodic differential operators. For the sake of illustration consider a simple example—the Schrödinger operator $H = -d^2/dx^2 + V$ in $L^2(\mathbb{R})$ with a periodic real-valued potential $V(x) = V(x + 2\pi)$, $x \in \mathbb{R}$. The spectral analysis of H is based on the Floquet decomposition (see [16]). The essence of this method is that the spectrum of H can be recovered from

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the spectra of the family of self-adjoint operators $H(k) = -d^2/dx^2 + V$, $k \in [0, 1)$, acting in $L^2(0, 2\pi)$, with the *quasi-periodic* boundary conditions

$$u(2\pi) = e^{i2\pi k} u(0), \qquad u'(2\pi) = e^{i2\pi k} u'(0).$$

The parameter k is called the *quasi-momentum*. Denote by $E_l(k)$ the eigenvalues of the operator $H(k)$. Then the spectrum of the initial operator H is given by the union of *spectral bands*:

$$\sigma(H) = \bigcup_l I_l, \qquad I_l = \bigcup_{k \in [0, 1)} E_l(k).$$

Similar representation holds for the multi-dimensional analog $-\Delta + V$ of H, when the quasi-momentum k is vector-valued. If one of the functions $E_l(\cdot)$ is constant in k, then the corresponding band degenerates into a point, which represents an eigenvalue of H. A central issue in the spectral theory of periodic problems is to find out whether such a degeneration really occurs for a given operator.

In the case of the operator $H = -d^2/dx^2 + V$ one can show, using ODE methods, that the eigenvalues $E_l(k)$ are analytic in k and there are no constants among them (see [16]). This implies that H is absolutely continuous. In the multi-dimensional case the standard (and the only) approach to absolute to absolute continuity is due to L. E. Thomas (see [17] and also [16]). The key idea is to check, using the analytic perturbation theory, that the eigenvalues E_l cannot be constant when one extends the operator to the complex plane in one of the components of the quasi-momentum k.

In this paper, instead of the analytic perturbation technique, we propose a different point of view. We regard the operators $H(k)$ as a family of self-adjoint extensions of the symmetric operator $A_0 = -d^2/dx^2 + V$ initially defined on the H^2-functions with the boundary conditions $u(0) = u(2\pi) = u'(0) = u'(2\pi) = 0$. Now the fact that $E_l(k)$ are non-constant can be interpreted as the absence of eigenvalues common to all $H(k)$, $k \in [0, 1)$. Having in mind applications to more general periodic operators, it is natural and useful to study common eigenvalues in the abstract setting. Namely, we start off with a symmetric operator A_0 in a Hilbert space $\mathscr{H}$ with purely residual spectrum, i.e., with $\sigma_r(A_0) = \mathbb{C}$. Then its deficiency indices are automatically equal: $n_+ = n_- = n$. Next we pick a subset $\mathfrak{A}$ of the family of all self-adjoint extensions of A_0 and describe the set

$$\Sigma_p = \Sigma_p(\mathfrak{A}) = \bigcap_{A \in \mathfrak{A}} \sigma_p(A)$$

of common eigenvalues for the family $\mathfrak{A}$ (see Theorem 3.6). To that end we make systematic use of the formalism of the *boundary value space* (BVS).

Essentially, this notion is based on an abstract version of Green's formula (see (2.1)) for the operator A_0^*. The BVS technique associates to every self-adjoint extension A a unique unitary operator $U = U_A$ acting in an auxiliary Hilbert space $\mathfrak{h}$ of dimension n, which can be interpreted as the space of "boundary values" of the elements of the domain $D(A_0^*)$. One can always take $\mathfrak{h} = \ker(A_0^* + i)$, but the choice of $\mathfrak{h}$ is not unique and as a rule it is more convenient to use a different realization of $\mathfrak{h}$. For instance, in the case $n \in \mathbb{N}$ one can assume that $\mathfrak{h} = \mathbb{C}^n$. In general the construction of a suitable BVS for a given symmetric operator may present an independent difficult problem (see [8]). The crucial point is that the spectral properties of A can be described in terms of those of the corresponding operator U and the *characteristic function* of the operator A_0 (see Theorem 3.1). This property constitutes the basis of our approach.

The boundary value space technique has been extensively used in the works of Soviet mathematicians since the 70's for investigating the spectral properties of ordinary and partial differential operators. However the idea to select self-adjoint extensions of symmetric operators by imposing "boundary conditions" was proposed much earlier by J. W. Calkin in [6] (see also [7]). Apparently the progress made in the Soviet School has not been widely known in the West partly because the relevant bibliography is not easy to find. Thus, to facilitate the reading and to make the paper self-contained, some results, that are available in the Soviet mathematical literature, are given with full proofs. At the same time we do not provide detailed bibliographical comments, but rather refer to the book [8] for a comprehensive account of the subject.

In this paper, we use our approach to analyze the spectrum of the one-dimensional periodic Schrödinger operator with local point interactions (see Section 4). Our motivation is two-fold: firstly, this operator is convenient for illustration of our method in view of its relative simplicity. Secondly, this problem seems to be worth studying, since the spectral properties of operators with point interactions have been intensively investigated in the literature (see [1], [15] and references therein). In particular, the operator with equidistant periodic δ- and δ'-interactions with $V = 0$ was shown to be absolute continuous. Our method allows us to consider the most general local point interactions, not necessarily equidistant, with a periodic potential $V \in L^1_{\text{loc}}(\mathbb{R})$. However, to avoid cumbersome technicalities we assume in this paper that $V \in L^2_{\text{loc}}(\mathbb{R})$. In contrast to δ- or δ'-interactions, in the general case the operator may break up in the infinite orthogonal sum of independent operators with discrete spectrum. In this case due to the periodicity the spectrum of the full operator consists of isolated eigenvalues of infinite multiplicity. If such a decoupling does not occur, we prove, relying upon the abstract results on common eigenvalues from Section 3, that the spectrum is absolutely continuous.

We believe that our approach can be also applied to higher order ordinary differential operators without any conceptual difficulties. However the multi-dimensional case apparently will contain much more serious obstructions.

2. BOUNDARY VALUE SPACE

2.1. *Boundary Value Space and Self-adjoint Extensions*

Here we provide general definitions and information on the boundary value space of a symmetric operator. Most of the results quoted in this subsection can be found in [8].

Below we systematically use the following notation.

We denote by $\mathscr{H}$ the underlying complex Hilbert space. Lower and upper case gothic letters denote various auxiliary Hilbert spaces: $\mathfrak{h}$, $\mathfrak{g}$, $\mathfrak{H}$. To distinguish scalar products in distinct spaces we use the subscripts: $(\cdot,\cdot)_{\mathscr{H}}$, $(\cdot,\cdot)_{\mathfrak{h}}$ etc, $\|f\| = \sqrt{(f,f)_{\mathscr{H}}}$. None of the above spaces is assumed to be separable. Throughout the paper we repeatedly use the fact that two Hilbert spaces have the same dimension if they are linearly homeomorphic.

The symbols "$\dot{+}$" and "$\oplus$" denote the *direct* and *orthogonal* sums respectively. The writing $T\colon \mathfrak{h} \to \mathfrak{g}$ means that T is a bounded operator acting between the Hilbert spaces $\mathfrak{h}$, $\mathfrak{g}$.

Everywhere below A_0 denotes a densely defined closed symmetric operator in the space $\mathscr{H}$ with equal (finite or infinite) deficiency indices.

DEFINITION 2.1. Let $\mathfrak{h}$ be a complex Hilbert space and let Γ_1, Γ_2 be two linear mappings from $D(A_0^*)$ into $\mathfrak{h}$. The triple $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ is called a *boundary value space* (BVS) of the operator A_0 if the following two conditions are satisfied:

1. For any $f, g \in D(A_0^*)$

$$(A_0^* f, g)_{\mathscr{H}} - (f, A_0^* g)_{\mathscr{H}} = (\Gamma_1 f, \Gamma_2 g)_{\mathfrak{h}} - (\Gamma_2 f, \Gamma_1 g)_{\mathfrak{h}}; \tag{2.1}$$

2. For any $F_1, F_2 \in \mathfrak{h}$ there is an $f \in D(A_0^*)$ such that $\Gamma_1 f = F_1$, $\Gamma_2 f = F_2$.

Note that (2.1) can be also rewritten in a different form:

$$\begin{aligned} 2i[(A_0^* f, g)_{\mathscr{H}} - (f, A_0^* g)_{\mathscr{H}}] \\ = ((\Gamma_1 - i\Gamma_2) f, (\Gamma_1 - i\Gamma_2) g)_{\mathfrak{h}} - ((\Gamma_1 + i\Gamma_2) f, (\Gamma_1 + i\Gamma_2) g)_{\mathfrak{h}}, \end{aligned} \tag{2.2}$$

which will be used later on.

The next proposition proclaims the existence of a BVS:

PROPOSITION 2.2. *For any densely defined closed symmetric operator with equal deficiency indices* (n, n) *there exists a BVS* $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ *with* $\dim \mathfrak{h} = n$.

The proof of this Proposition (see e.g. [8]) is constructive. In particular, it shows that $\ker(A_0^* + i)$ can be viewed as the space $\mathfrak{h}$ if one chooses appropriately the mappings Γ_1 and Γ_2. We emphasize however that the choice of a BVS is not unique.

Below we list without proofs some useful properties of the operators Γ_1, Γ_2:

- An element f belongs to $D(A_0)$ if and only if $\Gamma_1 f = \Gamma_2 f = 0$;
- The operators Γ_1, Γ_2 are bounded as mappings from the space $D(A_0^*)$ equipped with the graph scalar product $(A_0^* u, A_0^* v)_{\mathscr{H}} + (u, v)_{\mathscr{H}}$;
- The operators Γ_1 and Γ_2 induce continuous bijections

$$\Gamma_1 \oplus \Gamma_2 : D(A_0^*)/D(A_0) \to \mathfrak{h} \oplus \mathfrak{h},$$

$$\Gamma_1 \pm i\Gamma_2 : D(A_0^*)/D(A_0) \to \mathfrak{h}.$$

Using the last property we conclude that for any two BV spaces $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ and $(\mathfrak{g}, \Xi_1, \Xi_2)$ associated with the given operator A_0 we have $\dim \mathfrak{h} = \dim \mathfrak{g}$.

Since the deficiency indices of the operator A_0 are equal, it admits self-adjoint extensions. Using a BVS for A_0 we can actually establish a one-to-one correspondence between the elements A of the collection of all self-adjoint extensions $\mathfrak{S}$ and the set of unitary operators in the space $\mathfrak{h}$. Namely, for a unitary operator $U: \mathfrak{h} \to \mathfrak{h}$ denote by $D(A_0^*, U) \subset D(A_0^*)$ the subset of all vectors $f \in D(A_0^*)$ such that

$$U(\Gamma_1 + i\Gamma_2) f = (\Gamma_1 - i\Gamma_2) f. \tag{2.3}$$

Clearly $D(A_0^*, U)$ is dense in $\mathscr{H}$ for $D(A_0) \subset D(A_0^*, U)$. Observe that for any $f, g \in D(A_0^*, U)$ the r.h.s. of (2.2) equals zero. Moreover, the following Proposition holds:

PROPOSITION 2.3. *Let* $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ *be a BVS for the operator* A_0, *and let* U *be a unitary operator in the space* $\mathfrak{h}$. *Then the restriction of* A_0^* *on the set* $D(A_0^*, U)$ *defines a self-adjoint extension of* A_0.

Conversely, for any self-adjoint extension A *of the operator* A_0 *there exists a unique unitary operator* U *acting in* $\mathfrak{h}$ *such that* $D(A) = D(A_0^*, U)$.

From now on we sometimes use the notation A_U for a self-adjoint extension of the operator A_0 corresponding to the unitary operator $U: \mathfrak{h} \to \mathfrak{h}$. In this case we say that U *parametrizes* A in the BVS $(\mathfrak{h}, \Gamma_1, \Gamma_2)$. For any set $\mathfrak{A}$ of self-adjoint extensions of A we denote by $\mathfrak{U}_{\mathfrak{A}}$ the family of unitary operators in $\mathfrak{h}$ parametrizing the elements $A \in \mathfrak{A}$. Similarly, the unitary operator parametrizing given self-adjoint extension $A \in \mathfrak{S}$ will be denoted by U_A.

2.2. *Linear-Fractional Transformation and Parametrization*

As it was pointed out above, a construction procedure for a BVS is not unique. Let $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ and $(\mathfrak{g}, \Xi_1, \Xi_2)$ be two distinct BV spaces associated with a closed symmetric operator A_0. We shall show that the unitary operators U_A and W_A parametrizing an operator $A \in \mathfrak{S}$ in the BV spaces $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ and $(\mathfrak{g}, \Xi_1, \Xi_2)$ respectively, are related by a so-called *linear-fractional transformation*. To do so we shall need first of all the notion of a $\mathbb{J}$-unitary operator. Introduce in $\mathfrak{H} = \mathfrak{h} \oplus \mathfrak{h}$ the operator

$$\mathbb{J} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2.4}$$

An operator $\mathbb{T}$ acting in $\mathfrak{H}$ is said to be $\mathbb{J}$-unitary if

$$(\mathbb{J}X, \tilde{X})_{\mathfrak{H}} = (\mathbb{J}\mathbb{T}X, \mathbb{T}\tilde{X})_{\mathfrak{H}} \tag{2.5}$$

for any $X, \tilde{X} \in \mathfrak{H}$, and $\operatorname{ran} \mathbb{T} = \mathfrak{H}$. It is known (see e.g. [3]) that $\mathbb{T}$ is bounded and therefore can be represented as follows:

$$\mathbb{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

with bounded blocks $T_{jk} : \mathfrak{h} \to \mathfrak{h}$. It is easy to see that along with $\mathbb{T}$ the operator $(\mathbb{J}\mathbb{T})^*$ is also $\mathbb{J}$-unitary. Each $\mathbb{J}$-unitary operator gives rise to a linear-fractional transformation described in the next theorem:

THEOREM 2.4. *Let $\mathbb{T}$ be a $\mathbb{J}$-unitary operator and U be an arbitrary unitary operator in $\mathfrak{h}$. Then*

1. *The operators $T_{11} + T_{12}U$, $T_{21} + T_{22}U$ and $T_{22} - UT_{12}$, $T_{21} - UT_{11}$ are boundedly invertible*;
2. *The transformation*

$$\Omega(U) = \Omega(U, \mathbb{T}) = [T_{21} + T_{22}U][T_{11} + T_{12}U]^{-1} \tag{2.6}$$

maps the set of unitary operators in $\mathfrak{h}$ into itself:

3. *The mapping*

$$\Omega^*(U) = -\Omega(U, (\mathbb{J}\mathbb{T})^*) = -[T_{12}^* - T_{22}^*U][T_{11}^* - T_{21}^*U]^{-1} \tag{2.7}$$

is the inverse to Ω, *i.e.*

$$\Omega(\Omega^*(U)) = \Omega^*(\Omega(U)) = U. \tag{2.8}$$

4. *For any two unitary operators* U, W *in* $\mathfrak{h}$ *the equality takes place*:

$$\dim \ker(U - W) = \dim \ker(\Omega(U) - \Omega(W)). \tag{2.9}$$

The proof is postponed until the Appendix.

Since $\dim \mathfrak{h} = \dim \mathfrak{g}$, one can find an isometric map $V: \mathfrak{g} \to \mathfrak{h}$. As the next theorem shows, the correspondence between $U = U_A$ and $W = W_A$ is completely determined by the choice of such a map.

THEOREM 2.5. *Let* $(\mathfrak{h}, \Gamma_1, \Gamma_2)$, $(\mathfrak{g}, \Xi_1, \Xi_2)$ *be two BV spaces of a given operator* A_0, *and let* V *be an isometric map from* $\mathfrak{g}$ *onto* $\mathfrak{h}$. *Then there exists a* $\mathbb{J}$-*unitary operator* $\mathbb{T}$ *acting in* $\mathfrak{H} = \mathfrak{h} \oplus \mathfrak{h}$ *such that for any self-adjoint extension* A

$$U_A = \Omega(V W_A V^*), \tag{2.10}$$

with the transformation Ω *defined in* (2.6).

Proof. Along with the BVS $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ define a new BVS $(\mathfrak{h}, \tilde{\Gamma}_1, \tilde{\Gamma}_2)$ with $\tilde{\Gamma}_j = V\Xi_j$, $j = 1, 2$. To construct the operator $\mathbb{T}$ associate to every $X = (x', x'') \in \mathfrak{H}$ a vector $z \in D(A_0^*)$ such that

$$(\tilde{\Gamma}_1 + i\tilde{\Gamma}_2)\, z = x', \qquad (\tilde{\Gamma}_1 - i\tilde{\Gamma}_2)\, z = x''.$$

Set $Y = (y', y'')$ with

$$(\Gamma_1 + i\Gamma_2)\, z = y', \qquad (\Gamma_1 - i\Gamma_2)\, z = y''.$$

For a given X these relations define Y uniquely. Define the operator $\mathbb{T}$ to be the linear operator transforming X into Y. Obviously $\operatorname{ran} \mathbb{T} = \mathfrak{H}$. Moreover, in view of (2.2) for any $X, \tilde{X} \in \mathfrak{H}$ and $Y = \mathbb{T}X$, $\tilde{Y} = \mathbb{T}\tilde{X}$ one has

$$(x', \tilde{x}')_{\mathfrak{h}} - (x'', \tilde{x}'')_{\mathfrak{h}} = (y', \tilde{y}')_{\mathfrak{h}} - (y'', \tilde{y}'')_{\mathfrak{h}}.$$

This implies that

$$(\mathbb{J}X, \tilde{X})_{\mathfrak{H}} = (\mathbb{J}\mathbb{T}X, \mathbb{T}\tilde{X})_{\mathfrak{H}},$$

so that $\mathbb{T}$-is $\mathbb{J}$-unitary. Using the entries T_{jk} of $\mathbb{T}$ one can express Γ_1, Γ_2 in terms of $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$:

$$\Gamma_1 + i\Gamma_2 = T_{11}(\tilde{\Gamma}_1 + i\tilde{\Gamma}_2) + T_{12}(\tilde{\Gamma}_1 - i\tilde{\Gamma}_2);$$

$$\Gamma_1 - i\Gamma_2 = T_{21}(\tilde{\Gamma}_1 + i\tilde{\Gamma}_2) + T_{22}(\tilde{\Gamma}_1 - i\tilde{\Gamma}_2).$$

Let $\tilde{U}$ be the unitary operator parametrizing A in the BVS $(\mathfrak{h}, \tilde{\Gamma}_1, \tilde{\Gamma}_2)$. Thus by (2.3)

$$(\tilde{\Gamma}_1 - i\tilde{\Gamma}_2)\, f = \tilde{U}(\tilde{\Gamma}_1 + i\tilde{\Gamma}_2)\, f$$

for all $f \in D(A)$, which ensures that

$$(\Gamma_1 + i\Gamma_2)\, f = (T_{11} + T_{12}\tilde{U})(\tilde{\Gamma}_1 + i\tilde{\Gamma}_2)\, f,$$

$$(\Gamma_1 - i\Gamma_2)\, f = (T_{21} + T_{22}\tilde{U})(\tilde{\Gamma}_1 + i\tilde{\Gamma}_2)\, f.$$

According to definition (2.6) this implies that

$$(\Gamma_1 - i\Gamma_2)\, f = \Omega(\tilde{U})(\Gamma_1 + i\Gamma_2)\, f$$

with the unitary operator $\Omega(\cdot)$ defined in (2.6). By (2.3) this means that $\Omega(\tilde{U})$ parametrizes the operator A in the BVS $(\mathfrak{h}, \Gamma_1, \Gamma_2)$. In view of Proposition 2.3 a parametrizing operator is unique, so that $U = \Omega(\tilde{U})$. Noticing that $\tilde{U} = VWV^*$, we arrive at (2.10). ∎

Corollary 2.6. *Let $V\colon \mathfrak{g} \to \mathfrak{h}$ be an isometric map, and let $\mathfrak{U} = \mathfrak{U}_{\mathfrak{A}}$ and $\mathfrak{M} = \mathfrak{M}_{\mathfrak{A}}$ be the sets of unitary operators parametrizing the family $\mathfrak{A} \subset \mathfrak{S}$ in the BV spaces $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ and $(\mathfrak{g}, \Xi_1, \Xi_2)$ respectively. Then the transformation* (2.10) *defines a bijection of $\mathfrak{M}$ onto $\mathfrak{U}$.*

3. COMMON EIGENVALUE PROBLEM

3.1. *Characteristic Function*

In this subsection we investigate self-adjoint extensions of a *regular symmetric operator* A_0, i.e., an operator A_0 with $\sigma_r(A_0) = \mathbb{C}$. This property is equivalent to the fact that for any $\lambda \in \mathbb{R}$ there is a positive constant $c = c_\lambda$ such that

$$\|(A_0 - \lambda)\, f\| \geqslant c_\lambda \|f\|, \qquad \forall f \in D(A_0). \tag{3.1}$$

This requirement implies in particular that the operator A_0 has no eigenvalues and is simple. Recall that a symmetric operator A_0 is said to be *simple* if for any decomposition

$$A_0 = A_0^1 \oplus A_0^2 \tag{3.2}$$

the operators A_0^1, A_0^2 are not self-adjoint.

Among all self-adjoint extensions we single out the following one-parameter set. For any $\lambda \in \mathbb{R}$ define the extension $A(\lambda)$ to be a restriction of the operator A_0^* on the domain

$$D(A(\lambda)) = D(A_0) \dot{+} \ker(A_0^* - \lambda).$$

Using (3.1) one can show (see e.g. [10]) that $A(\lambda) \in \mathfrak{S}$. The extensions $A(\lambda)$ were introduced by J. von Neumann in [14]. They are fundamental for the Krein's theory of semi-bounded extensions of symmetric operators (see [12], [2]).

The unitary operator $U: \mathfrak{h} \to \mathfrak{h}$ parametrizing $A(\lambda)$ will be denoted by $U(\lambda)$. We shall call $U(\lambda)$ *the characteristic function* of the regular symmetric operator A_0 associated with the BVS $(\mathfrak{h}, \Gamma_1, \Gamma_2)$. Note that a self-adjoint extension A of the operator A_0 coincides with $A(\lambda)$ for some $\lambda \in \mathbb{R}$ if and only if

$$\ker(A - \lambda) = \ker(A_0^* - \lambda).$$

The special role of the characteristic function is seen from the following result which was establish essentially in [10] (see also [11]).

THEOREM 3.1. *A number $\lambda_0 \in \mathbb{R}$ is an eigenvalue of the operator A_U of multiplicity m if and only if the point $\mu = 0$ is an eigenvalue of the operator $U - U(\lambda_0)$ of the same multiplicity.*

For the sake of completeness we provide the proof.

Proof. We shall show that the map $\Gamma_1 + i\Gamma_2$ is a homeomorphism of $\ker(A - \lambda_0)$ onto $\ker(U - U(\lambda_0))$.

In view of (3.1) every element $f \in \ker(A - \lambda_0)$ belongs to $\ker(A_0^* - \lambda_0) \subset D(A(\lambda_0))$, and therefore

$$U(\lambda_0)(\Gamma_1 + i\Gamma_2) f = (\Gamma_1 - i\Gamma_2) f.$$

On the other hand by (2.3) the element f obeys the relation

$$U(\Gamma_1 + i\Gamma_2) f = (\Gamma_1 - i\Gamma_2) f. \tag{3.3}$$

The last two equalities imply that $g = (\Gamma_1 + i\Gamma_2) f \in \ker(U - U(\lambda_0))$. Moreover, since $\Gamma_1 + i\Gamma_2$ is invertible on $D(A_0^*)/D(A_0)$, the mapping $f \to g$ is an injection.

Conversely, let $g \in \ker(U - U(\lambda_0))$. Then by Definition 2.1 one can find a vector $\tilde{f} \in D(A_0^*)$ such that

$$(\Gamma_1 + i\Gamma_2) \tilde{f} = g, \qquad (\Gamma_1 - i\Gamma_2) \tilde{f} = Ug.$$

Thus (3.3) is fulfilled which implies that $\tilde{f} \in D(A)$. Moreover,

$$U(\lambda_0)(\Gamma_1 + i\Gamma_2)\,\tilde{f} = U(\lambda_0)\,g = Ug = (\Gamma_1 - i\Gamma_2)\,\tilde{f},$$

so that $\tilde{f} \in D(A(\lambda_0)) = D(A_0) \dotplus \ker(A_0^* - \lambda_0)$ as well. Let f be the projection of $\tilde{f}$ on $\ker(A_0^* - \lambda_0)$. Then $(\Gamma_1 + i\Gamma_2)\,f = g$.

Therefore the mapping $\Gamma_1 + i\Gamma_2 : \ker(A - \lambda_0) \to \ker(U - U(\lambda_0))$ is an injection and surjection at the same time.

To prove the continuity of $\Gamma_1 + i\Gamma_2$ it suffices to recall that this map is continuous on $D(A_0^*)$ equipped with the graph-norm of A_0^*, and to notice that the graph-norm is equivalent to the ordinary norm on $\ker(A - \lambda_0)$ for any $\lambda_0 \in \mathbb{R}$. ▮

This result will play a crucial role in the next subsection.

Without going into details note that there are several definitions for the characteristic function of an arbitrary symmetric operator with equal deficiency indices (see [11] and references therein). The closest to our definition is the one given in [11].

3.2. *Common Eigenvalues*

Let A_0 be a regular symmetric operator. Let $\mathfrak{A} \subset \mathfrak{S}$ be a non-empty subset. Our objective is to describe the set of common eigenvalues of the family $\mathfrak{A}$, i.e.

$$\Sigma_p = \Sigma_p(\mathfrak{A}) = \bigcap_{A \in \mathfrak{A}} \sigma_p(A).$$

Note that for a non-simple symmetric operator A_0 the presence of eigenvalues common to all $A \in \mathfrak{A}$ may be due to the existence of eigenvalues of the relevant self-adjoint part of A_0, i.e., A_0^1 or A_0^2 in the decomposition (3.2). However, we always assume that A_0 is regular, and consequently simple. Therefore such "trivial" common eigenvalues are ruled out.

If $n \in \mathbb{N}$ the operators from $\mathfrak{S}$ have purely discrete spectrum. If $n \notin \mathbb{N}$ then $\mathfrak{S}$ always contains operators with non-empty essential spectrum.

To describe Σ_p in terms of the characteristic function introduce the following

Definition 3.2. Let $\mathfrak{U}$, $\mathfrak{U}^d$ be two sets of unitary operators in $\mathfrak{h}$, the set $\mathfrak{U}$ being non-empty. Then $\mathfrak{U}^d$ is said to be *dual* to $\mathfrak{U}$ if

$$\mathfrak{U}^d = \{W : \ker(W - U) \neq \{0\},\ \forall U \in \mathfrak{U}\}.$$

In view of Theorem 3.1 this definition immediately leads to the formula

$$\Sigma_p(\mathfrak{A}) = \{\lambda \in \mathbb{R} : U(\lambda) \in \mathfrak{U}_{\mathfrak{A}}^d\}. \tag{3.4}$$

This formula has a drawback: it contains an object—$\mathfrak{U}^d_{\mathfrak{A}}$—which depends on the choice of a BVS. To remedy this we extend the notion of duality to the class $\mathfrak{S}$:

DEFINITION 3.3. Let $\mathfrak{A}$, $\mathfrak{A}^d$ be two sets of self-adjoint operators in $\mathfrak{S}$, $\mathfrak{A}$ being non-empty. Then $\mathfrak{A}^d$ is said to be *dual* to $\mathfrak{A}$ if

$$\mathfrak{U}_{\mathfrak{A}^d} = \mathfrak{U}^d_{\mathfrak{A}},$$

or, in other words,

$$\mathfrak{A}^d = \{A_W \in \mathfrak{S} \colon W \in \mathfrak{U}^d_{\mathfrak{A}}\}. \tag{3.5}$$

Let us show that the duality in $\mathfrak{S}$ is indeed invariant with respect to the choice of a BVS for the initial symmetric operator. To this end we first prove a statement analogous to Corollary 2.6:

LEMMA 3.4. *Let $V\colon \mathfrak{g} \to \mathfrak{h}$ be an isometric map, and let $\mathfrak{U} = \mathfrak{U}_{\mathfrak{A}}$ and $\mathfrak{M} = \mathfrak{M}_{\mathfrak{A}}$ be the sets of unitary operators parametrizing the family $\mathfrak{A} \subset \mathfrak{S}$ in the BV spaces $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ and $(\mathfrak{g}, \Xi_1, \Xi_2)$ respectively. Then the transformation $U = \Omega(VWV^*)$ defines a bijection of $\mathfrak{M}^d$ onto $\mathfrak{U}^d$.*

Proof. Applying (2.9) to Ω and Ω^* we obtain that

$$\begin{aligned} \dim\ker(U - U') &= \dim\ker(V^*\Omega^*(U)\,V - V^*\Omega^*(U')\,V), \\ \dim\ker(W - W') &= \dim\ker(\Omega(VWV^*) - \Omega(VW'V^*)) \end{aligned} \tag{3.6}$$

for any unitary U, U' and W, W' acting in the spaces $\mathfrak{h}$ and $\mathfrak{g}$ respectively. According to part 3 of Theorem 2.4 the transformations Ω and Ω^* are mutually inverse. Thus in view of Corollary 2.6 the equalities (3.6) mean that the inclusions $U \in \mathfrak{U}^d$, $W \in \mathfrak{M}^d$ imply the inclusions $V^*\Omega^*(U)\,V \in \mathfrak{M}^d$, $\Omega(VWV^*) \in \mathfrak{U}^d$. Using (2.8) again these relations lead to the required conclusion. ∎

This lemma in combination with Theorem 2.5 shows that the dual family (3.5) does not depend on the choice of a BVS. With the help of this family one can give another description of the set Σ_p, which immediately follows from (3.4):

$$\Sigma_p(\mathfrak{A}) = \{\lambda \colon A(\lambda) \in \mathfrak{A}^d\}. \tag{3.7}$$

For the use in the next section we shall rephrase this statement in terms of the eigenvalues.

DEFINITION 3.5. A number λ_0 is called an eigenvalue of the *maximal multiplicity* of the operator $A \in \mathfrak{S}$ if $A = A(\lambda_0)$. The set of all eigenvalues of maximal multiplicity of the operator A is denoted by $\sigma_{\max}(A)$.

If the deficiency indices of A_0 are equal to $n \in \mathbb{N}$ then for any self-adjoint extension A the set $\sigma_{\max}(A)$ coincides with the set of all eigenvalues of multiplicity n.

It is now easy to conclude that

$$\Sigma_p(\mathfrak{A}) = \bigcap_{A \in \mathfrak{A}^d} \sigma_{\max}(A). \tag{3.8}$$

Note that in contrast to (3.4) this formula does not contain the characteristic function of the operator A_0, which is, as a rule, difficult to find.

Let us summarize what has been done in this section:

THEOREM 3.6. *Let $\mathfrak{A} \subset \mathfrak{S}$ be a non-empty family of self-adjoint extensions of a regular symmetric operator A_0 in the Hilbert space $\mathscr{H}$. Let $\mathfrak{U}^d_{\mathfrak{A}}$ and $\mathfrak{A}^d$ be the dual sets as defined above. Then the set $\Sigma_p(\mathfrak{A})$ of common eigenvalues of the family $\mathfrak{A}$ can be found by any of the three formulae* (3.4), (3.7), *or* (3.8).

This theorem allows us to give another interpretation of the known fact that if $n = 1$ and card $\mathfrak{A} \geqslant 2$, then $\Sigma_p(\mathfrak{A}) = \varnothing$ (see e.g. [4]). Indeed, according to Propositions 2.2 and 2.3 all self-adjoint extensions of the operator A_0 can be parametrized by unitary operators in a one-dimensional space, i.e., by numbers $\phi \in [0, 2\pi)$. In case if $\mathfrak{A}$ contains more than one extension, the corresponding numbers ϕ are all distinct, and therefore the class $\mathfrak{U}^d$ is empty, and hence $\Sigma_p = \varnothing$.

4. PERIODIC SCHRÖDINGER OPERATORS

4.1. *Schrödinger Operators with Point Interactions*

We are interested in spectral properties of the Schrödinger operators in $L^2(\mathbb{R})$ having the form $-d^2/dx^2 + V$, with some boundary conditions on a discrete set of points in $\mathbb{R}$. Such operators are called in [5] the operators with *local point interactions.*

Let $X = \{x_j\}$, $j = 1, 2, ..., N$ be a finite collection of distinct points on the interval $(0, 2\pi)$ enumerated in the increasing order. Define the infinite sequence of points $Y = \{y_j\}$, $j \in \mathbb{Z}$ by setting

$$y_{j+nN} = x_j + 2\pi n, \qquad j = 1, 2, ..., \quad N, n \in \mathbb{Z}.$$

The domain of the operator will be defined by boundary conditions relating the values $u(y_j-)$, $u'(y_j-)$ and $u(y_j+)$, $u'(y_j+)$. As in [5] we give the appropriate definitions in terms of unitary 2×2 matrices M_j associated with the points y_j:

DEFINITION 4.1. Let the interval Δ be either $\mathbb{R}$ or $[0, 2\pi)$ and let $\mathfrak{M}$ be a collection of unitary 2×2-matrices M_j, $j\in\mathbb{Z}$. A function $u\in\mathbf{H}^2(\Delta\backslash Y)$ is said to belong to the class $\mathscr{B}(\Delta)$ if it obeys the following condition:

$$\begin{pmatrix} u(y_j+)-iu'(y_j+) \\ u(y_j-)+iu'(y_j-) \end{pmatrix} = M_j \begin{pmatrix} u(y_j-)-iu'(y_j-) \\ u(y_j+)+iu'(y_j+) \end{pmatrix} \tag{4.1}$$

for any j such that $y_j\in\Delta$.

Although the functions $u\in\mathscr{B}(\Delta)$ are allowed to have discontinuities at the points $y_j\in\Delta$, the unitary of M_j ensures that the bracket

$$[u, v](x) = u(x)\,\overline{v'(x)} - u'(x)\,\overline{v(x)}$$

is continuous everywhere for any two functions $u, v\in\mathscr{B}(\Delta)$. Indeed, denote by $u^{\pm}_{-}(y_j)$ and $u^{-}_{+}(y_j)$ the 2-vectors in the l.h.s. and the r.h.s. of the equality (4.1) respectively. Then

$$\begin{aligned} &2([u, v](y_j+)-[u, v](y_j-)) \\ &\quad = i(u^{-}_{+}(y_j), v^{-}_{+}(y_j))_{\mathbb{C}^2} - i(u^{+}_{-}(y_j), v^{+}_{-}(y_j))_{\mathbb{C}^2} \\ &\quad = i(M_j u^{-}_{+}(y_j), M_j v^{-}_{+}(y_j))_{\mathbb{C}^2} - i(u^{+}_{-}(y_j), v^{+}_{-}(y_j))_{\mathbb{C}^2} = 0. \end{aligned} \tag{4.2}$$

We also point out that the boundary condition (4.1) at each point y_j can be of two types depending on the entries of the matrix M_j:

$$M_j = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

(here we omit the subscript j to avoid cumbersome notation). Due to the unitarity of M_j the coefficients m_{11}, m_{22} both either equal zero or are distinct from zero. In the latter case the requirement (4.1) establishes a one-to-one correspondence between $u(y_j-)$, $u'(y_j-)$ and $u(y_j+)$, $u'(y_j+)$. In this case, following [5] we shall call the boundary condition (4.1) *connecting*. If $m_{11}=m_{22}=0$ then the values of $u(y_j-)$, $u'(y_j-)$ and $u(y_j+)$, $u'(y_j+)$ are independent. More precisely, under this condition $m_{12}=e^{2i\theta_+}$, $m_{21}=e^{2i\theta_-}$ with some $\theta_\pm\in[0, \pi)$, and (4.1) reads as

$$\sin\theta_+\, u(y_j+) + \cos\theta_+\, u'(y_j+) = 0,$$

$$\sin\theta_-\, u(y_j-) - \cos\theta_t\, u'(y_j-) = 0.$$

In this case it is natural to say that the boundary condition at y_j is *decoupling*.

As was mentioned in the Introduction, the δ- or δ'-interactions are special cases of the condition (4.1). For example the matrix

$$M_j = e^{-i\theta} \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

realizes a δ-interaction of "strength" $2\tan\theta$ at the point y_j.

Let us now define the Schrödinger operators we shall be working with. To begin with, we are interested in periodic operators, so that the matrices $M_j \in \mathfrak{M}$ are supposed to satisfy the periodicity condition

$$M_j = M_{j+nN}, \qquad \forall j, n \in \mathbb{Z}. \tag{4.3}$$

Further, let $V \in L^2_{\mathrm{loc}}(\mathbb{R})$ be a real-valued periodic function with the period 2π. Let H be the operator in $L^2(\mathbb{R})$ defined as

$$Hu = H_0 u + Vu, \qquad H_0 u = -u'', \tag{4.4}$$

on the domain $D(H) = \mathscr{B}(\mathbb{R})$. Using the property that $\inf_j |y_{j+1} - y_j| > 0$, it is quite straightforward to check that H_0 is self-adjoint on this domain. The perturbation V is infinitesimally small with respect to H_0, which guarantees the self-adjointness of H on $D(H)$ and essential self-adjointness of H on any core domain of H_0.

It is interesting to remark that in contrast to ordinary Sturm–Liouville problems and problems with δ-, δ'-interactions, the domains of operators with general local point interactions are not invariant under the complex conjugation $u \to \bar{u}$.

The spectrum of H strongly depends on the boundary conditions (4.1). If at least one of them is decoupling, let us say, at x_1, then the operator H splits into the infinite orthogonal sum of Sturm–Liouville type operators $H_l = -d^2/dx^2 + V$ acting on the intervals $(x_1 + 2\pi l, x_1 + 2\pi(l+1))$, $l \in \mathbb{Z}$. Due to periodicity of V and M_j the spectra $\sigma(H_l)$ all coincide. Consequently the spectrum of H consists of isolated eigenvalues of infinite multiplicity. Our concern in this section is the spectrum of H under the condition that all M_j are connecting:

THEOREM 4.2. *Let $\mathfrak{M}$ be a sequence of unitary 2×2-matrices satisfying* (4.3) *and let $V \in L^2_{\mathrm{loc}}(\mathbb{R})$ be a real-valued 2π-periodic function. If the boundary conditions at the points $x_j \in X$ are connecting for all $j = 1, 2, ..., N$, then the spectrum of H is absolutely continuous.*

4.2. *Direct Integral*

The proof of Theorem 4.2 will be based on the standard Bloch (or Floquet) analysis of the operator H. Associate to the operator H the family of operators $H(k)$, $k \in [0, 1)$, acting in $L^2(0, 2\pi)$ and defined by the same differential expression as (4.4) on the domain

$$D(k) = \{u \in \mathscr{B}(0, 2\pi) : u(2\pi) = e^{2\pi i k} u(0),\ u'(2\pi) = e^{2\pi i k} u'(0)\}. \tag{4.5}$$

The functions $u \in D(k)$ are said to be *quasi-periodic*. Define the operator

$$(\mathscr{U}u)(x, k) = \sum_{n=-\infty}^{\infty} e^{-i2\pi k n} u(x + 2\pi n),$$

acting from $L^2(\mathbb{R})$ into the space

$$\hat{\mathscr{H}} = \int_{(0,1)}^{\oplus} L^2(0, 2\pi)\, dk.$$

It can be easily shown that the mapping $\mathscr{U}$ is isometric (see [16]). Repeating the standard argument from [16] one can also show that the following decomposition takes place:

$$\mathscr{U} H \mathscr{U}^* = \int_{(0,1)}^{\oplus} H(k)\, dk. \tag{4.6}$$

It is quite straightforward to see that the resolvent of $H(k)$ is compact, so that the operator $H(k)$ has discrete spectrum for each $k \in [0, 1)$. One of the central conclusions of the Bloch analysis is that the spectrum of the operator H can be described in terms of the spectra of $H(k)$:

$$\Sigma(H) = \bigcup_{k \in [0,1)} \sigma_d(H(k)).$$

To prove the absolute continuity of $\sigma(H)$ we need to study $\sigma_d(H(k))$ more closely.

First of all we shall prove that the eigenvalues of $H(k)$ are analytic functions of k. To this end we have to modify the operators $H(k)$ to ensure that their domains do not depend on k. Let $\phi \in C^\infty([0, 2\pi])$ be a function such that

$$\begin{aligned} &\phi(0) = 0, \qquad \phi(2\pi) = 2\pi, \\ &\phi'(x_j) = 0, \qquad \forall j = 1, 2, ..., N, \quad \text{and} \quad \phi'(0) = \phi'(2\pi) = 1. \end{aligned} \tag{4.7}$$

Define in $L^2(0, 2\pi)$ the operator

$$\tilde{H}(k) = e^{-ik\phi} H(k)\, e^{ik\phi}$$

with the domain consisting of all functions u such that $e^{ik\phi}u \in D(k)$. This operator is self-adjoint. Note that due to the conditions (4.7) the domain of $\tilde{H}(k)$ is

$$\tilde{D} = \{u \in \mathscr{B}(0, 2\pi) : u(2\pi) = u(0),\ u'(2\pi) = u'(0)\}.$$

In particular it does not depend on the parameter k. On the other hand a direct calculation shows that

$$\tilde{H}(k)\, u = (-id/dx + k\phi')^2\, u + Vu.$$

Therefore $\tilde{H}(k)$ is an analytic family of type A (see [9]). The resolvent of $\tilde{H}(k)$ is compact and therefore the results of analytic perturbation theory are applicable.

In accordance with the analytic perturbation theory (see [9], Theorem VII.3.9) there exists a sequence of scalar functions $\lambda_l(k)$, $l \in \mathbb{N}$, and a sequence of vector-functions $\psi_l(\cdot, k)$ such that

1. The functions λ_l, ψ_l are real-analytic on $(0, 1)$;
2. For each $k \in (0, 1)$ and $l \in \mathbb{N}$

$$\tilde{H}(k)\, \psi_l(k) = \lambda_l(k)\, \psi_l(k);$$

3. The sequence $\psi_l(k)$ forms an orthonormal basis in $L^2(0, 2\pi)$ for each $k \in (0, 1)$.

Absolute continuity of H results from the following theorem:

THEOREM 4.3. *Under the conditions of Theorem 4.2 there are no constants among the functions $\lambda_l(k)$, $k \in (0, 1)$.*

Indeed, this property is equivalent to the fact that there are no constants among the eigenvalues of the operators $H(k)$. Now the decomposition (4.6) yields the required absolute continuity due to the properties 1–3 and [16], Theorem XIII.86.

In the next subsection we view the operators $H(k)$ as extensions of one symmetric operator and use the results of Section 3 to prove Theorem 4.3.

4.3. *Quasi-Periodic Problem*

In the Hilbert space $\mathscr{H} = L^2(0, 2\pi)$ define the operator A_0 by the conditions

$$A_0 = H \upharpoonright D(A_0), \qquad D(A_0) = \mathscr{B}(0, 2\pi) \bigcap \mathring{\mathbf{H}}^2(0, 2\pi).$$

The operator A_0 is densely defined and closed in the space $\mathscr{H}$. Using (4.2) one can show that A_0 is symmetric and the adjoint A_0^* is defined by the same differential expression on the domain $D(A_0^*) = \mathscr{B}(0, 2\pi)$. Also, as in [5] one proves that the deficiency indices of A_0 are (2, 2). All self-adjoint extensions $A \in \mathfrak{S}$ of A_0 have discrete spectrum (see [13]). To apply to A_0 the results of the previous section we check that the requirement (3.1) is fulfilled:

LEMMA 4.4. *Suppose that the boundary conditions at the points $x_j \in X$, $j = 1, 2, ..., N$, are connecting. Then for any $\lambda \in \mathbb{R}$ there exists a positive constant c_λ such that* (3.1) *holds.*

Proof. Suppose that the converse is true. Then one can find a point $\lambda_0 \in \mathbb{R}$ and a sequence of functions $\phi_n \in D(A_0)$, $n = 1, 2, ...$ such that

$$\|\phi_n\| = 1, \qquad \|(A_0 - \lambda_0)\, \phi_n\| \to 0, \quad n \to \infty.$$

Let us analyze separately the following two options:

- The sequence ϕ_n is pre-compact in $\mathscr{H}$;
- The sequence ϕ_n is not pre-compact in $\mathscr{H}$.

If ϕ_n is pre-compact then extract from ϕ_n a convergent subsequence and denote its limit by ϕ. Since A_0 is closed we have $\phi \in D(A_0)$ and $(A_0 - \lambda_0)\, \phi = 0$. Thus on the interval $(0, x_1)$ the function ϕ obeys the differential equation $-\phi'' + V\phi - \lambda_0 \phi = 0$ with initial conditions $\phi(0+) = \phi'(0+) = 0$, which implies that $\phi = \phi' = 0$ for all $x \in (0, x_1)$. Since we assumed that the boundary conditions are connecting this implies that $\phi(x_1+) = \phi'(x_1+) = 0$. Repeating the argument for the interval (x_1, x_2) and then for remaining intervals, we conclude that $\phi(x) = 0$ for all $x \in (0, 2\pi)$. This contradicts the equality $\|\phi\| = \lim \|\phi_n\| = 1$.

If ϕ_n is not pre-compact, then λ_0 is a point of the essential spectrum of A_0 and hence of any self-adjoint extension of A_0. This is impossible for the operators from the class $\mathfrak{S}$ have purely discrete spectrum. ∎

In the next lemma we construct a convenient BVS for the operator A_0. This BVS depends neither on the potential V nor on the set X or the unitary matrices M_j, $j = 1, ..., N$.

LEMMA 4.5. *The triple* $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ *with*

$$\mathfrak{h} = \mathbb{C}^2, \qquad \Gamma_1 u = \{u'(0+), u'(2\pi-)\}, \quad \Gamma_2 u = \{u(0+), -u(2\pi-)\}$$

forms a BVS for the symmetric operator A_0.

Proof. Integrating by parts for any $u, v \in D(A_0^*) = \mathscr{B}(0, 2\pi)$, we find that

$$(A_0^* u, v)_{\mathscr{H}} - (u, A_0^* v)_{\mathscr{H}} = [u, v](2\pi-) - [u, v](0+) \\ - \sum_{j=1}^{N} ([u, v](x_j+) - [u, v](x_j-)).$$

In view of (4.2) this equality reduces to

$$(A_0^* u, v)_{\mathscr{H}} - (u, A_0^* v)_{\mathscr{H}} = [u, v](2\pi-) - [u, v](0+) \\ = (\Gamma_1 u, \Gamma_2 v)_{\mathfrak{h}} - (\Gamma_2 u, \Gamma_1 v)_{\mathfrak{h}}.$$

Hence Condition 1 of Definition 2.1 is fulfilled. Condition 2 is trivially satisfied. ∎

Having found a BVS for the operator A_0, we can now study the family

$$\mathfrak{A} = \{H(k), k \in [0, 1)\}$$

of quasi-periodic self-adjoint extensions of A_0. We begin with describing the family $\mathfrak{U}_{\mathfrak{A}}$ of unitary 2×2-matrices parametrizing $\mathfrak{A}$ and its dual $\mathfrak{U}^d$ (see Definition 3.2). It is easily seen from the definitions (2.3) and (4.5) that the unitary matrix U_k parametrizing the operator $H(k)$ has the form

$$U_k = \begin{pmatrix} 0 & e^{-2\pi i k} \\ e^{2\pi i k} & 0 \end{pmatrix}, \qquad k \in [0, 1).$$

The dual family $\mathfrak{U}^d$ is constructed in the next lemma:

Lemma 4.6. *The dual family $\mathfrak{U}^d$ consists of the operators of the form*

$$W = W_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \qquad \phi \in [0, 2\pi).$$

Proof. For a matrix

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

denote

$$\Delta(k) = \det(W - U_k) \\ = w_{11} w_{22} - w_{12} w_{21} - 1 + w_{12} e^{2\pi i k} - w_{21} e^{-2\pi i k}.$$

Then the condition

$$\ker(W - U_k) \neq \{0\}, \qquad \forall k \in [0, 1),$$

is equivalent to the fact that the analytic function $\Delta(\cdot)$ equals zero for all $k \in [0, 1)$. This implies that $w_{12} = w_{21} = 0$. Thus $w_{11} w_{22} = 1$. By unitary of W one has $w_{11} = e^{i\phi}$, $w_{22} = e^{-i\phi}$, $\phi \in [0, 2\pi)$. ∎

Denote by A_ϕ, $\phi \in [0, 2\pi)$ the operators from the set $\mathfrak{A}^d$ constructed by $\mathfrak{U}^d$ in accordance with Definition 3.3.

Lemma 4.7. *Suppose that the boundary conditions at the points $x_j \in X$, $j = 1, 2, ..., N$, are connecting. Then the spectrum of each $A_\phi \in \mathfrak{A}^d$ consists only of eigenvalues of multiplicity* 1, *and hence $\sigma_{\max}(A_\phi) = \varnothing$.*

Proof. By Lemma 4.6 and (2.3) the boundary conditions defining the operator A_ϕ can be rewritten in the form of two decoupled equations,

$$(e^{i\phi} - 1)\, u'(0+) + i(e^{i\phi} + 1)\, u(0+) = 0,$$
$$(e^{i\phi} - 1)\, u'(2\pi-) - i(e^{i\phi} + 1)\, u(2\pi-) = 0,$$

which reduce to

$$\begin{aligned} \sin\theta\, u'(0+) + \cos\theta\, u(0+) &= 0, \\ \sin\theta\, u'(2\pi-) - \cos\theta\, u(2\pi-) &= 0 \end{aligned} \tag{4.8}$$

with $\theta = \phi/2$. Thus we need to prove that the eigenvalues of this Sturm–Liouville-type problem are of multiplicity one. If $X = \varnothing$ then this result is well known. In the case $X \neq \varnothing$ the proof is basically the same.

Suppose that there is an eigenvalue λ_0 with two linearly independent eigenfunctions u_1, u_2. Consider their linear combination

$$u = c_1 u_1 + c_2 u_2, \qquad |c_1| + |c_2| > 0.$$

If $\phi \neq 0$ choose the constants c_1, c_2 in such a way that $u(0+) = 0$. Then the condition (4.8) implies that $u'(0+) = 0$. By uniqueness and due to the fact that the boundary conditions at all x_j's are connecting, we have $u(x) = 0$ for all $x \in [0, 2\pi)$. This contradicts the linear independence of u_1, u_2. In the case $\phi = 0$ the condition (4.8) yields $u_1(0+) = u_2(0+) = u(0+) = 0$. Choosing c_1, c_2 in such a way that $u'(0+) = 0$ and repeating the argument above, we arrive at the required conclusion.

Recall that the deficiency indices of A_0 are (2, 2), so that the eigenvalues of maximal multiplicity must have multiplicity 2. Thus $\sigma_{\max}(A_\phi) = \varnothing$. ∎

Now Lemma 4.7 and Theorem 3.6 immediately imply that the set of common eigenvalues of the family $H(k)$, $k \in [0, 1)$ is empty, and hence Theorem 4.3 holds. This in its turn proves the absolute continuity of H proclaimed in Theorem 4.2.

APPENDIX: SOME PROPERTIES OF $\mathbb{J}$-UNITARY OPERATORS

Here we briefly describe the properties of $\mathbb{J}$-unitary operators used in the proof of Theorem 2.5.

Let $\mathfrak{h}$ be a Hilbert space and let $\mathfrak{H} = \mathfrak{h} \oplus \mathfrak{h}$. Below we denote by blackboard letters the operators in $\mathfrak{H}$. Let $\mathbb{J}: \mathfrak{H} \to \mathfrak{H}$ be the operator (2.4). Then for a $\mathbb{J}$-unitary operator $\mathbb{T}$ we obtain from (2.5) that

$$\mathbb{T}^*\mathbb{J}\mathbb{T} = \mathbb{J}, \qquad \operatorname{ran} \mathbb{T} = \mathfrak{H}. \tag{A.1}$$

It is known (see [3]) that any $\mathbb{J}$-unitary operator is bounded and therefore can be represented in the form

$$\mathbb{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \tag{A.2}$$

with bounded blocks $T_{jk}: \mathfrak{h} \to \mathfrak{h}$, $j, k = 1, 2$. Moreover, (A.1) yields that

$$T_{12}^* T_{11} - T_{22}^* T_{21} = T_{11}^* T_{12} - T_{21}^* T_{22} = 0; \tag{A.3}$$

$$T_{11}^* T_{11} - T_{21}^* T_{21} = T_{22}^* T_{22} - T_{12}^* T_{12} = I. \tag{A.4}$$

Since $\operatorname{ran} \mathbb{T} = \mathfrak{H}$ the equality (A.1) also implies that $\mathbb{T}$ has a bounded inverse $\mathbb{T}^{-1} = \mathbb{J}\mathbb{T}^*\mathbb{J}$ defined on the whole of $\mathfrak{H}$, whence $\mathbb{T}\mathbb{J}\mathbb{T}^* = \mathbb{J}$. This means that

$$T_{21} T_{11}^* - T_{22} T_{12}^* = T_{11} T_{21}^* - T_{12} T_{22}^* = 0; \tag{A.5}$$

$$T_{11} T_{11}^* - T_{12} T_{12}^* = T_{22} T_{22}^* - T_{21} T_{21}^* = I. \tag{A.6}$$

Let us prove some useful properties of the operator $\mathbb{T}$.

THEOREM A.1. *Let $\mathbb{T}$ be a $\mathbb{J}$-unitary operator and let T_{jk}, $j, k = 1, 2$, be the entries in the representation* (A.2). *Then*

1. *The operator* $(\mathbb{J}\mathbb{T})^*$ *is $\mathbb{J}$-unitary in $\mathfrak{H}$;*
2. *The operators T_{11}, T_{22} are boundedly invertible*:
3. *Let $\|T_{12}\| = \omega_1$, $\|T_{21}\| = \omega_2$. Then*

$$\begin{aligned} \|T_{11}^{-1} T_{12}\| &= \|T_{21} T_{11}^{-1}\| = \omega_2(1 + \omega_2^2)^{-1/2}, \\ \|T_{22}^{-1} T_{21}\| &= \|T_{12} T_{22}^{-1}\| = \omega_1(1 + \omega_1^2)^{-1/2}. \end{aligned} \tag{A.7}$$

Proof. Statement 1 follows immediately from the equality $\mathbb{T}\mathbb{J}\mathbb{T}^* = \mathbb{J}$. Statement 2 is a direct consequence of (A.4). To prove part 3 represent T_{12} and T_{21} in the polar form:

$$T_{12} = V_1 T_1, \qquad T_{21} = V_2 T_2,$$

with $T_1 = \sqrt{T_{12}^* T_{12}}$, $T_2 = \sqrt{T_{21}^* T_{21}}$ and some partially isometric operators V_1, V_2. It follows from (A.4) that the polar decompositions of the operators T_{11}, T_{22} have the form

$$T_{11} = W_1 (I + T_2^2)^{1/2}, \qquad T_{22} = W_2 (I + T_1^2)^{1/2}$$

with some unitary operators W_1, W_2. Furthermore, (A.6) ensures that

$$W_1 (I + T_2^2)\, W_1^* = V_1 T_1^2 V_1^* + I, \qquad W_2 (I + T_1^2)\, W_2^* = V_2 T_2^2 V_2^* + I.$$

This yields

$$T_1^2 = W_2^* V_2 T_2^2 V_2^* W_2, \qquad T_2^2 = W_1^* V_1 T_1^2 W_1^* W_1. \tag{A.8}$$

Now we are in position to prove the equalities (A.7). The first pair of them results from the relation $\omega_2 = \|T_{21}\| = \|T_2\|$ and the following chains of identities:

$$\begin{aligned}
\|T_{21} T_{11}^{-1}\| &= \|V_2 T_2 (I + T_2^2)^{-1/2}\, W_1^*\| \\
&= \|T_2 (I + T_2^2)^{-1/2}\|; \\
\|T_{11}^{-1} T_{12}\|^2 &= \|(I + T_2^2)^{-1/2}\, W_1^* V_1 T_1\|^2 \\
&= \|(I + T_2^2)^{-1/2}\, W_1^* V_1 T_1^2 V_1^* W_1 (I + T_2^2)^{-1/1}\| \\
&= \|(I + T_2^2)^{-1/2}\, T_2^2 (I + T_2^2)^{-1/2}\|.
\end{aligned}$$

In the last equality we have used (A.8). The second pair of equalities in (A.7) is proved in the same way. ▮

Proof of Theorem 2.4. Statements 2 and 3 of Theorem A.1 lead directly to Part 1 in view of the obvious formulae

$$\begin{aligned}
&T_{11} + T_{12} U = T_{11} (I + T_{11}^{-1} T_{12} U), \\
&T_{21} - U T_{11} = U (U^* T_{21} T_{11}^{-1} - I)\, T_{11}, \\
&T_{22} + T_{21} U = T_{22} (I + T_{22}^{-1} T_{21} U), \\
&T_{22} - U T_{12} = (I - U T_{12} T_{22}^{-1})\, T_{22},
\end{aligned}$$

and the fact that the r.h.s. in (A.7) is strictly less than 1.

To verify Statement 2 note that by virtue of (A.3) and (A.4) we have

$$(T_{11}^* + U^*T_{12}^*)(T_{11} + T_{12}U) = (T_{21}^* + U^*T_{22}^*)(T_{21} + T_{22}U).$$

This ensures the equalities $\Omega(U)^*\,\Omega(U) = \Omega(U)\,\Omega(U)^* = I$.

The operator (2.7) is well-defined since $(\mathbb{J}\mathbb{T})^*$ is $\mathbb{J}$-unitary. The equality (2.8) results from (A.3), (A.4), and (A.5), (A.6). To prove (2.9) write using the resolvent identity

$$\Omega(U) - \Omega(W) = B_1(U - W)\,B_2^{-1},$$
$$B_1 = (T_{22} - \Omega(U)\,T_{12}), \qquad B_2 = (T_{11} + T_{12}W).$$

The operators B_1, B_2 are boundedly invertible by virtue of Part 1. Thus the subspaces $\ker(U - W)$ and $\ker(\Omega(U) - \Omega(W))$ are linearly homeomorphic and therefore they have the same dimension. ∎

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