Europ. J. Combinatorics (1993) **14**, 461–477

Packing and Decomposition Problems for Polynomial Association Schemes

V. I. Levenshtein

We consider P- and Q-polynomial association schemes and introduce definitions of Delsarte codes and decomposable schemes. Many known combinatorial notions can be defined as Delsarte codes in suitable association schemes, and almost all classical association schemes turn out to be decomposable. For decomposable association schemes we prove some packing bounds, which were proven before only for antipodal schemes. We also prove that any Delsarte code consists of maximal possible numbers of points for its minimal distance. Some statements about the connection between designs in decomposable schemes and designs in their projections are also given. Detailed proofs of some of our results will be published in the longer paper [24], where analogous problems for a wider class of finite and infinite polynomial metric spaces are considered.

1. Introduction: Bounds for Extremal Subsets in Polynomial Metric Space

Considering packing problems for association schemes, it is convenient to use a definition which is analogous to the definition of metric space. A (symmetrical) *association scheme* (with D classes) is a finite set X with a given function $d(x, y)$ which is defined for any pair $x, y \in X$, taking values $0, 1, \ldots, D$, and has the following properties:
(i) $d(x, y) = 0$ iff $x = y$;
(ii) $d(x, y) = d(y, x)$ for any $x, y \in X$;
(iii) for any $x, y \in X$ and any $i, j \in \{0, 1, \ldots, D\}$, the number of points z such that $d(x, z) = i$, $d(z, y) = j$ depends on $d(x, y)$ only (usually this number is denoted by $p^k_{i,j}$, where $k = d(x, y)$). Using the adjacency matrices A_i, $i = 0, 1, \ldots, D$, defined by

$$(A_i)_{x,y} = \begin{cases} 1, & \text{if } d(x, y) = i, \\ 0, & \text{otherwise,} \end{cases}$$

the definition of association scheme can be expressed by

$$\sum_{i=0}^{D} A_i = J, \qquad A_0 = I, \qquad A_i = A_i^T, \qquad A_iA_j = \sum_{i=0}^{D} p^k_{i,j}A_k,$$

where J is the matrix the entries of which are all equal to one, I is the unit matrix and A^T is the transpose of A. The matrices A_i are linearly independent and generate a $(D + 1)$-dimensional commutative algebra $\mathfrak{U}$ of symmetric matrices, which is called the Bose–Mesner algebra. We consider the $|X|$-dimensional Hermitian space $V = \{f(x): X \Rightarrow \mathbb{C}\}$ of complex functions on X with the inner product

$$\langle u, v \rangle = \frac{1}{|X|} \sum_{x \in X} u(x)\overline{v(x)}. \tag{1.1}$$

It is known (Delsart [5], Bannai and Ito [1]) that for an association scheme with D classes there exists a decomposition

$$V = V_0 + V_1 + \cdots + V_D \tag{1.2}$$

0195-6698/93/050461 + 17 $08.00/0

of V into a direct sum of pairwise orthogonal subspaces V_i, where V_i is a maximal common eigenspace of $A_0, A_1, \ldots, A_D$ and V_0 consists of constants only. Let $r_i = \dim V_i$, $i = 0, 1, \ldots, D$, and $\{v_{i,j}(x), j = 1, \ldots, r_i\}$ be any orthonormal basis of V_i. The matrices

$$E_i(x, y) = \frac{1}{|X|} \sum_{j=1}^{r_i} v_{ij}(x)\overline{v_{ij}(y)}, \qquad i = 0, 1, \ldots, D, \tag{1.3}$$

do not depend on the choice of the basis of V_i *and form the basis of minimal idempotents of the algebra* $\mathfrak{U}$. The eigenmatrices $P = (P_{i,j})$ and $Q = (Q_{i,j})$ of an association scheme are defined by

$$A_j = \sum_{i=0}^{D} P_{i,j} E_i, \qquad E_j = \frac{1}{|X|} \sum_{i=0}^{D} Q_{i,j} A_i. \tag{1.4}$$

Furthermore, $Q_{0,j} = \operatorname{rank} E_j = r_j$ and $P_{0,j} = p_{j,j}^{0}$ (that is, the number of points z such that $d(x, z) = j$ for a fixed $x \in X$; we denote this number by k_j).

It is clear that an association scheme is a metric space with distance $d(x, y)$ when for this function the triangle inequality holds. Delsarte proved [5] that it holds if there exist polynomials $p_i(t)$ of degree i, $i = 1, \ldots, D$, such that $A_i = p_i(A_1)$ or, in other words, $P_{i,j} = p_j(P_{i,1})$. Such an association scheme is called a *P-polynomial*, or *metric*.

There is another description of metric association schemes in terms of graphs. The vertex set X of any undirected graph Γ can be considered as a metric space with metric $d_\Gamma(x, y)$ equal to the number of edges in the shortest path from x to y. An undirected connected graph Γ with the vertex set X is called *distance-regular* if or any $x, y \in X$ the number of vertices z such that $d_\Gamma(x, z) = 1$, $d_\Gamma(y, z) = d_\Gamma(x, y) - 1$ and the number of vertices z such that $d_\Gamma(x, z) = 1$, $d_\Gamma(y, z) = d_\Gamma(x, y) + 1$ depend on $d_\Gamma(x, y)$ only. Delsarte proved [5] that for any distance-regular graph Γ with vertex set X, $\{X, d_\Gamma(x, y)\}$ is a metric association scheme, and for any metric association scheme $\{X, d(x, y)\}$ the graph Γ with vertex set X and the adjacency matrix A_1 is a distance-regular graph, and $d(x, y) = d_\Gamma(x, y)$. Thus, there is one-to-one correspondence between metric association schemes with D classes and distance-regular graphs of diameter D.

An association scheme $\{X, d(x, y)\}$ with D classes is called *Q-polynomial*, or *cometric*, if there exist polynomials $q_j(\sigma)$ of degree j, $j = 0, 1, \ldots, D$, of a real variable σ such that $Q_{d,j} = q_j(Q_{d,1})$ for any $d = 0, 1, \ldots, D$. For any Q-polynomial association scheme we fix some function

$$\sigma(d) = c_1 Q_{d,1} + c_2, \qquad \text{where } c_1, c_2 \in \mathbb{R} \text{ and } c_1 \neq 0$$

and consider polynomials

$$Q_j(\sigma) = \frac{1}{r_j} q_j\left(\frac{\sigma - c_2}{c_1}\right), \qquad j = 0, 1, \ldots, D,$$

so that

$$Q_{d,j} = r_j Q_j(\sigma(d)) \qquad \text{for any } d = 0, 1, \ldots, D. \tag{1.5}$$

Without loss of generality, we can assume that

$$\sigma(0) = 0 \leq \sigma(d) \leq 1 \qquad \text{for } d = 0, 1, \ldots, D, \tag{1.6}$$

because the function $\sigma(d)$ is determined only up to a linear transformation. From (1.1)–(1.6), it follows that for any $x, y \in X$ and any $i = 0, 1, \ldots, D$,

$$Q_i(\sigma(d(x, y))) = \frac{1}{r_i} \sum_{j=1}^{r_i} v_{ij}(x)\overline{v_{ij}(y)}, \tag{1.7}$$

and for the polynomial system $\{Q_j(\sigma), j = 0, 1, \ldots, D\}$ the following conditions of orthogonality and normalization hold:

$$\frac{r_i}{|X|} \sum_{d=0}^{D} Q_i(\sigma(d))Q_j(\sigma(d))k_d = \delta_{i,j}, \tag{1.8}$$

$$Q_i(0) = 1, \qquad i = 0, 1, \ldots, D. \tag{1.9}$$

Note that the polynomial

$$Q_{D+1}(\sigma) = (\sigma(0) - \sigma)(\sigma(1) - \sigma) \cdots (\sigma(D) - \sigma) \tag{1.10}$$

is orthogonal to all polynomials $Q_j(\sigma), j = 0, 1, \ldots, D$. The polynomial system $\{Q_j(\sigma), j = 0, 1, \ldots, D\}$ has the following property of 'positivity': all coefficients $q_{i,j}^k$ (Krein parameters), defined uniquely by

$$Q_i(\sigma)Q_j(\sigma) = \sum_{k=0}^{D} q_{i,j}^k Q_k(\sigma) (\operatorname{mod} Q_{D+1}(\sigma)), \qquad i, j = 0, \ldots, D, \tag{1.11}$$

are non-negative, because

$$q_{i,j}^k = \frac{r_k}{|X|} \sum_{d=0}^{D} Q_i(\sigma(d))Q_j(\sigma(d))Q_k(\sigma(d))k_d$$

$$= \frac{1}{|X|^2 r_i r_j} \sum_{r=1}^{r_i} \sum_{s=1}^{r_j} \sum_{t=1}^{r_k} \left| \sum_{x \in X} v_{i,r}(x) v_{j,s}(x) v_{k,t}(x) \right|^2.$$

Furthermore,

$$q_{i,j}^k > 0, \qquad \text{if } i + j = k. \tag{1.12}$$

Hereafter we consider Q-polynomial association schemes $\{X, d(x, y)\}$ with D classes for which $d(x, y)$ is a metric on X^2. Such Q-polynomial association schemes and metric spaces are referred to as (finite) *polynomial metric spaces* (of diameter D with the function $\sigma(d)$). In particular, P- and Q-polynomial association schemes are polynomial metric spaces. For a polynomial metric space of diameter D we call the function $\sigma(d)$ *standard* if $\sigma(d)$ is an increasing function such that

$$\sigma(0) = 0 \leqslant \sigma(d) \leqslant \sigma(D) = 1. \tag{1.13}$$

Note that, in fact, we use the standardization $\sigma(d) = (r_1 - Q_{d,1})/(r_1 - Q_{D,1})$, which is distinct from the standardization $\sigma(d) = Q_{d,1}/r_1$ used in [2]. We do not consider in detail here infinite polynomial metric spaces, which are the same as connected compact two-point homogeneous spaces (see Wang [37], Kabatjansky and Levenshtein [18] and Sloane [31]); namely, the Euclidean sphere, the real, complex, quaternionic projective spaces and the Cayley elliptic plane.

We recall some classical metric spaces which are polynomial (Delsarte [5, 7], Delsarte and Goethals [9] and Stanton [32, 33]) with the standard function $\sigma(d) = d/D$ or $\sigma(d) = [\alpha^D(\alpha^d - 1)]/[\alpha^d(\alpha^D - 1)]$, where D is the diameter and α is a power of a prime. Let q be a power of a prime, let F_q be the finite field with q elements, and let F_q^n be the n-dimensional vector space over F_q. We denote the Hamming space of all vectors of length n over the alphabet $\{0, 1, \ldots, r-1\}$ by $H(n, r)$; the space of all matrices of size $r \times n$ over F_q, $r \geqslant n$, by $H(n, r, q)$; the Johnson space of all n-subsets of a v-set, $2n \leqslant v$, by $J(v, n)$; the Grassmann space of all n-dimensional linear subspaces of F_q^v, $n \leqslant v$, by $J(v, n, q)$; the space of all alternating matrices of order N over F_q by $A(N, q)$, and the dual polar spaces of all maximal (n-dimensional) isotropic subspaces of a non-singular bilinear form of one of six types over F_α, where $\alpha = q$ or $\alpha = q^2$, by $S_i(n, q)$, $i = 1, \ldots, 6$. The metrics are: $d(x, y) = n - |x \cap y|$ for $J(v, n)$; $d(x, y) = n -$

$\dim(x \cap y)$ for $J(v, n, q)$ and $S_i(n, q)$; $d(x, y) = \text{rank}(x - y)$ for $H(n, r, q)$, and $d(x, y) = \frac{1}{2}\text{rank}(x - y)$ for $A(N, q)$. The diameter D of the above-mentioned metric spaces is equal to n, if $[N/2] = n$ for $A(N, q)$. We do not consider the space $He(n, q)$ of all Hermitian matrices of order n over F_α, where $\alpha = q^2$, because the function $\sigma(d)$ is not monotone in this case, nor the Egawa [14] space of quadratic forms over F_q^N which has the same parameters as $A(N + 1, q)$.

Now we give some definitions and notations for any subset or, as we prefer to say, code W of a polynomial metric space X (with the function $\sigma(d)$). Let $l(W)$ be the *number of distances* between distinct points of W, let $d(W)$ be the *minimal distance* between distinct points of W, and let $D(W)$ be the *maximal distance* between points of W (that is, the *diameter* W). A code $W \subseteq X$ is called *diametrical* if $D(W) = D(X)$, and called *d-code* if $d(W) \geq d$. We call a code $W \subseteq X$ *maximum* if, for any $W' \subseteq X$ such that $d(W') \geq d(W)$, the inequality $|W| \geq |W'|$ holds. A polynomial $f(\sigma)$ is called *annihilating* for W if

$$f(\sigma(d(x, y))) = 0 \qquad \text{for any distinct } x, y \in W.$$

An annihilating polynomial for W of minimal degree (that is, $l(W)$) is defined up to a constant factor and denoted by $f_W(\sigma)$. A code W is called a *t-design*, $t = 0, 1, \ldots, D$, if (see (1.2))

$$\sum_{x \in W} v(x) = 0 \qquad \text{for any } v(x) \in \bigcup_{i=1}^{t} V_i.$$

Let $t(W)$ be the maximal t, $t = 0, 1, \ldots, D$, such that W is a t-design.

For the formalization of the known bounds on the cardinality of codes with given parameters $t(W)$, $l(W)$ and $d(W)$ in polynomial metric space X of diameter D (with the standard function $\sigma(d)$), it is convenient to introduce for any $a, b = 0, 1, \ldots$ 'adjacent' systems $\{Q_i^{a,b}(\sigma)\}$ of orthogonal polynomials. Let $c^{a,b}$ be a positive constant (for normalization of a new measure), so that

$$\frac{c^{a,b}}{|X|} \sum_{d=0}^{D} (\sigma(d))^a (1 - \sigma(d))^b k_d = 1, \qquad c^{0,0} = 1. \tag{1.14}$$

Then due to (1.13) the polynomials $Q_j^{a,b}(\sigma)$ and positive constants $r_j^{a,b}$ are determined uniquely by the following conditions of orthogonality and normalization:

$$\frac{r_i^{a,b} c^{a,b}}{|X|} \sum_{d=0}^{D} Q_i^{a,b}(\sigma(d)) Q_j^{a,b}(\sigma(d)) (\sigma(d))^a (1 - \sigma(d))^b k_d = \delta_{i,j}, \tag{1.15}$$

$$Q_j^{a,b}(0) = 1. \tag{1.16}$$

Denote the smallest root d of equation $Q_i^{a,b}(\sigma(d)) = 0$ by $d_i^{a,b}$ and notice that some statements on separation of these values hold; in particular [22],

$$d_k^{1,1} < d_k^{1,0} < d_{k-1}^{1,1}, \quad k = 1, \ldots, D - 1, \qquad \text{where } d_0^{1,1} = D, \quad d_D^{1,0} = d_{D-1}^{1,1} = 1.$$

Also, introduce the kernel

$$K_i^{a,b}(s, t) = \sum_{j=0}^{i} r_j^{a,b} Q_j^{a,b}(s) Q_j^{a,b}(t). \tag{1.17}$$

Using (1.15), (1.16) and the Cristoffel–Darboux formula [35]

$$(s - t) K_i^{a,b}(s, t) = r_i^{a,b} m_i^{a,b} (Q_{i+1}^{a,b}(s) Q_i^{a,b}(t) - Q_i^{a,b}(s) Q_{i+1}^{a,b}(t)), \tag{1.18}$$

where $m_i^{a,b}$ is the ratio of the highest coefficient of $Q_i^{a,b}(\sigma)$ to that of $Q_{i+1}^{a,b}(\sigma)$ ($m_i^{a,b} < 0$

for our normalization), we have

$$Q_i^{0,1}(\sigma)=\frac{K_i^{0,0}(1,\sigma)}{K_i^{0,0}(1,0)},\quad Q_i^{1,0}(\sigma)=\frac{K_i^{0,0}(0,\sigma)}{K_i^{0,0}(0,0)},\qquad i=0,1,\ldots,D-1, \tag{1.19}$$

$$Q_i^{1,1}(\sigma)=\frac{K_i^{0,1}(0,\sigma)}{K_i^{0,1}(0,0)},\qquad i=0,1,\ldots,D-2. \tag{1.20}$$

It is convenient to assume (cf. (1.10)) that

$$Q_{D+1}^{0,0}(\sigma)=-\sigma Q_D^{1,0}(\sigma)=(1-\sigma)Q_D^{0,1}(\sigma)=-\sigma(1-\sigma)Q_{D-1}^{1,1}(\sigma), \tag{1.21}$$

so that $m_{D-1}^{1,0}$, $m_{D-1}^{0,1}$ and $m_{D-2}^{1,1}$ will also be negative.

From (1.7) and (1.13) we can see that for any $W\subseteq X$ and any polynomial

$$f(\sigma)=\sum_{i=0}^{t} f_iQ_i(\sigma),\qquad t=0,1,\ldots,D, \tag{1.22}$$

the equality

$$|W|f(0)+\sum_{\substack{x,y\in W\\ x\neq y}} f(\sigma(d(x,y)))=|W|^2f_0+\sum_{i=1}^{t}\frac{f_i}{r_i}\sum_{j=1}^{r_i}\left|\sum_{x\in W} v_{i,j}(x)\right|^2 \tag{1.23}$$

holds. This equality and the decomposition (1.2) are the main tools for obtaining bounds on extremal codes $W\subseteq X$. We exclude from consideration the case $W=X$ in which $l(W)=t(W)=D(W)=D$, $f_W(\sigma)=Q_D^{1,0}(\sigma)=(1-\sigma)Q_{D-1}^{1,1}(\sigma)$ and $d(W)=1=d_D^{1,0}=d_{D-1}^{1,1}$.

Corollary 1.1 (Delsarte [5]). *If W is a t-design, $f(\sigma)\geq 0$ for $0\leq\sigma\leq 1$ and $f_0>0$, then*

$$|W|\geq \Omega(f)=f(0)/f_0. \tag{1.24}$$

Moreover, this bound is attained iff $f(\sigma)$ is a annihilating polynomial for W.

Corollary 1.2 (Delsarte [5]). *If W is a d-code and the polynomial $f(\sigma)$ has the following properties:*

$$\begin{aligned} f_i&>0, & i&=0,1,\ldots,t,\\ f(\sigma)&\leq 0 & &\textit{for } \sigma(d)\leq\sigma\leq 1,\end{aligned} \tag{1.25}$$

then

$$|W|\leq \Omega(f)=f(0)/f_0. \tag{1.26}$$

Moreover, this bound is attained iff $f(\sigma)$ is an annihilating polynomial for W, and W is a t-design.

The problem is to find the permissible polynomials $f(\sigma)$ which optimize the right-hand sides of (1.24) and (1.26). Notice that all bounds given below depend only on the diameter D, the function k_d, and the standard function $\sigma(d)$, $d=0,1,\ldots,D$. Furthermore, from results of Leonard [19] and Terwilliger [36] it follows that the bounds depend only on the first values of these functions: k_1, k_2, $\sigma(1)$, $\sigma(2)$, and $\sigma(3)$. It should be observed in advance that the bounds (1.31) and (1.33) given below have been proved for some, but not yet for all, polynomial metric spaces.

Bounds for t-designs (Delsarte [5] and Dunkl [13]). For any $W \subset X$,

$$|W| \geq \sum_{i=0}^{k} r_i \qquad \text{if } t(W) = 2k, \tag{1.27}$$

$$|W| \geq \left(1 - \frac{Q_{k-1}^{1,0}(1)}{Q_k(1)}\right) \sum_{i=0}^{k-1} r_i \qquad \text{if } t(W) = 2k-1. \tag{1.28}$$

These bounds are attained iff $f_W(\sigma) = Q_k^{1,0}(\sigma)$ and $f_W(\sigma) = (1-\sigma)Q_{k-1}^{1,1}(\sigma)$ respectively.

For a proof of (1.27), Delsarte [5] used Corollary 1.1 for polynomials $f(\sigma) = (Q_k^{1,0}(\sigma))^2$ ((without the assumption on monotonicity of $\sigma(d)$). For a proof of (1.28), Dunkl [13] used the polynomial $f(\sigma) = (1-\sigma)(Q_{k-1}^{1,1}(\sigma))^2$ and monotonicity of $\sigma(d)$ (see (1.13)). Moreover, Delsarte proved that $t(W) \leq 2l(W)$ for any $W \subseteq X$ and that every code from the class

$$\mathrm{DD}(X) = \{W \subset X : t(W) \geq 2l(W) - 2\}$$

(*Delsarte design* in X) is a Q-polynomial association scheme. The codes W, for which the bounds (1.27) or (1.28) are attained, are called *tight* designs. The class $\mathrm{TD}(X)$ of tight designs in X can be defined by

$$\mathrm{TD}(X) = \{W \subset X : t(W) = 2l(W) \text{ or } t(W) = 2l(W) - 1 \text{ and } D(W) = D(X)\}.$$

Now we introduce a class $\mathrm{DC}(X)$ of *Delsarte codes* in X by

$$\mathrm{DC}(X) = \{W \subset X : t(W) \geq 2l(W) - 1 \text{ or } t(W) = 2l(W) - 2 > 0 \text{ and } D(W) = D(X)\}. \tag{1.29}$$

The class $\mathrm{DC}(X)$ is intermediate between $\mathrm{TD}(X)$ and $\mathrm{DD}(X)$. In Section 3, we will see that Delsarte codes are maximum. This is surprising, since in the definition of Delsarte codes we say nothing about its minimal distance. It is also interesting that many known combinatorial notions can be defined as Delsarte codes in certain polynomial metric spaces (see, for example, Table 1, a list of Delsarte codes in Hamming space).

Absolute bounds (Delsarte [5]). For any $W \subset X$,

$$|W| \leq \sum_{i=0}^{k} r_i \qquad \text{if } l(W) = k, \tag{1.30}$$

$$|W| \leq \left(1 - \frac{Q_{k-1}^{1,0}(1)}{Q_k(1)}\right) \sum_{i=0}^{k-1} r_i \qquad \text{if } l(W) = k \text{ and } D(W) = D(X). \tag{1.31}$$

These bounds are attained iff $t(W) = 2k$ and $f_W(\sigma) = Q_k^{1,0}(\sigma)$ or $t(W) = 2k-1$ and $f_W(\sigma) = (1-\sigma)Q_{k-1}^{1,1}(\sigma)$ respectively.

For the proof of (1.30), Delsarte [5] used the decomposition (1.2) and the annihilating polynomial $f_W(\sigma)$. The bound (1.31) was proved by Delsarte, Goethals and Seidel [10, 11], and by Hoggar [17] for infinite polynomial metric spaces. Neumaier [26] has published (1.31) as a conjecture for finite polynomial metric spaces, and has noted that it is true for antipodal spaces. Notice that, by Theorem 8.2.4 in Brouwer, Cohen and Neumaier [2], an antipodal polynomial metric space can be defined as a polynomial metric space X of diameter D with a standard function $\sigma(d)$ such that $\sigma(d) + \sigma(D-d) = 1$ for any $d = 0, 1, \ldots, D$; and that for any $x \in X$ there exists one and only one point at distance D from x.

The following bounds are an improvement of the McEliece–Rodemich–Rumsey–Welch bounds for d-codes [25], which were obtained in 1977 for the Hamming and Johnson spaces by using Corollary 1.2 for the polynomials

$$(\sigma(d)-\sigma)(K_{k-1}^{0,0}(\sigma,\sigma(d))^2, \qquad \text{if } d_k^{0,0}<d<d_{k-1}^{0,0}.$$

Bounds for d-codes (Levenshtein [20]). Let $W\subset X$ and $d(W)=d$. Then

$$|W|\leqslant B(d)=\begin{cases}\left(1-\dfrac{Q_{k-1}^{1,0}(\sigma(d))}{Q_k(\sigma(d))}\right)\displaystyle\sum_{i=0}^{k-1} r_i & \text{if } d_k^{1,0}\leqslant d\leqslant d_{k-1}^{1,1}, \quad (1.32)\\ \left(1-\dfrac{Q_k^{1,0}(\sigma(d))}{Q_k^{0,1}(\sigma(d))}\right)\displaystyle\sum_{i=0}^{k} r_i & \text{if } d_k^{1,1}<d<d_k^{1,0}, \quad (1.33)\end{cases}$$

and, in particular,

$$|W|\leqslant B(d_k^{1,0})=\sum_{i=0}^{k} r_i \qquad \text{if } d=d_k^{1,0}, \tag{1.34}$$

$$|W|\leqslant B(d_{k-1}^{1,1})=\left(1-\frac{Q_{k-1}^{1,0}(1)}{Q_k(1)}\right)\sum_{i=0}^{k-1} r_i \qquad \text{if } d=d_{k-1}^{1,1}. \tag{1.35}$$

Furthermore, the function $B(d)$ is a decreasing continuous function, and the bounds (1.32)–(1.35) are attained iff

$$f_W(\sigma)=\begin{cases}(\sigma(d)-\sigma)K_{k-1}^{1,0}(\sigma,\sigma(d)) & \text{if } d_k^{1,0}\leqslant d\leqslant d_{k-1}^{1,1},\\ (1-\sigma)(\sigma(d)-\sigma)K_{k-1}^{1,1}(\sigma,\sigma(d)) & \text{if } d_k^{1,1}<d<d_k^{1,0},\end{cases}$$

and

$$t(W)=\begin{cases}2k-1 & \text{if } d_k^{1,0}<d\leqslant d_{k-1}^{1,1},\\ 2k & \text{if } d_k^{1,1}<d\leqslant d_k^{1,0}.\end{cases}$$

These bounds are obtained [20] by using Corollary 1.2 for the polynomials

$$f^{(d)}(\sigma)=\begin{cases}(\sigma(d)-\sigma)(K_{k-1}^{1,0}(\sigma,\sigma(d)))^2 & \text{if } d_k^{1,0}\leqslant d\leqslant d_{k-1}^{1,1}, \quad (1.36)\\ (1-\sigma)(\sigma(d)-\sigma)(K_{k-1}^{1,1}(\sigma,\sigma(d)))^2 & \text{if } d_k^{1,1}<d<d_k^{1,0}. \quad (1.37)\end{cases}$$

It was proved in [20–22] that $\Omega(f^{(d)})=B(d)$ for any d, $1\leqslant d\leqslant D$. On the other hand, Sidelnikov's result [30] can be reformulated [24] as follows: for any polynomial $f(\sigma)$ (see (1.22)) such that $f(\sigma)\leqslant 0$ for $\sigma(d)\leqslant\sigma\leqslant 1$, $f_0>0$ and the degree of $f(\sigma)$ does not exceed the degree of $f^{(d)}(\sigma)$, the inequality $\Omega(f)\geqslant B(d)$ holds (cf. (1.26)). The condition (1.25) for the polynomials (1.36) has been proved [20–22] for any (finite and infinite) polynomial metric spaces, and hence the bound (1.32) and its special cases (1.34) and (1.35) are true. But the condition (1.25) for the polynomials (1.37) and hence the bound (1.33) was proved in these works only for antipodal spaces. Later, the author found [23] a proof of the bound (1.33) for *all* infinite polynomial metric spaces. In [22] one can find the explicit forms of the above-mentioned bounds for many finite and infinite polynomial metric spaces, and numerous cases of their attainability.

It should be noted that, for $k=1$, (1.35) gives the upper bound

$$B=B(D)=1-\frac{1}{Q_1(1)} \tag{1.38}$$

for the maximal cardinality of a code W with minimal distance $d(W)=D(X)=D$. *Using this notation it is possible to show (see [24]) that the value*

$$B(d_{k-1}^{1,1})=\left(1-\frac{Q_{k-1}^{1,0}(1)}{Q_k(1)}\right)\sum_{i=0}^{k-1} r_i$$

(see (1.28), (1.31) and (1.35)) may also be expressed as

$$B(d_{k-1}^{1,1})=B\sum_{i=0}^{k-1} r_i^{0,1}.$$

Some of the geometrical meaning of this will be clear from the next section. Note that $B=2$ for antipodal spaces, $B=r$ for $\mathrm{H}(n, r)$, and $B=q^r$ for $\mathrm{H}(n, r, q)$.

The main incentive for this work was the desire to prove the bounds (1.31) and (1.33) for all finite polynomial metric spaces with the standard $\sigma(d)$. Later, we introduce a natural definition of decomposable polynomial metric spaces, and prove that for such spaces all the above-mentioned bounds and the conditions of their attainability are true. Many classical polynomial metric spaces, in particular, Hamming, Johnson and Grassman spaces, turn out to be decomposable. It is easy to see that our bounds for d-codes can be attained only for Delsarte codes. We prove the converse statement that all Delsarte codes in decomposable spaces are maximum. Furthermore, we give some statements about the connection between designs in decomposable spaces and designs in their projections. In conclusion, we formulate some open problems.

This paper is the extended lecture at the Conference 'Algebraic Combinatorics', held at Vladimir in August 1991. The limited extent of the paper does not allow us to give detailed proofs of all statements. Some of them will be be published in the longer paper [24], where analogous problems for a wide class of finite and infinite polynomial metric spaces are considered. Some of our results were announced earlier in [23].

2. Decomposable Polynomial Metric Spaces

A polynomial metric space X with the standard function $\sigma(d)$ is called *decomposable* if for some h there exist h metric subspaces $X_1, \ldots, X_h$ of X such that:

(i)
$$X=\bigcup_{i=1}^{h} X_i; \tag{2.1}$$

(ii) all the subspaces X_i are isometric to a single metric space $\tilde{X}$ which is polynomial with the same $\sigma(d)$;
(iii) for any $x, y \in X$ the number of subspaces $X_1, \ldots, X_h$ containing both x and y is equal to

$$(1-\sigma(d(x, y)))h\frac{|\tilde{X}|}{|X|}. \tag{2.2}$$

Let X be a decomposable metric space and let $X_1, \ldots, X_h$ be the subspaces mentioned in its definition. For any $W \subseteq X$ we say that $W \cap X_j$ is the *projection* of W onto X_j and, in particular, that X_j is projection of X (onto X_j). Notice that the space $\tilde{X}$ (and any X_j) is a polynomial metric space with respect to $\sigma(d)$, which is not standard for $\tilde{X}$, since from (1.13) and (2.2) it follows that

$$D(\tilde{X})=D(X)-1.$$

The parameters of the space $\tilde{X}$, which are analogous to the parameters k_i, r_i and $Q_i(\sigma)$, $i=0, 1, \ldots, D$, of the space X, are denoted by $\tilde{k}_i$, $\tilde{r}_i$ and $\tilde{Q}_i(\sigma)$, $i=0, 1, \ldots, D-1$, respectively.

THEOREM 2.1. *Let X be a decomposable polynomial metric space with diameter D.*

Then

$$\tilde{k}_d = (1-\sigma(d))k_d, \qquad d=0,1,\ldots,D-1; \tag{2.3}$$

$$|X| = B\,|\tilde{X}|; \tag{2.4}$$

$$\tilde{r}_i = r_i^{0,1}, \quad \tilde{Q}_i(\sigma) = Q_i^{0,1}(\sigma), \qquad i=0,1,\ldots,D-1. \tag{2.5}$$

Proof. It is clear that, for $d=0,1,\ldots,D-1$,

$$\sum_{j=1}^{h} |\{x,y\}: d(x,y)=d,\ x\in X_j,\ y\in X_j| = h\,|\tilde{X}|\,\tilde{k}_d.$$

We can calculate this sum in another way by multiplying (2.2), for $d(x,y)=d$, by $|X|\,k_d$. That gives (2.3). Using (2.3), the equality

$$Q_1(\sigma) = 1-(1-Q_1(1))\sigma, \tag{2.6}$$

the orthogonality condition (1.8) and the definition (1.38) of the number B, we have

$$|\tilde{X}| = \sum_{d=0}^{D-1} \tilde{k}_d = \sum_{d=0}^{D} (1-\sigma(d))k_d = \frac{|X|\,Q_1(1)}{Q_1(1)-1} = \frac{|X|}{B}, \tag{2.7}$$

which proves (2.4). From (2.7) and (1.14)–(1.16) it follows that $c^{0,1}=B$, and the parameters $r_i^{0,1}$ and $Q_i^{0,1}(\sigma)$, $i=0,1,\ldots,D-1$, are determined uniquely by the following orthogonality and normalization conditions:

$$\frac{r_i^{0,1}}{|\tilde{X}|}\sum_{d=0}^{D} Q_i^{0,1}(\sigma(d))Q_j^{0,1}(\sigma(d))(1-\sigma(d))k_d = \delta_{ij}, \qquad Q_i^{0,1}(0)=1.$$

But because of (2.3) these conditions determine uniquely the parameters $\tilde{r}_i$ and $\tilde{Q}_i(\sigma)$ as well, and (2.5) holds. □

Below we use the fact that for a decomposable space both orthogonal systems $\{Q_i(\sigma)\}$ and $\{Q_i^{0,1}(\sigma)\}$ have the 'positivity' property. Denote by $F[\sigma]$ the set of all polynomials of real variable σ. For every $g\in F[\sigma]$ there exists a unique polynomial $\tilde{g}(\sigma)=\sum_{i=0}^{l} g_i Q_i^{0,1}(\sigma)$ of some degree l, $0\leqslant l\leqslant D-1$, such that $g_l\neq 0$ if $g\neq 0$ and

$$\tilde{g}(\sigma)\equiv g(\sigma) \qquad (\mathrm{mod}\ Q_D^{0,1}(\sigma)).$$

Denote the set of all polynomials $g\in F[\sigma]$, for which all coefficients $g_0,\ldots,g_l$ of $\tilde{g}(t)$ are positive, by $F_+^{0,1}$. It follows from (1.11) and (1.12) (for the system $\{Q_i^{0,1}(\sigma\}$) that

$$g_1\cdot g_2\in F_+^{0,1} \qquad \text{if } g_1\in F_+^{0,1} \text{ and } g_2\in F_+^{0,1}. \tag{2.8}$$

Theorem 2.2. *For any decomposable polynomial metric space X the bounds* (1.32)–(1.35) *for d-codes and the conditions of their attainability are valid.*

Proof. As noted before, we only have to prove that the condition (1.25) is satisfied for the polynomials (1.37) of degree $t=2k$ with $k\leqslant D-1$. First, using (1.17) and (1.18) we have

$$K_{k-1}^{1,1}(s,\sigma) = \sum_{i=0}^{k-1} r_i^{1,1}Q_i^{1,1}(s)Q_i^{1,1}(\sigma),$$

$$(s-\sigma)K_{k-1}^{1,1}(s,\sigma) = r_{k-1}^{1,1}m_{k-1}^{1,1}(Q_k^{1,1}(s)Q_{k-1}^{1,1}(\sigma)-Q_{k-1}^{1,1}(s)Q_k^{1,1}(\sigma)),$$

where, in particular, $s=\sigma(d)$. It is known [22, (2.53)] that for $d_k^{1,1}<d<d_k^{1,0}$ the inequalities

$$Q_i^{1,1}(\sigma(d)>0, \quad i=0,1,\ldots,k-1, \qquad Q_k^{1,1}(\sigma(d))<0$$

hold. Since $Q_i^{1,1}(\sigma) \in F_+^{0,1}$, $i = 0, 1, \ldots, D-2$, by (1.20) and

$$Q_{D-1}^{1,1}(\sigma) = \frac{Q_D^{0,1}(\sigma)}{-\sigma} = (-r_{D-1}^{0,1} m_{D-1}^{0,1})^{-1} K_{D-1}^{0,1}(0, \sigma) \in F_+^{0,1}$$

due to (1.18), (1.21) and $Q_D^{0,1}(0) = 0$, it follows that

$$(\sigma(d) - \sigma) K_{k-1}^{1,1}(\sigma(d), \sigma) \in F_+^{0,1}, \qquad K_{k-1}^{1,1}(\sigma(d), \sigma) \in F_+^{0,1};$$

hence, by (2.8),

$$(\sigma(d) - \sigma)(K_{k-1}^{1,1}(\sigma(d), \sigma))^2 \in F_+^{0,1}.$$

Now note that, by (1.19), (1.18) and (1.21),

$$(1-\sigma) Q_i^{0,1}(\sigma) = \frac{Q_{i+1}(1) Q_i(\sigma) - Q_{i+1}(\sigma) Q_i(1)}{Q_{i+1}(1) - Q_i(1)}, \qquad i = 0, 1, \ldots, D-1,$$

$(1-\sigma) Q_D^{0,1}(\sigma) = Q_{D+1}(\sigma)$ and $\operatorname{sgn} Q_i(1) = (-1)^i$. This completes the proof. □

THEOREM 2.3. *Let W be a code in a decomposable metric space $X = \bigcup_{j=1}^h X_j$. If every projection $W_j = W \cap X_j$, $j = 1, \ldots, h$, is a $(t-1)$-design in X_j of cardinality $|W|/B$, then W is a t-design in X.*

PROOF. We use the following known statement (see (1.23)): a code W in a polynomial metric space X is a t-design iff, for every polynomial $f(\sigma)$ of degree at most t,

$$\frac{1}{|W|^2} \sum_{x, y \in W} f(\sigma(d(x, y))) = f_0, \qquad \text{where } f_0 = \frac{1}{|X|} \sum_{d=0}^{n} f(\sigma(d)) k_d.$$

Let $f(\sigma)$ be an arbitrary polynomial of degree at most t and

$$f(\sigma) = (1-\sigma) g(\sigma) + f(1). \tag{2.9}$$

Since W_j is a $(t-1)$-design in X_j and $|W_j|\, B = |W|$,

$$\frac{B^2}{|W|^2} \sum_{x, y \in W_j} g(\sigma(d(x, y))) = \frac{1}{|\bar{X}|} \sum_{d=0}^{D-1} g(\sigma(d)) \bar{k}_d. \tag{2.10}$$

Averaging (2.10) over all j, $j = 1, \ldots, h$, and using (2.2)–(2.4), we have

$$\frac{1}{|W|^2} \sum_{x, y \in W} (1 - \sigma(d(x, y))) g(\sigma(d(x, y))) = \frac{1}{|X|} \sum_{d=0}^{D} (1 - \sigma(d)) g(\sigma(d)) k_d.$$

This completes the proof due to (2.9). □

The author has not been able to find a proof of necessity of the theorem conditions. Note, however, that by the Roos theorem every projection W_j of a t-design W in a decomposable metric space $X = \bigcup_{j=1}^h X_j$ has size $|W|/B$, since by (2.3) each X_j is an antidesign with dual diameter 1 (see [27]).

THEOREM 2.4. *The classical polynomial metric spaces* J(v, n), J(v, n, q), S$_i(n, q)$, $i = 1, \ldots, 6$, H(n, r), H(n, r, q) *and* A$(2n, q)$ *of diameter n are decomposable.*

SKETCH OF PROOF. We can consider the above-mentioned spaces as spaces of mappings, namely: identity mappings of the n-subsets of a v-set for J(v, n), of the

n-dimensional subspaces of F_q^v for $J(v, n, q)$, and of the n-dimensional maximal isotropic subspaces of a non-singular bilinear form for $S_i(n, q)$; or mappings f: $\{1, \ldots, n\} \Rightarrow \{0, \ldots, r-1\}$ for $H(n, r)$, matrix transformations of F_q^r into F_q^n for $H(n, q, r)$, and alternating matrix transformations of F_q^N into F_q^N for $A(N, q)$. If we fix some element or some 1-dimensional space respectively from the domain of these mappings and consider all mappings which take some fixed value at this element, then we obtain some number h of metric subspaces; namely, $h = v$, $h = (q^v - 1)/(q-1)$, $h = (1 + \alpha^n)(\alpha^n - 1)/(\alpha - 1)$, $h = nr$, $h = q^r(q^n - 1)/(q-1)$ and $h = q^{N-1}(q^N - 1)/(q-1)$ in order of consideration. Using the transitivity of the isometry group of each initial space, it is easy to show that all these h subspaces are isometric to $J(v-1, n-1)$, $J(v-1, n-1, q)$, $S_i(n-1, q)$, $H(n-1, r)$, $H(n-1, r, q)$ or $A(N-1, q)$ respectively, which are polynomial with respect to the same function $\sigma(d)$. Then, using Delsarte [5, 6, 8] and Stanton [32, 33], we check that the condition (iii) (see (2.2)) of the definition of a decomposable space is satisfied for the above-mentioned spaces, except $A(N, q)$ for odd N.

THEOREM 2.5. *Let X be one of decomposable spaces $J(v, n)$, $J(v, n, q)$, $S_i(n, q)$, $i = 1, \ldots, 6$, $H(n, r)$ or $H(n, r, q)$. Then a code $W \subset X$ is a t-design in X iff every projection $W_j = W \cap X_j$, $j = 1, \ldots, h$, is a $(t-1)$-design in X_j of cardinality $|W|/B$.*

PROOF. In each case the space X can be embedded (Delsarte [6], Stanton [32, 34]) as nth level set of a ranked poset $P(X)$. Elements of the ith level of $P(X)$ are mappings, defined on i-subsets or i-dimensional subspaces depending on the space under consideration, and elements of $P(X)$ are ordered by the extension of mappings. The Delsarte–Stanton theorem tells that W is a t-design in X iff there exists a constant c such that for any element α of the tth level of $P(X)$ the equality

$$|\{x \in X : x \supset \alpha\}| = c\, |\{x \in W : x \supset \alpha\}| \tag{2.11}$$

holds; moreover, $c = |X|/|W|$ for a t-design W in X. If $X_1, \ldots, X_h$ are all projections of the decmposable space X, then from the proof of Theorem 2.4 it follows that the first level of $P(X)$ consists of exactly h elements $\alpha_1, \ldots, \alpha_h$ such that

$$X_j = \{x \in X : x \supset \alpha_j\}, \qquad j = 1, \ldots, h. \tag{2.12}$$

Furthermore, every element α of tth level of $P(X)$, such that $\alpha \supset \alpha_j$, can be considered as an element of the $(t-1)$-level of the poset $P(X_j)$. Therefore, the code $W_j = W \cap X_j$ is a $(t-1)$-design in X_j iff there exists a constant C such that for any element α of the tth level of $P(X)$ satisfying $\alpha \supset \alpha_j$, the equality

$$|\{x \in X_j : x \supset \alpha\}| = C\, |\{x \in W_j : x \supset \alpha\}| \tag{2.13}$$

holds; moreover, $C = |X_j|/|W_j|$ for a t-design W_j in X_j. Let W be a t-design in X. Then, for any X_j and any element α of the tth level of $P(X)$ satisfying $\alpha \supset \alpha_j$ from (2.11) and (2.12), (2.13) follows with $C = c = |X|/|W|$. This means that W_j is a $(t-1)$-design in X_j and $|W_j| = |W|\,|X_j|/|X| = |W|/B$. On the other hand, the conditions of the theorem are sufficient by Theorem 2.3. □

In the general case, we could only prove the following weaker statements.

THEOREM 2.6. *Let W be a diametrical code in a decomposable metric space $X = \bigcup_{j=1}^h X_j$ such that $l(W) = k$ and $f_W(\sigma) = (1-\sigma)g(\sigma)$, where $g \in F_+^{0,1}$ and $g(0) = 1$, and let t be an integer such that $k \leq t \leq 2k-1$. Then W is a t-design in X iff every projection $W \cap X_j$, $j = 1, \ldots, h$, is a $(t-1)$ design in X_j of cardinality $|W|/B$.*

COROLLARY 2.1. *A diametrical code W in a decomposable metric space $X = \bigcup_{j=1}^{h} X_j$ is a Delsarte code iff every projection $W \cap X_j$, $j = 1, \ldots, h$, is a non-diametrical Delsarte code in X_j of cardinality $|W|/B$.*

COROLLARY 2.2. *A code W in a decomposable metric space $X = \bigcup_{j=1}^{h} X_j$ is a tight $(2k+1)$-design iff every projection $W \cap X_j$, $j = 1, \ldots, h$, is a tight $2k$-design in X_j.*

Now, as an example, we give bounds for d-codes in the Hamming space $H(n, r)$, which are true by Theorems 2.2 and 2.4. In this case we have the diameter $D = n$, $k_i = r_i = \binom{n}{i}(r-1)^i$ and

$$Q_i(\sigma) = \frac{1}{r_i} P_i^{n,r}(\sigma n),$$

where

$$P_k^{n,r}(d) = \sum_{j=0}^{k} (-1)^j (r-1)^{k-j} \binom{d}{j}\binom{n-d}{k-j} \tag{2.14}$$

is the Krawtchouk polynomial of degree k. Using (1.14)–(1.20) and [22] it is possible to show that

$$c^{0,1} = r, \qquad c^{1,0} = \frac{r}{r-1}, \qquad c^{1,1} = \frac{nr^2}{(n-1)(r-1)},$$

$$r_k^{0,1} = \binom{n-1}{k}(r-1)^k, \qquad r_k^{1,0} = \frac{\left(\sum_{i=0}^{k}\binom{n}{i}(r-1)^i\right)^2}{\binom{n-1}{k}(r-1)^k},$$

$$r_k^{1,1} = \frac{\left(\sum_{i=0}^{k}\binom{n-1}{i}(r-1)^i\right)^2}{\binom{n-2}{k}(r-1)^k}, \qquad Q_k^{0,1}(\sigma) = \frac{P_k^{n-1,r}(\sigma n)}{\binom{n-1}{k}(r-1)^k},$$

$$Q_k^{1,0}(\sigma) = \frac{P_k^{n-1,r}(\sigma n - 1)}{\sum_{i=0}^{k}\binom{n}{i}(r-1)^i}, \qquad Q_k^{1,1}(\sigma) = \frac{P_k^{n-2,r}(\sigma n - 1)}{\sum_{i=0}^{k}\binom{n-1}{i}(r-1)^i},$$

and hence, denoting the smallest zero of (2.14) by $d_k(n, r)$ (or $d_k(n)$ for short), we have

$$d_k^{0,1} = d_k(n-1, r), \qquad d_k^{1,0} = d_k(n-1, r) + 1, \qquad d_k^{1,1} = d_k(n-2, r) + 1.$$

The bounds (1.32)–(1.35) take the following form. If $W \subset H(n, r)$ and $d = d(W) > 1$, then

$$|W| \leq B(d) = \begin{cases} \displaystyle\sum_{i=0}^{k-1}\binom{n}{i}(r-1)^i - \binom{n}{k}(r-1)^k \frac{P_{k-1}^{n-1,r}(d-1)}{P_k^{n,r}(d)} \\ \text{if } d_k(n-1) \leq d-1 \leq d_{k-1}(n-2), \\ \displaystyle r\left(\sum_{i=0}^{k-1}\binom{n-1}{i}(r-1)^i - \binom{n-1}{k}(r-1)^k \frac{P_{k-1}^{n-2,r}(d-1)}{P_k^{n-1,r}(d)}\right) \\ \text{if } d_k(n-2) < d-1 < d_k(n-1); \end{cases} \tag{2.15}$$

and, in particular,

$$|W| \leq B(d_k^{1,0}) = \sum_{i=0}^{k} \binom{n}{i}(r-1)^i \qquad \text{if } d \geq d_k(n-1, r)+1,$$

$$|W| \leq B(d_{k-1}^{1,1}) = r\sum_{i=0}^{k-1} \binom{n-1}{i}(r-1)^i \qquad \text{if } d \geq d_{k-1}(n-2, r)+1.$$

The bound (2.15) is attained iff the code W is a k-distance set, with the distances being the zeros x of the polynomial

$$(x-d)\sum_{i=0}^{k-1} \frac{P_i^{n-1,r}(d-1)P_i^{n-1,r}(x-1)}{\binom{n-1}{i}(r-1)^i}$$

if $d_k(n-1) \leq d-1 \leq d_{k-1}(n-2)$, or a $(k+1)$-distance set with the distances being the zeros x of the polynomial

$$(x-d)(x-n)\sum_{i=0}^{k-1} \frac{P_i^{n-2,r}(d-1)P_i^{n-2,r}(x-1)}{\binom{n-2}{i}(r-1)^i}$$

if $d_k(n-2) < d-1 < d_k(n-1)$, and the code W forms a $(2k-1)$-design if $d_k(n-1) < d-1 \leq d_{k-1}(n-2)$, or a $2k$-design if $d_k(n-2) < d-1 \leq d_k(n-1)$. The parameters of the known (to the author) codes for which the bounds (2.15) are attained, are given in Table 1.

In particular, $d_1(n) = (r-1)n/r$, $d_2(n) = (2(r-1)n - r + 2 - \sqrt{4(r-1)n + (r-2)^2})/2r$, and if $d_2(n-1) \leq d-1 \leq d_1(n-2)$, then

$$|W| \leq \frac{rd((n(r-1)+1)(n(r-1)-rd+2)-r)}{rd(2n(r-1)-r+2-rd)-n(n-1)(r-1)^2}; \tag{2.16}$$

moreover, the equality in (2.16) is attained iff W is a two-distance code with distances

$$d \qquad \text{and} \qquad \frac{(r-1)(n-1)}{r}\left(1+\frac{1}{(r-1)(n-1)-r(d-1)}\right)+1$$

and W forms a 3-design or a 4-design depending on whether $d > d_2(n-1)+1$ or $d = d_2(n-1)+1$. It is interesting that in the case $r = q$ the class of linear (n, k)-codes over F_q attaining the bound (2.16) coincides with the class of all projective $(n, k, n-w_1, n-w_2)$ caps, investigated by Calderbank [3] and Calderbank and Kantor [4].

Let

$$\delta_r(n, M) = \max_{W \subseteq H(n,r),\, |W| \geq M} d(W).$$

Since the function $B(d)$ is decreasing, $\delta_r(n, B(d)) \geq d$ for any d. As proved in [22] for fixed k and r,

$$d_k(n, r) = \frac{r-1}{r} n\left(1 - \sqrt{\frac{2}{n(r-1)}}\, h_k\right) + O(1)$$

as n tends to infinity, where h_k is the greatest zero of the Hermite polynomial $H_k(z)$ of degree k, $h_1 = 0$, $h_2 = 1/\sqrt{2}$, $h_3 = \sqrt{3}/\sqrt{2}$ [35]. This implies, for example, that

$$\delta_r\left(n, \sum_{i=0}^{k} \binom{n}{i}(r-1)^i\right) \leq \frac{r-1}{r} n\left(1 - \sqrt{\frac{2}{n(r-1)}}\, h_k\right) + O(1)$$

TABLE 1
The parameters of known Delsarte codes in Hamming space $H(n, r)$.

n	r	$l(W)$	$t(W)$	Tight or not	List of distances	$\|W\| = M$	Notes
n	r	1	1	T	$d = n$	r	
n	r	1	1		$\frac{(r-1)(n-1)}{r} + 1 < d < n$	$\frac{rd}{rd - n(r-1)}$	Coexistence [28] with resolvable block-designs $2-(M, M/r, n-d)$
n	r	1	2	T	$d = \frac{(r-1)(n-1)}{r} + 1$	$n(r-1)+1$	Coexistence [28] with affine resolvable block-designs $2-(M, M/r, n-d)$
$p^l q$	$q = p^m$	2	2		$n - p^l, n$	nq	$l, m = 1, 2, \ldots$; Semakov, Zinovév and Zaitsev [29]
$q(h-1)+h$	q	2	2		$n-h, n$	q^3	$2 \mid q, h \mid q$, and $2 < h < q$; Denniston [12]
q^2+1	q	2	3		$n-q-1, n-1$	q^4	Ovoid in $PG(3, q)$; see [4]
56	3	2	3		36, 45	3^6	Projective cap; Hill [15]
78	4	2	3		56, 64	4^6	Projective cap; Hill [16]
$4l$	2	2	3	T	$n/2, n$	$2n$	Hadamard codes
$q+2$	q	2	3	T	$n-2, n$	q^3	$2 \mid q$, hyperoval in $PG(2, q)$; see [4]
11	3	2	4	T	6, 9	3^5	Projection of Golay code
12	3	3	5	T	6, 9, 12	3^6	Golay code
22	2	3	5		8, 12, 16	2^{10}	Projection of Golay code
23	2	3	6	T	8, 12, 16	2^{11}	Projection of Golay code
24	2	4	7	T	8, 12, 16, 24	2^{12}	Golay code
n	2	$[n/2]$	$n-1$	T	All even	2^{n-1}	Even-weight code

in the asymptotical process under consideration. However, for fixed r the asymptotical behaviour of the value $(1/n)\log_r B(d)$ as n increases to infinity and d/n tends to a certain limit is the same as for the McEliece–Rodemich–Rumsey–Welch bound for d-codes [25].

3. Maximality of Delsarte Codes

We defined the Delsarte codes in (1.29) and the function $B(d)$ in (1.32) and (1.33). A proof of maximality of Delsarte codes is based on the following statement, which is true for any (finite or infinite) polynomial metric space X with a standard function $\sigma(d)$.

THEOREM 3.1. *Let $W \subset X$, $l(W) = k \geq 1$ and $d(W) = d$. Then*:
(i) $t(W) = 2k$ *iff*

$$d = d_k^{1,0}, \qquad |W| = B(d) \qquad \textit{and} \qquad f_W(\sigma) = Q_k^{1,0}(\sigma);$$

(ii) $t(W) = 2k - 1$ *iff*

$$d_k^{1,0} < d \leq d_{k-1}^{1,1}, \qquad |W| = B(d) \qquad \textit{and} \qquad f_W(\sigma) = (\sigma(d) - \sigma)K_{k-1}^{1,0}(\sigma, \sigma(d));$$

(iii) *for* $k \geq 2$, $t(W) = 2k - 2$ *and* $D(W) = D(X)$ *iff*

$$d_{k-1}^{1,1} < d < d_{k-1}^{1,0}, \qquad |W| = B(d) \qquad \textit{and} \qquad f_W(\sigma) = (1 - \sigma)(\sigma(d) - \sigma)K_{k-2}^{1,1}(\sigma, \sigma(d)).$$

The first part of this theorem is well known, and follows immediately from (1.27) and (1.30). For proof of the second and third parts we used in [24] the bounds (1.27) and (1.28) for t-designs, the absolute bound (1.30) (but not (1.31)), the bound (1.32) for d-codes (but not (1.33)) and Theorem 5.23 of Delsarte [5].

From the proof of Theorem 3.1 one can derive the following:

COROLLARY 3.1. *The annihilating polynomial $f_W(\sigma)$ for any Delsarte code W, normalized by the condition $f_W(0) = 1$, decomposes over the basis $\{Q_j(\sigma), j = 0, 1, \ldots, D\}$ with positive coefficients.*

It is interesting to note that from Theorem 3.1 and monotonicity of $B(d)$ it follows that:

COROLLARY 3.2. *Any of the values $d(W)$ or $|W|$ of a Delsarte code W in a polynomial metric space X uniquely determines the other value and also the values $l(W)$ and $t(W)$, the list of distances between code points and the parameters of the association scheme formed by the code W.*

THEOREM 3.2. *Any Delsarte code in a decomposable polynomial metric space is maximum.*

The theorem follows from Theorems 2.2 and 3.1. In the case of infinite polynomial metric spaces, all Delsarte codes are maximum [23, 24]. In the general case of finite polynomial metric spaces, from the above statements it only follows that a Delsarte code W is maximum if $t(W) \geq 2l(W) - 1$.

THEOREM 3.3. *For any decomposable polynomial metric space X the absolute bounds* (1.30) *and* (1.31) *and the conditions of their attainability are valid.*

PROOF. As was noted before, we have to prove only the bound (1.31) and conditions of its attainability. Let W be a diametrical code in a decomposable space X and let $l(W)=k$. Since the function $\sigma(d)$ is standard there exists an annihilating polynomial $f_W(\sigma)=(1-\sigma)f(\sigma)$, where $f(\sigma)$ is a polynomial of degree $k-1$ such that $f(0)=1$. From (2.2) and (2.4) it follows that every point $x\in X$ belongs to h/B projections among h ones $X_1,\ldots,X_h$ of the space X, and hence

$$h\,|W|=B\sum_{j=1}^{h}|W_j|, \tag{3.1}$$

where $W_j=W\cap X_j$ is the projection of W onto X_j, $j=1,\ldots,h$. Since $l(W_j)\leq k-1$, using Theorem 2.1 and absolute bound (1.30) for space X_j, we have that

$$|W_j|\leq\sum_{i=0}^{k-1}r_i^{0,1}; \tag{3.2}$$

moreover, if the bound (3.2) is attained then $f(\sigma)=\tilde{Q}_{k-1}^{1,0}(\sigma)$ and hence (see (1.20)) $f(\sigma)=Q_{k-1}^{1,1}(\sigma)$. From (3.1) and (3.2) it follows that

$$|W|\leq B\sum_{i=0}^{k-1}r_i^{0,1} \tag{3.3}$$

and if the bound (3.3) is attained then for any projection W_j the bound (3.2) is attained and $f_W(\sigma)=(1-\sigma)Q_{k-1}^{1,1}(\sigma)$. Using Theorem 3.1 for $d=d_{k-1}^{1,1}$ we obtain that $t(W)=2k-1$ if the bound (3.3) is attained. The sufficiency of the condition for attainability of (1.31) is a direct consequence of the bound (1.28) for $(2k-1)$-designs. □

In conclusion, we would like to formulate the following open problems.
(i) Describe all decomposable metric spaces.
(ii) Describe all Delsarte codes in any (finite and infinite) polynomial metric space. In Table 1 we give a list of the known Delsarte codes in the Hamming space $\mathrm{H}(n, r)$. Similar lists for some other polynomial metric spaces can be found in [24].
(iii) Prove the above-mentioned bounds for all finite polynomial metric spaces with the standard function $\sigma(d)$. In connection with the Bannai–Ito conjecture [1] on the structure of all P- and Q-polynomial association schemes, it should be noted that we proved in [24] that the bounds (1.31) and (1.35) hold for the folded space of any antipodal polynomial metric space of even diameter, and that they hold for 'halves' of any bipartite polynomial metric space of even diameter if its is true for the whole space.

REFERENCES

1. E. Bannai and T. Ito, *Algebraic Combinatorics I, Association Schemes*, Benjamin/Cummings, London, 1984.
2. A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-regular graphs*, Springer-Verlag, Berlin, 1989.
3. R. Calderbank, On uniformly packed $[n, n-k, 4]$ codes over $GF(q)$ and a class of caps in $PG(k-1, q)$, *J. Lond. Math. Soc.*, **26** (1982), 365–384.
4. R. Calderbank and W. M. Kantor, The geometry of two-weight codes, Bull. Lond. Math. Soc., **18** (1986), 97–122.
5. Ph. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Res. Rep.*, Suppl. 10 (1973).
6. Ph. Delsarte, Association schemes and t-designs in regular semilattices, *J. Combin. Theory, Ser. A*, **20** (1976), 230–243.
7. Ph. Delsarte, Bilinear forms over a finite field with application to coding theory, *J. Combin. Theory, Ser. A*, **25** (1978), 226–241.
8. Ph. Delsarte, Hahn polynomials, discrete harmonics, and t-designs, *SIAM J. Appl. Math.*, **34** (1978), 157–166.

9. Ph. Delsarte and J.-M. Goethals, Alternating bilinear forms over $GF(q)$, *J. Combin. Theory, Ser. A*, **19** (1975), 26–50.
10. Ph. Delsarte, J.-M. Goethals and J. J. Seidel, Bounds for systems of lines, and Jacobi polynomials, *Philips Res. Rep.*, **30** (1975), 91*–105*.
11. Ph. Delsarte, J.-M. Goethals and J. J. Seidel, Spherical codes and designs, *Geom. Ded.*, **6** (1977), 363–388.
12. R. H. F. Denniston, Some maximal arcs in finite projective planes, *J. Combin. Theory*, **6** (1969), 317–319.
13. C. F. Dunkl, Discrete quadrature and bounds on t-design, *Mich. Math. J.*, **26** (1979), 81–102.
14. Y. Egawa, Association schemes of quadratic forms, *J. Combin. Theory Ser. A*, **38** (1985), 1–14.
15. R. Hill, On the largest size cap in $S_{5,3}$, *Rend. Acad. Naz. Lincei* (8), **54** (1973), 378–384.
16. R. Hill, Caps and groups, in: *Atti dei Convegni Lincei, Colloquio Intern. sulle Theorie Combinatorie* (Rome 1973), no. 17, Acad. Naz. Lincei, 1976, pp. 384–394.
17. S. G. Hoggar, t-designs in projective spaces, *Europ. J. Combin.* **3** (1982), 233–254.
18. G. A. Kabatjansky and V. I. Levenshtein, Bounds for packings on a sphere and in space, *Problemy Peredachi Informacii*, **14,** N1 (1978), 3–25 (in Russian). English translation in *Problems of Information Transmission*, **14,** N1 (1978), 1–17.
19. D. A. Leonard, Orthogonal polynomials, duality and association schemes, *SIAM J. Math.*, **13** (1982), 656–663.
20. V. I. Levenshtein, On choosing polynomials to obtain bounds in packing problems, in: *Proceedings of the Seventh All-Union Conference on Coding Theory and Information Transmission, Part II*, Moscow–Vilnius, 1978, pp. 103–108 (in Russian).
21. V. I. Levenshtein, On bounds for packings in n-dimensional Euclidean space, *Dokl. Akad. Nauk SSSR*, **245,** N6 (1979), 1299–1303 (in Russian). English translation in *Sov. Math. Dokl.*, **20,** N2 (1979), 417–421.
22. V. I. Levenshtein, Bounds for packings of metric spaces and some of their applications, in: *Problemy Kiberneticki*, Issue 40, Moscow, Nauka, 1983, pp. 43–110 (in Russian).
23. V. I. Levenshtein, Packing of polynomial metric spaces, in *Proceedings of the Third International Workshop on Information Theory*, Sochi, 1987, pp. 271–274.
24. V. I. Levenshtein, Designs as maximal codes in polynomial metric spaces, *Acta Appl. Math.*, **29** (1992), 1–82.
25. R. J. McEliece, E. R. Rodemich, H. Rumsey Jr. and L. R. Welch, New upper bounds on the rate of a code via the Delsarte–MacWilliams inequalities, *IEEE Trans. Inform. Theory*, **IT-23** (1977), 157–166.
26. A. Neumaier, Combinatorial configurations in terms of distances, *Memorandum 81-09* (*Wiskunde*), Eindhoven University of Technology, 1981.
27. C. Roos, On antidesigns and designs in an association scheme, *Delft Progr. Rep.*, **7** (1982), 98–109.
28. N. V. Semakov and V. A. Zinov'ev, Equidistant q-ary codes and resolved balanced incomplete block designs, *Problemy Peredachi Informacii*, **4,** N2 (1968), 3–10 (in Russian). English translation in *Problems of Information Transmission*, **4,** N2 (1968), 000–000.
29. N. V. Semakov, V. A. Zinov'ev and G. V. Zaitsev, Class of maximal equidistant codes, *Problemy Peredachi Informacii*, **5,** N2 (1969), 84–87 (in Russian). English translation in *Problems of Information Transmission*, **5,** N2 (1969), 000–000.
30. V. M. Sidelnikov, On extremal polynomials used to estimate the size of codes, *Problemy Peredachi Informacii*, **16,** N3 (1980), 17–30 (in Russian). English translation in *Problems of Information Transmission*, **16,** N3 (1980), 174–186.
31. N. J. A. Sloane, Recent bounds for codes, sphere packing and related problems obtained by linear programming and others methods, *Contemp. Math.*, **9** (1982), 153–185.
32. D. Stanton, Some q-Krawtchouk polynomials on Chevalley groups, *Am. J. Math.*, **102** (1980), 625–662.
33. D. Stanton, A partially ordered set and q-Krawtchouk polynomials, *J. Combin. Theory, Ser. A*, **30** (1981), 276–284.
34. D. Stanton, t-designs in classical association schemes, *Graphs Combin.*, **2** (1986), 283–286.
35. G. Szego, *Orthogonal Polynomials*, American Mathematical Society, New York, 1959.
36. P. Terwilliger, A characterization of P- and Q-polynomial schemes, *J. Combin. Theory, Ser. A*, **45** (1987), 8–26.
37. H.-C. Wang, Two-point homogeneous spaces, *Ann. Math.*, **55** (1952), 177–191.

Received 26 December 1991 and in revised form 7 December 1992

V. I. Levenshtein

Keldysh Institute for Applied Mathematics,
Miusskaya sq. 4,
125 047, Moscow, Russia