Gorenstein Associated Graded Rings of Analytic Deviation Two Ideals

Shiro Goto

Department of Mathematics, School of Science and Technology, Meiji University, Kawasaki 214-8571, Japan
E-mail: goto@math.meiji.ac.jp

and

Shin-ichiro Iai

Mathematics Laboratory, Sapporo College, Hokkaido University of Education, Sapporo 002-8502, Japan
E-mail: iai@sap.hokkyodai.ac.jp

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This paper studies the question of when the associated graded ring \( \mathcal{G}(I) := \bigoplus_{n=0}^{\ell} \mathcal{R}(I) \) of a certain ideal \( I \) in a local ring is Gorenstein. The main result implies, for example, that if \( A \) is a regular local ring, \( \mathfrak{p} \) is a prime ideal in \( A \) with \( \dim A/\mathfrak{p} = 2 \), and \( A/\mathfrak{p} \) is a complete intersection in codimension one, then the associated graded ring \( \mathcal{G}(\mathfrak{p}) \) is Gorenstein if and only if the reduction number of \( \mathfrak{p} \) is at most 1.

1. INTRODUCTION

In this article we will give a characterization of Gorenstienness in the graded ring associated to an ideal having analytic deviation two.

In what follows, let \( A \) be a Gorenstein local ring with the maximal ideal \( \mathfrak{m} \) and \( \dim A = d \). Assume that the field \( A/\mathfrak{m} \) is infinite. Let \( I \neq A \) be an ideal in \( A \) of height \( s \). We write \( \ell = \lambda(I) := \dim A/\mathfrak{m} \otimes_A \mathcal{G}(I) \) and call it the analytic spread of \( I \). We define \( \mathcal{R}(I) = A[I, t] \subseteq A[t, t^{-1}] \) (\( t \) is an indeterminate over \( A \)) and \( \mathcal{G}(I) = \mathcal{R}(I)/t^{-1}\mathcal{R}(I) \). Let \( \operatorname{ad}(I) := \ldots \)
$\ell - s$ that we call the analytic deviation of $I$ ([HH]). Let $J$ be a minimal reduction of $I$. Hence $J \subseteq I$ and $I^{n+1} = J I^n$ for some $n \geq 0$. We put 
\[ r_J(I) := \min \{ n \geq 0 \mid I^{n+1} = J I^n \} \]
and call it the reduction number of $I$ with respect to $J$. Suppose that $a_1, a_2, \ldots, a_\ell$ is a minimal system of generators for the minimal reduction $J$ of $I$ satisfying the following two conditions.

\[ (* \ ) \quad J_\ell A_p \text{ is a reduction of } I A_p \text{ for any } p \in V(I) \text{ with } i = \text{ht}_A p \leq \ell. \]

\[ (** \ ) \quad a_i \not\in p \text{ if } p \in \text{Ass}_A A / J_{i-1} \backslash V(I) \text{ for any } 1 \leq i \leq \ell. \]

Here $V(I)$ denotes the set of prime ideals in $A$ containing $I$, and we set $J_i = (a_1, a_2, \ldots, a_i)$ for $0 \leq i \leq \ell$. According to [GNN2], 2.1, there always exists a minimal system of generators $a_1, a_2, \ldots, a_\ell$ for $J$ satisfying conditions $(*)$ and $(**)$.

We put 
\[ r_i = \max \{ r_{J_i}(I_p) \mid p \in V(I) \text{ and } \text{ht}_A p = i \} \text{ for any } s \leq i < \ell. \]

Our ideal $I$ is said to be generically a complete intersection if $r_s = 0$. Let 
\[ U = U(I) := \bigcap_{p \in \text{Ass}_A A / I} (I A_p \cap A), \]
where $\text{Ass}_A A / I = \{ p \in V(I) \mid \dim A / I = \dim A / p \}$. If $I = U$, then we say $I$ is unmixed. We denote the $a$-invariant of $\mathcal{G}(I)$ by a $(\mathcal{G}(I))$ ([GW], 3.1.4). With this notation the main result of this paper is stated as follows.

**Theorem 1.1.** Assume that $\text{ad}(I) \leq 2$ and $\text{ht}_A p < s + 2$ for any $p \in \text{Ass}_A A / I$. Suppose that $\mathcal{G}(I)$ is a Cohen-Macaulay ring. Then the following two conditions are equivalent.

\[ (1) \quad \mathcal{G}(I) \text{ is a Gorenstein ring and } a(\mathcal{G}(I)) = -s. \]

\[ (2) \quad \begin{array}{l}
(a) \quad r_J(I) \leq \text{ad}(I), \\
(b) \quad r_i \leq i - s \text{ for any } s \leq i < \ell, \text{ and} \\
(c) \quad I = J_{s+1} U :_U (J_{s+1} I :_I I). 
\end{array} \]

When this is the case, if $\text{ad}(I) = 2$, then the following two assertions are satisfied:

\[ (i) \quad I = U \text{ if and only if } r_{s+1} = 0; \]

\[ (ii) \quad r_{J_i}(I_p) \leq 1 \text{ for any } p \in V(I) \text{ such that } \text{ht}_A p = \ell. \]

The Cohen–Macaulay property of the associated graded ring $\mathcal{G}(I)$ have been studied closely, and we have a satisfactory criterion given by the first author, Y. Nakamura, and K. Nishida ([GNN2]). If $\mathcal{G}(I)$ is a Cohen–Macaulay ring, then we have depth $A / I^n \geq d - \ell$ for all $n \geq 1$, so that $\text{ht}_A p \leq \ell$ for any $p \in \text{Ass}_A A / I^n$. Hence when $\text{ad}(I) \leq 1$, the Cohen-Macaulayness of $\mathcal{G}(I)$ require the condition that $\text{ht}_A p < s + 2$ for any $p \in \text{Ass}_A A / I$. 


Many authors have studied the Gorensteinness of the associated graded rings of ideals having positive analytic deviation (cf. [GI], [GNa1], [GNa3], [GNN1], [GNN3], [HHR], [HSV], [JU], [SUV], [T], [U]). However, almost all authors assumed the ring $A/I$ is Cohen-Macaulay and we lack satisfactory references analyzing ideals for which the rings $A/I$ are not necessarily Cohen-Macaulay. In our Theorem 1.1 we have assumed that $\text{ht}_A \mathfrak{p} < s + 2$ for any $\mathfrak{p} \in \text{Ass}_A A/I$ but the assumption that the ring $A/I$ is Cohen-Macaulay is removed. The first author and Y. Nakamura [GNa1] explored generically complete intersection ideals $I$ (which are not necessarily Cohen-Macaulay ideals) with $\text{ad}(I) = 1$ in a Gorenstein local ring $A$ of $\dim A = 1$ and gave in that case a criterion for the Gorensteinness in $\mathcal{G}(I)$. We have recently succeed in generalizing their result to the case where the dimension of $A$ is arbitrary ([GI]). By our Theorem 1.1 we have overcome the assumption that $\text{ad}(I) = 1$ in [GI] also, and get a characterization in the case where $\text{ad}(I) = 2$. The strongest point of the theorem is that the unmixedness of $I$ is not required. However, it complicates the conditions in Theorem 1.1. This complication, at least in terms of the method of proof, has a lot to do with the main reason we must assume $\text{ad}(I) \leq 2$.

We now briefly mention the contents of the paper. The proof of Theorem 1.1, which will be given in Section 3, is based on the case where $d = \ell = 2$ and $s = 0$. In Section 2, we shall devote to the theorem in such a case and give an example of a mixed ideal satisfying the conditions. In Section 4 we will summarize some corollaries for unmixed ideals derived from our theorem.

We end this section by collecting some more of the notation which we will be using throughout this paper. We denote $\mathcal{G}(I)$ simply by $G$ and put $a = a(G)$. Let $\mathfrak{m} = \mathfrak{m}G + G_+$. We denote by $H^n_{\mathfrak{m}}(*) (i \in \mathbb{Z})$ the $i$th local cohomology functor of $G$ with respect to $\mathfrak{m}$. For each graded $G$-module $E$, let $[E]_n$ stand for the homogeneous component of $E$ of degree $n$ and let $a_i(E) = \sup \{ n \in \mathbb{Z} \mid [H^n_{\mathfrak{m}}(E)_n] \neq (0) \} (i \in \mathbb{Z})$. Let $\mathcal{A} = \{ \mathfrak{p} \in V(I) \mid \text{ht}_A \mathfrak{p} = \dim G_\mathfrak{p}/\mathfrak{p}G_\mathfrak{p} \}$. We denote by $K_G$ the graded canonical modules of $G$. We shall freely refer to [BH], [GN], [HK], and [HIO] for details of the theory on canonical modules.

2. THE CASE WHERE $d = \ell = 2$ AND $s = 0$

Throughout this section we always assume that $d = \ell = 2$ and $s = 0$. Let $I$ be generically a complete intersection. The purpose of this section is to prove the next proposition.

PROPOSITION 2.1. Assume that $\text{depth} A/I > 0$. Then the following two conditions are equivalent.
(1) G is a Gorenstein ring.

(2) 
   (i) \( r_I \leq 1 \),
   (ii) \( r_1 \leq 1 \), and
   (iii) \( I = a_1 U :_U (a_1 I :_I) \).

When this is the case, \( I = U \) if and only if \( r_1 = 0 \).

When the conditions (i) and (ii) are satisfied, we get \( G \) is a Cohen-Macaulay ring if depth \( A/I > 0 \) (see [GNN2]). Then from the \( a \)-invariant formula:

\[
a(G) = \max \{ \{ r_i - i \mid s \leq i < \ell \} \cup \{ r_I(I) - \ell \} \}
\]

(cf. [U], 1.4) we obtain that \( a = 0 \). Conversely when \( G \) is a Gorenstein ring, we have \( a = a(G) \) for any \( p \in \mathcal{V}(I) \), so that \( a = 0 \) (recall that \( r_0 = 0 \)). Therefore we get \( r_1 \leq 1 \) by the \( a \)-invariant formula. Hence in the rest of this section we consider the case where depth \( A/I > 0 \) and \( I = a_1 U :_U (a_1 I :_I) \).

Lemma 2.2. \( K_B \cong a_1 U :_U (a_1 I :_I) \).

Proof. Let \( D = a_1 I \bar{A} :_{\bar{A}} I \bar{A} \) and let \( \alpha = \bar{a}_1 \), which denotes the coset of \( a \) containing \( a_1 \). Then we have

\[
B = \frac{D}{\alpha} := \left\{ \frac{\beta}{\alpha} \mid \beta \in D \right\}
\]

in \( Q(\bar{A}) \). In fact, it is routine to check \( B \supseteq \frac{D}{\alpha} \). Conversely, we take \( \gamma \in B \). Then \( \alpha \gamma \in I \bar{A} \) and hence \( \gamma = \beta/\alpha \) for some \( \beta \in I \bar{A} \). Since \( I \bar{A} : \beta = (I \bar{A} : \gamma) : \alpha \subseteq I \bar{A} : \alpha \), we have \( \beta \in D \). Therefore \( \gamma \in \frac{D}{\alpha} \).

We obtain that

\[
K_B := U \bar{A} :_{Q(\bar{A})} B = U \bar{A} :_{\bar{A}} \frac{D}{\alpha} = aU \bar{A} :_{\bar{A}} D = \frac{(a_1 U + \alpha) :_A (a_1 I + \alpha) :_I}{\alpha}.
\]
Since $I \cap \alpha = (0)$, we have $a_1U : A (a_1I : \mathfrak{I} I) = (a_1U + \alpha : A ((a_1I + \alpha) : \mathfrak{I} I)$
and hence

$$K_B = \frac{a_1U : A (a_1I : \mathfrak{I} I)}{\alpha}. $$

And we obtain that the natural homomorphism

$$a_1U : U (a_1I : \mathfrak{I} I) \rightarrow \frac{a_1U : A (a_1I : \mathfrak{I} I)}{\alpha}$$

is bijective. In fact, since $U \cap \alpha = (0)$, we have the map $\varphi$ is injective. Let

$x \in a_1U : A (a_1I : \mathfrak{I} I)$. We get $a_1x \in a_1U$, as $a_1 \in a_1I : \mathfrak{I} I$. Hence $x \in U + \alpha$

because $\alpha = (0) : a_1$ (see [GNa1], 2.1). Therefore we get $a_1U : A (a_1I : \mathfrak{I} I) \subseteq U + \alpha$ and hence it is surjective. \hfill \Box

We put $T = \mathcal{O}(I \mathcal{A})$ and $S = \mathcal{O}(IB)$ for short. Look at the natural exact sequence

$$0 \rightarrow \mathcal{A} \rightarrow B \rightarrow B/I \mathcal{A} \rightarrow 0. $$

Let $C = B/I \mathcal{A}$. Then $\dim C \leq 1$. Since $I A = IB$, we get the exact sequence

$$0 \rightarrow T \rightarrow S \rightarrow C \rightarrow 0 \quad (\varepsilon)$$

of graded $G$-modules. Moreover we have the exact sequence

$$0 \rightarrow \alpha \rightarrow G \rightarrow T \rightarrow 0 \quad (\#)$$

of graded $G$-modules by [GNa1], 2.3. We note $a_1$ is $B$-regular element and hence $\lambda(IB_n) > 0$ for all maximal ideal in $n$ in $B$. Let $\mathcal{A}$ denote the $m$-adic completion of $A$. Notice that $\mathcal{A} \otimes_A S \cong \prod_{j=1}^\infty \mathcal{O}(IB_j)$ is the direct product of associated graded rings $S_j := \mathcal{O}(IB_j)$ of ideals $IB_j$ (with positive analytic spread) in Cohen-Macaulay local rings $B_j$, which are finite as $\mathcal{A}$-modules.

**Lemma 2.3.** $S$ is a maximal Cohen-Macaulay $G$-module.

**Proof.** Since depth $\alpha = 2$, we get depth $T > 0$ by ($\#$). We apply the local cohomology functors $\overset{\mathcal{O}}{H}_m^i (\ast) (i \in \mathbb{Z})$ to the graded exact sequences ($\varepsilon$) and ($\#$). Then by the resulting graded exact sequence

$$0 \rightarrow H^1_{20}(T) \rightarrow H^2_{20}(\alpha) \rightarrow H^2_{20}(\mathcal{O}) \rightarrow H^2_{20}(T) \rightarrow 0 $$

of local cohomology modules from ($\varepsilon$), we have $H^1_{20}(T) = [H^1_{20}(T)]_0$ and $a(T) \leq 0$ because $H^2_{20}(\alpha) = [H^2_{20}(\alpha)]_0$ (see [GH], 2.2) and $a = 0$. And by the resulting graded exact sequence

$$0 \rightarrow H^0_{20}(S) \rightarrow H^0_{20}(C) \rightarrow H^1_{20}(T) \rightarrow H^1_{20}(S) \rightarrow H^1_{20}(C) \rightarrow H^2_{20}(T) \rightarrow H^2_{20}(S) \rightarrow 0 $$

of local cohomology modules from ($\#$), we have $H^i_{20}(S) = [H^i_{20}(S)]_0$ for any integers $i = 0, 1$ and $a_2(S) \leq 0$ because $H^1_{20}(T), H^2_{20}(C)$, and $H^2_{20}(C)$ are concentrated in degree 0 (see [GH], 2.2) and $a(T) \leq 0$. 


Now assume that $S$ is not a Cohen-Macaulay $G$-module. Let $t = \text{depth } S$. Then $t = 0$ or 1. Because $\widehat{A} \otimes_A H^0_G(S) \cong \bigoplus_{j=1}^{n} H^j_{A \otimes_A S}(S_j)$ as graded $\widehat{A} \otimes_A G$-modules, we can find $1 \leq j \leq n$ such that $(0) \neq H^j_{A \otimes_A S}(S_j) = [H^j_{A \otimes_A S}(S_j)]_0$. From [KN], 3.1 we obtain that $a_i(S_j) < a_{i+1}(S_j)$. However this is impossible since $a_i(S_j) = 0$ and $a_{i+1}(S_j) \leq a_{i+1}(S) \leq 0$.

Apply the functor $\text{Hom}_{G/\mathbb{Z}}(\ast, K_G)$ to the graded exact sequences (â) and (ââ), and we get the following commutative and exact diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & K_S & \rightarrow & K_G & \rightarrow & \text{Coker } \varepsilon^* \circ \varphi^* & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K_T & \rightarrow & K_G & \rightarrow & \text{Hom}_G(\alpha, K_G) & \rightarrow & \text{Ext}^1_{G/\mathbb{Z}}(T, K_G) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{Ext}^1_{G/\mathbb{Z}}(C, K_G) & \rightarrow & 0 & & & & & & & & & & \\
\downarrow & & & & & & & & & & & & \\
0 & & & & & & & & & & & & \\
\end{array}
$$

of graded $G$-modules where $K_S$ denotes formally $\text{Hom}_G(S, K_G)$. Notice that $\text{Hom}_G(\alpha, K_G)$ and $\text{Ext}^1_{G/\mathbb{Z}}(C, K_G)$ are concentrated in degree 0 (see [BH], 3.6.19 and [GH], 2.2). Now let $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ stand for the canonical $I$-filtration of $A$ ([GI]). Hence we have $I^{i+1} \subseteq \omega_i$ for all $i \in \mathbb{Z}$ and $K_G \cong \bigoplus_{i \geq 0} \omega_{i-1}/\omega_i$ as graded $G$-modules.

**Lemma 2.4.** Coker $\varepsilon^* \cong A/U$.

**Proof.** We put $Z = \text{Coker } \varepsilon^*$. Since $Z \subseteq \text{Hom}_G(\alpha, K_G)$, we have $Z$ is concentrated in degree 0. Therefore $Z \cong A/E$ for some ideal $E$ in $A$ because we have a surjective homomorphism $A/\omega_0 \cong [K_G]_0 \rightarrow Z$. Hence $U \subseteq E$, as $\alpha = (0) : U$. Assume $U \subseteq E$ and choose $v \in \text{Ass}_A A/U$ so that $U_v \subseteq E_v$. Since $v \not\supseteq \alpha$, we have $T_v = (0)$, so that $(0) = K_{T_v} \cong [K_T]_v$. Thus we get $[K_G]_v \cong [A/E]_0$ be the diagram above. From $[K_G]_v \cong K_{G_v} = A_v$, we obtain $E_v = (0)$, which is a contradiction. ■
Lemma 2.5. Suppose \( \mathfrak{p} \in V(I) \) such that \( \text{ht}_A \mathfrak{p} = 1 \) and \( \mathfrak{p} \supseteq \mathfrak{a} \). Then \( IB_\mathfrak{p} = a_1B_\mathfrak{p} \).

Proof. We have \( I A_\mathfrak{p} \) is \( \mathfrak{p}A_\mathfrak{p} \)-primary ideal with \( (IA_\mathfrak{p})^2 = a_1IA_\mathfrak{p} \) because \( r_1 \leq 1 \). From [GIW], 4.1 we obtain that \( [IA_\mathfrak{p}] = A_\mathfrak{p}B_\mathfrak{p} \). Since \( IA = IB \), we get \( IB_\mathfrak{p} = a_1B_\mathfrak{p} \).

We put \( X = \text{Coker} \phi^* \circ \phi^* \). Since \( \text{Hom}_G(\mathfrak{a}, K_G) \) and \( \text{Ext}_G^1(\mathfrak{c}, K_G) \) are concentrated in degree 0, we get \( X = [X]_0 \). Hence there exists an ideal \( F \) in \( A \) such that \( X \cong A/F \) (recall that \( A/\omega_0 \cong [K_G]_0 \to X \)).

Lemma 2.6. Suppose \( \omega_0 = 1 \). Then \( I = F \) and hence \( r_1(I) \leq 1 \).

Proof. We have a surjective homomorphism \( A/I \to X \) and hence \( I \subseteq F \).

Assume \( I \not\subseteq F \) and choose \( \mathfrak{p} \in \text{Ass}_A A/I \) so that \( I_\mathfrak{p} \not\subseteq F_\mathfrak{p} \). We have \( \mathfrak{p} \not\supseteq \mathfrak{m} \) because depth \( A/I > 0 \). If \( \mathfrak{p} \supseteq \mathfrak{u} \), then \( S_\mathfrak{p} = (0) \), so that \( (0) = K_\mathfrak{S}_0 \cong [K']_0 \).

Thus we get \( [K_\mathfrak{S}]_0 \cong [A/F]_0 \) by the diagram above. Since \( [K_\mathfrak{S}]_0 \cong A/\omega_0 = A/I \), we have \( [A/F]_0 \cong [A/I]_0 \). Therefore \( I_\mathfrak{p} = F_\mathfrak{p} \), which is impossible. If \( \mathfrak{p} \supseteq \mathfrak{a} \), then \( \text{ht}_A \mathfrak{p} = 1 \) and hence \( S_\mathfrak{p} = \mathfrak{g}(a_1B_\mathfrak{p}) \) is the polynomial ring in one variable by Lemma 2.5. Then we have \( a_2(S_\mathfrak{p}) = -1 \) and hence \( (0) = [K_\mathfrak{S}]_0 \cong [K_\mathfrak{S}]_0 \) (see [BH], 3.6.19). Therefore we have \( [A/F]_0 \cong [A/F]_0 \), whence \( I_\mathfrak{p} = F_\mathfrak{p} \). This is a contradiction. Thus we get \( I = F \).

Then we get the exact sequence

\[
0 \to K_S \to K_G \to A/I \to 0
\]

of graded \( G \)-modules. Since \( [K_G]_0 \cong A/I \), we have \( [K_S]_0 = (0) \) and hence \( a_2(S) < 0 \) (see [BH], 3.6.19). Let \( \mathfrak{n} \) be an maximal ideal in \( B \). Then \( S_\mathfrak{n} = \mathfrak{g}(IB_\mathfrak{n}) \) is a Cohen-Macaulay ring with \( a(S_\mathfrak{n}) < 0 \) (recall that \( a_j(S) \geq a(S_j) = a(S_n) \) for some \( j = 1, 2, \ldots, \mathfrak{n} \)). We have \( \lambda(IB_\mathfrak{n}) > 0 \). Suppose \( \lambda(IB_\mathfrak{n}) = 1 \). There is an element \( b \in J\mathfrak{B}_\mathfrak{n} \) such that \( bB_\mathfrak{n} \) is a minimal reduction of \( IB_\mathfrak{n} \). Thanks to the \( a \)-invariant formula, we get \( r_{bB_\mathfrak{n}}(IB_\mathfrak{n}) \leq a(S_\mathfrak{n}) + 1 \leq 0 \) and hence \( bB_\mathfrak{n} = JB_\mathfrak{n} = IB_\mathfrak{n} \). Suppose \( \lambda(IB_\mathfrak{n}) = 2 \). Then we obtain \( r_{bB_\mathfrak{n}}(IB_\mathfrak{n}) \leq a(S_\mathfrak{n}) + 2 \leq 1 \) from the \( a \)-invariant formula. Therefore in any case we get \( r_{bB_\mathfrak{n}}(IB_\mathfrak{n}) \leq 1 \), whence \( I^2B = JIB \). So \( I^2A = JA \), as \( IB = JA \). Then \( I^2 \subseteq JI + \mathfrak{a} \), and hence we have \( I^2 = JI \) because \( I \cap \mathfrak{a} = (0) \).

We now come to the proof of our proposition.

Proof of Proposition 2.1 (1) \( \Rightarrow \) (2). Assume \( G \) is a Gorenstein ring. Then \( \omega_0 = 1 \), so that \( r_1(I) \leq 1 \) by Lemma 2.6. We must show that \( I = a_1U :_U (a_1I :_I I) \). Let \( L = a_1U :_U (a_1I :_I I) \). Then we have \( I \subseteq L \).

Let \( \mathfrak{p} \in \text{Ass}_A A/I \). It is enough to prove \( I_\mathfrak{p} = L_\mathfrak{p} \). We have \( \mathfrak{p} \not\subseteq \mathfrak{m} \) (see, e.g., [GI], 3.1) and \( \mathfrak{p} \not\supseteq \mathfrak{m} \), as depth \( A/I > 0 \). If \( \text{ht}_A \mathfrak{p} = 0 \), then \( U_\mathfrak{p} = (0) \). Hence we may assume \( \text{ht}_A \mathfrak{p} = 1 \). Then \( a_1A_\mathfrak{p} \) is a minimal reduction of
If $r_{a_1 A_q}(I_p) \leq 1$ because $p \in \mathfrak{A}$ and $r_1 \leq 1$. So $I_p = a_1 I_p : l_{p} I_p$. If $ht_A I_p = 0$, then $I_p$ is generically a complete intersection and hence we get $I_p = a_1 U(I_p) : U(I_p) I_p$ by [GI], 6.3. Therefore $I_p = L_p$, as $U(I_p) = U(I)$.

If $ht_A I_p = 1$, then $I_p$ is an $p A_p$-primary ideal of $A_q$ with $r_{a_1 A_q}(I_p) = 1$ because $r_{a_1 A_q}(I_p) = a(\mathfrak{g}(I_p)) + 1$ and $a(\mathfrak{g}(I_p)) = a(\mathfrak{g}(I)) = a = 0$. Hence, according to [GI], 1.4, we get $I_p = a_1 A_q : I_p$. Therefore $I_p = L_p$, as $p \not\supseteq U$.

(2) $\Rightarrow$ (1). First of all, we obtain that $G_p = \mathfrak{g}(I_p)$ is a Gorenstein ring with $a(G_p) = 0$ for all $p \in \text{Ass}_A A/I$. In fact, let $p \in \text{Ass}_A A/I$. If $ht_A p = 0$, then $G_p = A_p$, so that we have nothing to prove. Hence we may assume $ht_A p = 1$. Then $a_1 A_q$ is a minimal reduction of $I_p$ with $r_{a_1 A_q}(I_p) \leq 1$, whence $I_p = a_1 I_p : l_{p} I_p$. When $ht_A I_p = 0$, we get $I_p = a_1 U(I_p) : U(I_p) I_p$.

And when $ht_A I_p = 1$, we get $I_p = a_1 A_q : I_p$ (recall that $p \not\supseteq U$) and hence $r_{a_1 A_q}(I_p) = 1$. Consequently, from [GI], 6.3 and 1.4 we obtain that $G_p$ is a Gorenstein ring with $a(G_p) = 0$ in any case.

Then we have $[\omega_0] = I_p$ for all $p \in \text{Ass}_A A/I$, therefore $\omega_0 = I$ (recall that $\omega_0 \subseteq I$). According to Lemma 2.6, we have $I = F$. Hence we get the exact sequence

$$0 \to K_S \to K_G \to A/I \to 0$$

of graded $G$-modules. Since $[K_G]_0 = A/I$, we get $[K_S]_0 = (0)$.

**Claim 2.7.** The graded $G$-module $K_S$ is generated by elements of degree 1.

**Proof.** Since $a_1$ is a $A \overline{A}$-regular element, $a_1$ is $B$-regular element. We put $\overline{B} = B/a_1 B$ and $S = \mathfrak{g}(IB)$. We have $B/IB = \overline{B}/IB$ is a Cohen-Macaulay ring and $ht_B IB = 1$, as $B$ and $IB$ are maximal Cohen-Macaulay $A$-modules. From Lemma 2.5 we obtain that $IB \subseteq (0)$ for any $\Sigma \in V(IB)$ such that $ht_B \Sigma = 0$. Thus we have $IB$ is not nilpotent since $IB \neq (0)$ and $IB \subseteq (0)$ for all $\Sigma \in \text{Ass}_B \overline{B}/IB$. We have $I^2 \overline{B} = a_2 IB$, therefore $a_2 t$ is $S_1 := \bigoplus_{t \geq 0} [S]$-regular element because $a_2$ is $IB$-regular element (see [GNa1], 2.1). Thus we get the graded exact sequence

$$0 \to \overline{S}_1(-1) \xrightarrow{a_2 t} \overline{S}_1 \to IB/I^2 \overline{B} \to 0.$$ 

Apply the local cohomology functors $H^i_{\mathfrak{g}}(*)$ ($i \in \mathbb{Z}$) to this, and we have the graded exact sequence

$$0 \to H^0_{\mathfrak{g}}(IB/I^2 \overline{B}) \to H^1_{\mathfrak{g}}(\overline{S}_1(-1)) \xrightarrow{a_2 t} H^1_{\mathfrak{g}}(\overline{S}_1)$$

of local cohomology modules. And furthermore, applying the functor $\text{Hom}_G(G/M, *)$ to this, we get isomorphism

$$\text{Hom}_G(G/M, H^0_{\mathfrak{g}}(IB/I^2 \overline{B})) \cong \text{Hom}_G(G/M, H^1_{\mathfrak{g}}(\overline{S}_1))(-1)$$

of graded $G$-modules because $a_2 t G \subseteq M$, and hence $\text{Hom}_G(G/M, H^1_{\mathfrak{g}}(\overline{S}_1))$ is concentrated in degree 0 because $H^0_{\mathfrak{g}}(IB/I^2 \overline{B}) = [H^0_{\mathfrak{g}}(IB/I^2 \overline{B})]_1$ (see [GH], 2.2). We apply the local cohomology functors $H^i_{\mathfrak{g}}(*)$ ($i \in \mathbb{Z}$) to
the graded exact sequence $0 \to \overline{S}_+ \to \overline{S} \to B/IB \to 0$. Then by the resulting graded exact sequence $0 \to H^1_{20}(\overline{S}_+) \to H^1_{20}(\overline{S}) \to H^1_{20}(B/IB) \to 0$ of local cohomology modules, we get the exact sequence

$$0 \to \text{Hom}_G(G/\mathcal{M}, H^1_{20}(\overline{S}_+)) \to \text{Hom}_G(G/\mathcal{M}, H^1_{20}(\overline{S}))$$

$$\to \text{Hom}_G(G/\mathcal{M}, H^1_{20}(B/IB))$$

of graded $G$-modules. We have $H^1_{20}(B/IB) = [H^1_{20}(B/IB)]_0$, hence we obtain that $\text{Hom}_G(G/\mathcal{M}, H^1_{20}(\overline{S}))$ is concentrated in degree 0. We note $a_1t$ is $S$-regular element (see [GNN3], 2.3). Then we have the sequence $0 \to S(-1) \to S \to \overline{S} \to 0$ of graded $G$-modules, as $\overline{S} \cong S/a_1tS$ (see [VV]). Applying the local cohomology functors $H^i_{20}(*)$ ($i \in \mathbb{Z}$) to this, we have the graded exact sequence $0 \to H^1_{20}(\overline{S}) \to H^2_{20}(S)(-1) \to H^3_{20}(S)$ of local cohomology modules. Therefore we get the isomorphism

$$\text{Hom}_G(G/\mathcal{M}, H^1_{20}(\overline{S})) \cong \text{Hom}_G(G/\mathcal{M}, H^2_{20}(S)(-1))$$

of graded $G$-modules, and hence $\text{Hom}_G(G/\mathcal{M}, H^2_{20}(S))$ is concentrated in degree $-1$. This means that the graded $G$-module $K_S$ is generated by elements of degree 1.

By the condition (iii) and Lemma 2.2 we may assume $K_R = I$. Using Claim 2.7 and [HSV], 2.4 we obtain $K_S = G_+$ because $S$ is a Cohen-Macaulay ring. Therefore we have the exact sequence

$$0 \to G_+ \to K_G \to A/I \to 0$$

of graded $G$-modules. Look at the homogeneous components

$$0 \to A/\omega_0 \to A/I \to 0$$

$$0 \to I^2/\omega_1 \to 0$$

$$0 \to I^3/\omega_2 \to 0$$

...,

of above, where $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ is the canonical $I$-filtration of $A$ ([GI]). By induction on $i$, we see that $\omega_i = I^{i+1}$ for all integers $i \geq 0$. In fact, we have $\omega_0 = I$. Let $i > 0$ and assume $\omega_{i-1} = I^i$. We note that $\omega_i \geq I^{i+1}$. From bijections above we obtain that $I^i/I^{i+1} \cong \omega_{i-1}/\omega_i = I^i/\omega_i$, and hence the natural surjective map $I^i/I^{i+1} \to I^i/\omega_i$ is bijective. Thus $\omega_i = I^{i+1}$ for all $i \geq 0$. This means $G$ is a Gorenstein ring.

Now let us check the last assertion. Suppose $I = U$. We take any $p \in V(I)$ such that $ht_{A_p} = 1$. If $p \nsubseteq a$, then $I_p = (0)$. And if $p \supseteq a$, then $B_p = A_p$ because $A_p$ is a Cohen-Macaulay ring with $K_{A_p} \cong I_p$. Therefore we have $IA_p = a_1A_p$ by Lemma 2.5, so that $IA_p \subseteq a_1A_p + a_p$. Since $IA_p \cap a_p = (0)$,
we get $IA_p = a_1 A_p$. Thus in any case we obtain $r_{a_1 A_p}(IA_p) = 0$. This establishes that $r_1 = 0$.

Conversely, assume $I \subseteq U$ and choose $p \in \text{Ass}_A A/I$ so that $I_p \subseteq U_p$. Then we have $\text{ht}_A p = 1$ because $U_p \neq (0)$ and depth $A/I > 0$, and hence $IA_p = a_1 A_p$ since $r_1 = 0$. Take the $G$-dual of the sequence (###), and we have the exact sequence

$0 \to K_T \to G \to A/U \to 0$

of graded $G$-modules by Lemma 2.4, and hence $[K_T]_0 \cong U/I$. So $[K_T]_0 \cong U_p/I_p \neq (0)$. However, we have $a(T_p) = 1$ because $T_p = \mathcal{G}(a_1 A_p)$ is a polynomial ring in one variable. This is a contradiction, which completes the proof of Proposition 2.1. ~\\

We would like to close this section with the following example of a mixed ideal $I$ satisfying the conditions (i), (ii), and (iii) in Proposition 2.1 and whose the associated ring is a Gorenstein ring.

Let $k[[X, Y, Z, W]]$ be the formal power series ring in 4 variables over an infinite field $k$. We put $A = k[[X, Y, Z, W]]/(X(Y + W), (X + Z)(Y + Z)W)$ that is a Gorenstein local ring with dimension 2. We denote by $x, y, z$, and $w$ the reduction of $X, Y, Z$, and $W$ mod $(X(Y + W), (X + Z)(Y + Z)W)$. Let $I := (x, z) \cap (x, y, w) = (x, yz, zw)$. Then $I$ is generically a complete intersection with height 0. Let $a_1 = x + yz + zw$ and $a_2 = zw$. We put $J = (a_1, a_2)$ that is a reduction of $I$ with $r_I(I) = 1$, as $I^2 =JI + (x^2)$ and $x^2 = a_1 x$. Moreover we get $J$ is minimal because $a_1, a_2$ is $d$-sequence, and hence $\lambda(I) = 2$. It is routine to check that $a_1, a_2$ satisfy ($*$) and (**) stated in Section 1 and $r_1 = 1$. We put $L = a_1 U :_U (a_1 I :_I I)$. Then $I \subseteq L$. Assume that $I \subseteq L$ and take $p \in \text{Ass}_A A/I$ so that $I_p \subseteq L_p$. Then $p = (x, y, w)$, which is height one. We have $r_{a_1 A_p}(I_p) = 1$. Consequently, $L_p = a_1 A_p :_A I_p \neq A_p$. Since $I_p$ is the maximal ideal in $A_p$, we have $I_p = L_p$. Therefore $I = L$. Hence we get $\mathcal{G}(I)$ is a Gorenstein ring by Proposition 2.1.

3. THE PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. To do this, we may assume that $I$ is generically a complete intersection and $a = -s$ because when $G$ is Cohen-Macaulay, we have $a = -s$ if and only if the conditions (a) and (b) stated in our Theorem 1.1 are satisfied (use the $a$-invariant formula: $a(G) = \text{max}\{t_i - i \mid s \leq i < \ell \} \cup \{t_j(I) - \ell\}$). If $\text{ad}(I) = 1$, then Theorem 1.1 is covered by [GI], 1.6. Hence we may assume furthermore $\text{ad}(I) = 2$. Thus it is enough to show the following.
THEOREM 3.1. Assume that $I$ is generically a complete intersection with $\text{ad}(I) = 2$ and $\text{ht}_A\varphi < s + 2$ for any $\varphi \in \text{Ass}_A A/I$. Suppose that $G$ is a Cohen-Macaulay ring. Then the following two conditions are equivalent.

(1) $G$ is a Gorenstein ring.

(2) (a) $r_I(I) \leq 2$,
(b) $r_x + 1 \leq 1$, and
(c) $I = J_{s+1}U :_U (J_{s+1} : I)$.

When this is the case $a_1(G) = -s$, and the following two assertions are satisfied:

(i) $I = U$ if and only if $r_x = 0$;

(ii) $r_I(I) \leq 1$ for any $\varphi \in V(I)$ such that $\text{ht}_A\varphi = s + 2$.

Proof. If $G$ is a Gorenstein ring, then $a = -s$ (recall that $r_x = 0$ and $a = a_0(I)$ for all $\varphi \in V(I)$) and hence conditions (a) and (b) stated in Theorem 3.1 are satisfied (use the $a$-invariant formula). Therefore, to prove Theorem 3.1, we may assume that conditions (a) and (b) are satisfied. Then we obtain that $a = -s$ (use $a$-invariant formula again). Here we put $K = (a_1, a_2, \ldots, a_s)$. Since $G$ is a Cohen-Macaulay ring, we get the sequence $a_1t, a_2t, \ldots, a_s$ is $G$-regular (see [GNa2], 3.3), so that

$$G/(a_1t, a_2t, \ldots, a_s)G \cong (I/K)$$

of graded $A$-algebras ([VV]). We note the following.

CLAIM 3.2. $J_{s+1}U :_U (J_{s+1} : I) = (J_{s+1}U + K) :_U ((J_{s+1}I + K) :_I I)$.

Proof. We put $D = J_{s+1} :_I I$, and then $D \subseteq (J_{s+1}I + K) :_I I$. Let $x \in (J_{s+1}I + K) :_I I$. Then $xI \subseteq (J_{s+1}I + K) \cap I^2 = J_{s+1}I + K \cap I^2$. Since $a_1t, a_2t, \ldots, a_st$ is $G$-regular, we get $K \cap I^2 = KI$, and hence $D = (J_{s+1}I + K) :_I I$. Moreover we have $J_{s+1}U :_U D \subseteq (J_{s+1}U + K) :_U D$. Let $x \in (J_{s+1}U + K) :_U D$. Then $xD \subseteq (J_{s+1}U + K) \cap IU = J_{s+1}U + K \cap IU$. Hence it suffices to show that $K \cap IU = KU$. Let $v \in \text{Ass}_A KU$. Then it is enough to prove $K_v \cap I_vU_v = K_vU_v$. We may assume $U \subseteq v$. Look at the exact sequence $0 \to K/KU \to A/KU \to A/K \to 0$ of $A$-modules. Then since $K/KU \cong (A/U) \otimes_{A/K} (K/K^2) \cong (A/U)^s$, we have $\text{Ass}_A A/KU \subseteq \text{Ass}_A A/U \cup \text{Ass}_A A/K$. Therefore $\text{ht}_A v = s$, whence $I_v = U_v = K_v$. 

According to [GNN2], 3.4 together with Claim 3.2, passing to the ring $A/K$, we may assume furthermore that $s = 0$. We must check that $G$ is a Gorenstein ring if and only if $I = a_1U :_U (a_1I : I)$. Assume $G$ is a Gorenstein ring. Then in the same way of the proof of Proposition 2.1, (1) $\Rightarrow$ (2), we get $I = a_1U :_U (a_1I : I)$.

Conversely, we take any $\varphi \in \mathcal{A}$. It suffices to show that $\mathcal{A}(I_\varphi)$ is a Gorenstein ring with $a(\mathcal{A}(I_\varphi)) = 0$ (see [GI], 1.2). Notice that
ht_\mathcal{A}p = \lambda(I_p) \leq 2$ because $J_p$ is a reduction of $I_p$. If $ht_\mathcal{A}p = 0$, then $I_p = (0)$ and hence we have nothing to prove.

Assume that $ht_\mathcal{A}p = 1$. Then $a_1 A_p$ is a minimal reduction of $I_p$ with $r_{a_1 A_p}(I_p) \leq 1$ because $ht_\mathcal{A}p = \lambda(I_p) = 1$ and $r_1 \leq 1$, so that $I_p = a_1 I_p : I_p$. If $ht_\mathcal{A}p(I_p) = 1$, then $I_p$ is a $p A_p$-primary ideal in $A_p$. Therefore $p \nsubseteq U$. So we have $I_p = a_1 A_p : I_p$ and hence $r_{a_1 A_p}(I_p) = 1$. If $ht_\mathcal{A}p(I_p) = 0$, then $U(I_p) = U(I_p)$ and hence we have $I_p = [a_1 U(I_p) : I_p]U(I_p)$. Consequently, thanks to [GI], 1.4 and [GI], 6.3, we get $\mathcal{G}(I_p)$ is a Cohen-Macaulay ring with $a(\mathcal{G}(I_p)) = 0$.

Suppose that $ht_\mathcal{A}p = 2$. Then $J_p$ is a minimal reduction of $I_p$ with $r_1(I_p) \leq 2$ because $ht_\mathcal{A}p = \lambda(I_p) = 2$ and $r_1(I) \leq 2$. If $ht_\mathcal{A}p(I_p) = 2$, then $p$ is a minimal prime ideal of $I$. So we have $p \in \text{Ass}_A A/I$, however this is a contradiction to the standard assumption. Hence we get $ht_\mathcal{A}p(I_p) \leq 1$.

Let $ht_\mathcal{A}p(I_p) = 1$. Hence $ad(I) = 1$. Since $p \notin \text{Ass}_A A/I$ we have $A_p/I_p$ is a Cohen-Macaulay ring. Thanks to [GNN3], it is enough to check the following three assertions:

1. $r_{[a_1 A_p]}([I_p]_Q) = 1$ for all $Q \in \text{Ass}_A A_p/I_p$;
2. $a_1 A_p : A_p I_p \subseteq I_p$;
3. $r_{a_1 A_p}(I_p) \leq 1$.

Since $ht_\mathcal{A}p(I_p) = 1$, we have $p \nsubseteq U$ and hence $I_p = a_1 A_p : A_p (a_1 I_p : I_p) \supseteq a_1 A_p : A_p I_p$. Thus we get the assertion 2. Let $Q \in \text{Ass}_A A_p/I_p$. Then $ht_\mathcal{A}p Q = 1$ because the ring $A_p/I_p$ is Cohen-Macaulay. Take $q \in V(I)$ such that $Q = \mathfrak{q} A_p$, and we have $r_{a_1 A_p}(I_p) \leq 1$, as $ht_\mathcal{A}p q = 1$. Since $a_1 A_p : A_p I_p \subseteq I_p \neq A_p$, we get $r_{a_1 A_p}(I_p) = 1$. We must show the assertion 3. Notice that $a(G_p) = 0$ (use the a-invariant formula: $a(G_p) = \max\{r_1(I_p) - 1, r_2(I_p) - 2\}$). Since $G_p$ is a Cohen-Macaulay ring, we have $a_1 I$ is $G_p$-regular element (see [GNN3], 2.3). Hence, passing to the ring $A_p/a_1 A_p$, it suffices to show the following claim, which is due to [GNN3] (see [GNN2], 3.4).

**Claim 3.3.** ([GNN3]). Let $A$ be a Gorenstein local ring. Let $I$ be an ideal in $A$ with $ht_A = 0$ and $bA$ a minimal reduction of $I$. Assume that $G$ is a Cohen-Macaulay ring with $a(G) = 1$. Then $r_{bA}(I) \leq 1$ if $A/I$ is Cohen-Macaulay and $(0) :_A I \subseteq I$.

**Proof.** We put $\alpha = (0) :_A I$, $\overline{A} = A/\alpha$ and $T = \mathcal{G}(\overline{I} \overline{A})$. According to [GNN3], 4.5, we have the exact sequence

$$0 \rightarrow K_T \rightarrow K_G \rightarrow A/I(1) \rightarrow 0$$

of graded $G$-modules. Let $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ be the canonical $I$-filtration of $A$ ([GI]), hence we have $\omega_i = A$ for $i \leq -2$ and $K_G \cong \bigoplus_{i \leq -1} \omega_{i-1}/\omega_i$ as graded $G$-modules. Look at the homogeneous component $0 \rightarrow [K_T]_{-1} \rightarrow A/\omega_{-1} \rightarrow A/I \rightarrow 0$ of degree $-1$ in the exact sequence above, and we get
\[ \omega_{-1} = I \] because \( \omega_{-1} \supseteq I \) and hence \([K_T]_{-1} = (0)\). Therefore \( a(T) = 1 \). So \( r_{b(\overline{A})}(I \overline{A}) \leq 1 \) by [GNN3], 4.4. Then \( I^2 \subseteq bI + \alpha \), so that \( I^2 = bI \) because \( \alpha \cap I^2 = (0) \) (see [GNN3], 4.2(1)).

Now let \( \text{ht}_I I_p = 0 \). To prove that \( \mathfrak{g}(I_p) \) is a Gorenstein ring with \( a(\mathfrak{g}(I_p)) = 0 \), we must show \( r_j(I_p) \leq 1 \) (see Proposition 2.1). It is enough to show \( \omega_0 = I_p \) by Lemma 2.6. We have \( \omega_0 \supseteq I_p \). Let \( q \in \text{Ass}_A A_p/I_p \). Then \( \text{ht}_q q \leq 1 \). We have discussed earlier that \( \mathfrak{g}(I_q) \) is a Gorenstein ring with \( a(\mathfrak{g}(I_q)) = 0 \). Therefore we get \( \omega_0 = I_q \), so that \( \omega_0 = I_p \).

Because \( G_p \) is Gorenstein for any \( p \in V(I) \) such that \( \text{ht}_p p = 2 \), the last assertion follows from Proposition 2.1. This completes the proof of Theorem 3.1.

As consequence of Theorem 3.1 with [GNN2], 6.6, we get the following.

**Corollary 3.4.** Assume that \( I \) is generically a complete intersection with \( \text{ad}(I) = 2 \). Suppose that \( \text{depth} A/I \geq d - s - 1 \). Then the following two conditions are equivalent.

1. \( G \) is a Gorenstein ring.
2. \( \begin{align*} & (a) \quad r_j(I) \leq 2, \\ & (b) \quad r_{s+1} \leq 1, \\ & (c) \quad I = J_{s+1}U : U(J_{s+1} : I), \quad \text{and} \\ & (d) \quad \text{depth} A/I^2 \geq d - s - 2. \end{align*} \)

When this is the case, \( a(G) = -s \), and the following two assertions are satisfied:

1. \( I = U \) if and only if \( r_{s+1} = 0 \);
2. \( r_I(I_p) \leq 1 \) for any \( p \in V(I) \) such that \( \text{ht}_p p = s + 2 \).

4. THE CASE WHERE \( I \) IS UNMIXED

In this section we summarize some corollaries derived from Theorem 1.1. When \( I \) is an unmixed ideal, we can simplify these conditions in 1.1 and readily have the following.

**Corollary 4.1.** Assume that \( \text{ad}(I) \leq 2 \) and \( I \) is an unmixed ideal. Suppose \( \mathfrak{g}(I) \) is a Cohen-Macaulay ring. Then the following two conditions are equivalent.

1. \( \mathfrak{g}(I) \) is a Gorenstein ring and \( a(\mathfrak{g}(I)) = -s \).
2. \( \begin{align*} & (a) \quad r_j(I) \leq \text{ad}(I), \\ & (b) \quad \mu_{A_p}(I_p) \leq \text{ht}_A p \quad \text{for any} \ p \in V(I) \quad \text{such that} \ \text{ht}_A p < \ell. \end{align*} \)
Here let $\mu(\ast)$ stand for the number of generators. The condition (b) in Corollary 4.1 is equivalent to saying that our ideal $I$ has a certain special reduction (cf. [N], (2.2)).

Next, let us consider the case where $d = \ell$. Then we get the following, which is due to [GNa1] when $ad(I) = 1$.

**Corollary 4.2.** Assume that $ad(I) \leq 2$ and $I$ is an unmixed ideal. Suppose $d = \ell$. Then the following two conditions are equivalent.

1. $\mathcal{G}(I)$ is a Gorenstein ring and $a(\mathcal{G}(I)) = -s$.
2. (a) $r_f(I) \leq \max\{0, ad(I) - 1\}$,
   (b) $\mu_{A_\ell}(I_\ell) \leq \text{ht}_A\mathfrak{p}$ for any $\mathfrak{p} \in V(I)$ such that $\text{ht}_A\mathfrak{p} < \ell$.

**Proof.** We may assume $G$ is Cohen-Macaulay by [GNN2], 6.5. Hence the sequence $a_1t$, $a_2t$, $\ldots$, $a_t$ is $G$-regular (see [GNN3], 2.3 and [GNa2], 3.3) and hence we may assume furthermore $s = 0$. Then by Proposition 2.1 with [GNa1], 2.11, we get Corollary 4.2.

Let us note the following corollary.

**Corollary 4.3.** Let $\mathfrak{p}$ be a prime ideal in a regular local ring $A$ with $\dim A/\mathfrak{p} = 2$. Assume that $A/\mathfrak{p}$ is a complete intersection in codimension one, then the associated graded ring $\mathcal{G}(\mathfrak{p})$ is Gorenstein if and only if there is a minimal reduction $J$ with $r_f(\mathfrak{p}) \leq 1$.

We conclude this paper with the following example of the case where $ad(\mathfrak{p}) = 2$. Let $k[X_1, X_2, X_3, Y_1, Y_2]$ and $k[s, t]$ be the polynomial rings over an infinite field $k$. Let $n \geq 4$ be an integer such that $3|n$ and let $\phi : k[X_1, X_2, X_3, Y_1, Y_2] \rightarrow k[s, t]$ be the homomorphism of $k$-algebras defined by $\phi(X_1) = s$, $\phi(X_2) = st$, $\phi(X_3) = st^2$, $\phi(Y_1) = t^3$, $\phi(Y_2) = t^n$. We put $S = k[X_1, X_2, X_3, Y_1, Y_2]$ and $A = S_{\mathfrak{m}}$, where $\mathfrak{m} = (X_1, X_2, X_3, Y_1, Y_2)S$. Let $\mathfrak{m} = \ker \phi$ and $\mathfrak{p} = \mathfrak{m}A$. We have the ring $A/\mathfrak{p}$ is a complete intersection in codimension one. Let $k > 0$. Then the ideal $\mathfrak{m}$ is minimally generated by the following seven elements:

\[f_1 = X_2^2 - X_1X_3, \quad f_2 = X_2X_3 - X_1^2Y_1,\]
\[f_3 = X_3^2 - X_1X_2Y_1, \quad f_4 = \begin{cases} X_1Y_2 - X_2Y_1^k & (n = 3k + 1) \\
X_1Y_2 - X_3Y_1^k & (n = 3k + 2), \end{cases} \]
\[f_5 = \begin{cases} X_2Y_2 - X_3Y_1^k & (n = 3k + 1) \\
X_2Y_2 - X_1Y_1^{k+1} & (n = 3k + 2), \end{cases} \]
\[f_6 = \begin{cases} X_3Y_2 - X_1Y_1^{k+1} & (n = 3k + 1) \\
X_3Y_2 - X_2Y_1^{k+1} & (n = 3k + 2), \end{cases} \]
\[g = Y_2^2 - Y_1^n.\]
When \( n = 3k + 1 \), they satisfy the relations:

\[
\begin{align*}
f_1 g &= -Y_1^{k+1}f_4^2 + Y_2f_3^2 - Y_2f_4f_6 + Y_1f_5f_6 \\
f_2 g &= -Y_1 Y_2 f_4^2 + Y_1^k f_6^2 - Y_1^{k+1} f_4 f_5 + Y_2 f_5 f_6 \\
f_1 f_6 &= f_2 f_5 - f_3 f_4
\end{align*}
\]

and hence, letting \( \mathfrak{S} = (f_3, f_4, f_5, f_2 + g, f_1 + f_6) S \), we get \( \mathfrak{S}^2 = \mathfrak{S} \mathfrak{S} \). When \( n = 3k + 1 \), they satisfy the relations:

\[
\begin{align*}
f_1 g &= -Y_2 f_5^2 + Y_1^k f_6^2 - Y_1^{k+1} f_4 f_5 + Y_2 f_4 f_6 \\
f_2 g &= -Y_1 Y_2 f_4^2 + Y_1^{k+1} f_5^2 - Y_1^{k+1} f_4 f_6 + Y_2 f_5 f_6 \\
f_2 f_6 &= f_3 f_5 - Y_1 f_5 f_6
\end{align*}
\]

and hence, letting \( \mathfrak{S} = (f_3, f_4, f_5, f_1 + g, f_2 + f_6) S \), we get \( \mathfrak{S}^2 = \mathfrak{S} \mathfrak{S} \). Hence by Corollary 4.3, the ring \( \mathfrak{S}(\nu) \) is a Gorenstein ring.

In particular, we have \( \text{ad}(\nu) = 2 \). In fact, assume \( \mathfrak{S}^{(2)} = \mathfrak{S}^2 \). Then \( \mathfrak{S}^2 \ni f_1 f_3 - f_2^2 = X_1(3X_1X_2X_3Y_1 - X_2^2Y_1 - X_3^2 - X_1^2Y_1^2) \), so that \( 3X_1X_2X_3Y_1 - X_2^2Y_1 - X_3^2 - X_1^2Y_1^2 \in \mathfrak{S}^2 \). However this is impossible. Therefore, thanks to [N], we have \( \lambda(\nu) = 5 \) and hence \( \text{ad}(\nu) = 2 \).

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