An approach for solving fuzzy implicit variational inequalities with linear membership functions

Heng-you Lan

Department of Mathematics, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, People’s Republic of China

Received 31 January 2007; received in revised form 24 May 2007; accepted 7 June 2007

Abstract

In this paper, we consider a class of fuzzy implicit variational inequalities with linear membership functions. By using the “tolerance approach”, we show that solving such problems can be reduced to a semi-infinite programming problem. A version of the “method of centres” with “entropic regularization” techniques, only used a quasi-Newton line search using MATLAB software is required in our implementation. We also give a numerical example to illustrate the validity of our approach.

Keywords: Fuzzy implicit variational inequality; Linear membership functions; Fuzzy mathematical programming; Tolerance approach; Entropic regularization

1. Introduction and preliminaries

It is well known that there are many numerical methods for solving nonlinear semi-infinite programming problems in recent years. For example, based on a penalty function, Teo et al. [1] and Yang and Teo [2] devised a computational algorithm for solving a class of functional inequality constrained optimization problems and semi-infinite programs, respectively. By using a discretization method and an adaptive scheme, Teo et al. [3] studied a functional inequality constrained optimization problem. For more relaxed works, see, for example, [4–8] and the references therein.

On the other hand, variational inequalities have been widely used as a mathematical programming tool in modeling many optimization and decision making problems. However, facing uncertainty is a constant challenge for optimization and decision making. In 1989, Chang and Zhu [9] introduced the concepts of the variational inequalities for fuzzy mappings which were later developed by many authors (see, for example, [10–13] and the references therein).

Moreover, Inuiguchi et al. [14] considered solving fuzzy linear programming problems in view of fuzzy linear inequalities. Recently, Hu and Fang [15] studied a system of fuzzy inequalities with linear membership functions which can be converted to a regular convex programming problem. Fang and Hu [16] and Hu [17–19] introduced and studied some fuzzy variational inequalities in a fuzzy environment, and proved the existence of the optimal solution for the
fuzzy variational inequalities by using the tolerance approach and the entropic regularization technique. Especially, in [18], Hu showed that solving the fuzzy variational inequalities is equivalent to solving a fuzzy generalized complementarity problem. Wang and Liao [20,21] studied the variational inequalities with fuzzy convex cone and fuzzy resolution on the infeasibility of variational inequalities, respectively. For some related works, we refer to [22–25].

Inspired and motivated by the above research work on this subject, in this paper, we consider a class of fuzzy implicit variational inequalities with linear membership functions. By using the “tolerance approach”, we show that solving such problems can be reduced to semi-infinite programming problems. A version of the “method of centres” with “entropic regularization” techniques, only using a quasi-Newton line search implemented in MATLAB software is required in our implementation. We also give a numerical example to illustrate the validity of our approach.

Throughout this paper, let $\mathbb{R}^n$ be a $n$-dimension real numeral set. We consider the following fuzzy implicit variational inequalities: Find $x$ such that

$$\begin{cases}
  x \in U \\
  \langle A(x), y - h(x) \rangle \gtrapprox 0, \quad \forall y \in U,
\end{cases}$$

(1.1)

where $U \subset \mathbb{R}^n$ is a convex set, $A : U \rightarrow \mathbb{R}^n, h : U \rightarrow U$ are two mappings, $\langle A(x), y - h(x) \rangle \gtrapprox 0$ are fuzzy inequalities, for all $y \in U$, and “$\gtrapprox$” denotes the fuzzified version of “$\geq$” with the linguistic interpretation “approximately greater than or equal to”.

If $h = I$, the identity mapping, then the problem (1.1) is equivalent to find $x$ such that

$$\begin{cases}
  x \in U \\
  \langle A(x), y - x \rangle \gtrapprox 0, \quad \forall y \in U,
\end{cases}$$

(1.2)

which was studied by Hu [18].

More precisely, given $y \in U$, each fuzzy inequality $\langle A(x), y - h(x) \rangle \gtrapprox 0$ actually determines a fuzzy set $\Omega_y$ in $\mathbb{R}^n$, whose membership function is denoted by $\mu_{\Omega_y}(\cdot)$, such that for each $x \in \mathbb{R}^n$, $\mu_{\Omega_y}(x)$ is the degree to which the regular inequality $\langle A(x), y - h(x) \rangle \geq 0$ is satisfied. To specify the membership function $\mu_{\Omega_y}(\cdot)$, it is commonly assumed that $\mu_{\Omega_y}(x)$ should be 0 if the regular inequality $\langle A(x), y - h(x) \rangle \geq 0$ is strongly violated, and 1 if it is satisfied. This “tolerance approach” leads to a membership function in the following form:

$$
\mu_{\Omega_y}(x) = \begin{cases} 1, & \text{if } \langle A(x), y - h(x) \rangle > 0, \\
\mu_y(\langle A(x), y - h(x) \rangle), & \text{if } -t_y < \langle A(x), y - h(x) \rangle \leq 0, \\
0, & \text{if } \langle A(x), y - h(x) \rangle \leq -t_y,
\end{cases}
$$

where $t_y \geq 0$ is the tolerance level which a decision maker can tolerate in the accomplishment of the fuzzy inequality $\langle A(x), y - h(x) \rangle \geq 0$. We usually assume that $\mu_{\Omega_y}(\cdot) \in (0, 1]$ and it is continuous and strictly increasing over $[-t_y, 0]$. Fig. 1 shows different shapes of such membership functions.

One motivation to study such a system is related to finding “almost optimal” solutions for a general convex minimization problem. Consider the following problem:

$$\begin{align*}
\min_x & \quad T(x) \\
\text{s.t.} & \quad x \in K,
\end{align*}$$

(1.3)

where $T(\cdot)$ is a smooth real-value function defined on a convex set $K \subset \mathbb{R}^n$. Solving this problem is equivalent to solving the following variational inequalities (see [26]):

Find $x$ such that

$$\begin{cases}
  x \in K \\
  \langle \nabla T(x), y - x \rangle \geq 0, \quad \forall y \in K.
\end{cases}$$

(1.4)

To find an “almost optimal” solution for problem (1.3), we consider solving problem (1.4) with $\langle \nabla T(x), y - x \rangle \gtrapprox 0$, for all $y \in K$, i.e., $\langle \nabla T(x), y - x \rangle \gtrapprox 0, \forall y \in K$. It can be shown that a solution satisfying the corresponding fuzzy inequality system to a degree $\alpha$ close to 1 is a near optimal solution to problem (1.3) (see [17]).
To find a solution to the fuzzy variational inequalities (1.1), we define a fuzzy decision \( \tilde{D} \) of problem (1.1) as the fuzzy set resulting from the intersection of fuzzy sets \( \Omega_y \), for all \( y \in U \). By choosing the commonly used “minimum operator” for the fuzzy set intersections [27], we can define the membership function for \( \tilde{D} \) as

\[
\mu_{\tilde{D}}(x) = \min_{y \in U} \{ \mu_{\Omega_y}(x) \}.
\]

Therefore, a solution, say \( x \) to the fuzzy variational inequalities (1.1) with some degree \( \alpha, 0 \leq \alpha \leq 1 \), should satisfy that the inner product \( \langle A(x), y - h(x) \rangle \) is greater than or equal to zero to some degree \( \alpha \in [0, 1] \), for all \( y \in U \). In this case, the solution set of the fuzzy variational inequalities (1.1) is a fuzzy solution set. Assuming that we are not interested in a fuzzy solution set but in a crisp “optimal” solution we could suggest the “maximizing solution” to (1.1), which can be taken as the solution with highest membership in the fuzzy decision set \( \tilde{D} \) and obtained by solving the following problem (see [27]):

\[
\max_{x \in U} \mu_{\tilde{D}}(x),
\]

or equivalently,

\[
\max_{x \in U} \min_{y \in U} \{ \mu_{\Omega_y}(x) \}.
\]

Introducing one new variable \( \alpha \) results in an equivalent problem:

\[
\begin{align*}
\max_{x \in U} \quad & \alpha \\
\text{s.t.} \quad & \mu_{\Omega_y}(x) \geq \alpha, \quad \forall y \in U, \\
& x \in U, \\
& 0 \leq \alpha \leq 1.
\end{align*}
\]  

(1.5)

Notice that problem (1.5) is a semi-infinite programming problem (see [28]) with finitely many variables, \( x_1, x_2, \ldots, x_n, \alpha \), and infinitely many constraints. From the above procedure, we see that a system of fuzzy variational inequalities (1.1) can eventually be reduced to a regular semi-infinite programming problem (1.5).
Moreover, when $\mu_{\Omega}$ is invertible and its inverse function is increasing, we have

$$
\begin{align*}
\max \quad & \alpha \\
\text{s.t.} \quad & x \geq \mu_{\Omega}^{-1}(\alpha), \quad \forall y \in U, \\
& x \in U, \\
& 0 \leq \alpha \leq 1.
\end{align*}
$$

(1.6)

Let $g_i : \mathbb{R}^n \to \mathbb{R}$ for all $i = 1, 2, \ldots, m$, and $Dx = (g_i(x))_{m \times 1} = (d_i)_{m \times n} x$. Then

$$
U \triangleq \{ x \in \mathbb{R}^n | Dx \geq 0, D = (d_i)_{m \times n} \in \mathbb{R}^{m \times n} \}
$$

is a convex cone [20]. By Theorem 1 of [18], we know that the fuzzy variational inequalities (1.1) are equivalent to the following fuzzy implicit complementarity problem:

Find $x$ such that

$$
\begin{align*}
x \in U, \\
\langle A(x), h(x) \rangle \gtrless 0, \\
A(x) \gtrless U^*,
\end{align*}
$$

(1.7)

where “$\gtrless$” denotes the fuzzified version of “$=$” with the linguistic interpretation “approximately equal to”, “$\gtrsim$” denotes the fuzzified version of “$=$” with the linguistic interpretation “approximately in” and $U^* = \{ z \in \mathbb{R}^n | \langle z, y \rangle \geq 0, \forall y \in U \}$ is a polar (dual) cone of $U$ in $\mathbb{R}^n$. It can be shown that $A(x) \in U^*$ if and only if there exists a nonnegative vector $v = (v_1, v_2, \ldots, v_m)^T \in \mathbb{R}^m$ such that $A(x) = v_1 d_1^T + v_2 d_2^T + \cdots + v_m d_m^T = D^T v$, that is $d_j^T A(x) \geq 0$, where $d_j^T$ is normal to $d_j$ for all $i = 1, 2, \ldots, m$ (see [18]). Therefore, the fuzzy implicit complementarity problem (1.7) can be written as follows:

Find $x$ such that

$$
\begin{align*}
x \in U, \\
\langle A(x), h(x) \rangle \gtrsim 0, \\
d_j^T A(x) \gtrsim 0, \quad \forall j = 1, 2, \ldots, m,
\end{align*}
$$

or find $x$ such that

$$
\begin{align*}
g_i(x) \gtrsim 0, \quad \forall i = 1, 2, \ldots, m, \\
\langle A(x), h(x) \rangle \gtrsim 0, \\
\langle -A(x), h(x) \rangle \gtrsim 0, \\
d_j^T A(x) \gtrsim 0, \quad \forall j = 3, 4, \ldots, m + 2.
\end{align*}
$$

(1.8)

We see that problem (1.8) is a system of fuzzy inequalities. Each fuzzy inequality in (1.7) can be represented by a fuzzy set $\bar{\Delta}_j$ with corresponding membership function $\mu_{\bar{\Delta}_j}(x)$ for $j = 1, 2, \ldots, m + 2$. To specify the membership functions $\mu_{\bar{\Delta}_j}$, $j = 1, 2, \ldots, m + 2$, a similar treatment for defining the membership function $\mu_{\Omega}$ of the fuzzy inequality $\langle A(x), y - h(x) \rangle \gtrsim 0$ can be applied, that is

$$
\begin{align*}
\mu_{\bar{\Delta}_1}(x) &= \begin{cases} 
1, & \text{if } \langle A(x), h(x) \rangle > 0, \\
\mu_1(\langle A(x), h(x) \rangle), & \text{if } -t_1 < \langle A(x), h(x) \rangle \leq 0, \\
0, & \text{if } \langle A(x), h(x) \rangle \leq -t_1.
\end{cases}
\\
\mu_{\bar{\Delta}_2}(x) &= \begin{cases} 
1, & \text{if } \langle -A(x), h(x) \rangle > 0, \\
\mu_2(\langle -A(x), h(x) \rangle), & \text{if } -t_2 < \langle -A(x), h(x) \rangle \leq 0, \\
0, & \text{if } \langle -A(x), h(x) \rangle \leq -t_2.
\end{cases}
\\
\mu_{\bar{\Delta}_j}(x) &= \begin{cases} 
1, & \text{if } d_{j-2}^T A(x) > 0, \\
\mu_j(d_{j-2}^T A(x)), & \text{if } -t_j < d_{j-2}^T A(x) \leq 0, \\
0, & \text{if } d_{j-2}^T A(x) \leq -t_j.
\end{cases}
\end{align*}
$$
where \( t_j \geq 0 \) for \( j = 3, 4, \ldots, m + 2 \), is the tolerance level which a decision maker can tolerate in the accomplishment of the fuzzy inequalities in (1.7).

**Remark 1.1.** We can find a solution to (1.1) and (1.2) by the above approach.

2. Solution theorems

Consider the case that the membership function of each fuzzy variational inequality \( \langle A(x), y - h(x) \rangle \sim 0 \) in (1.1) is continuous, strictly increasing, and linear over the tolerance interval \([-t_y, 0]\). A commonly used example in fuzzy set theory is that \( f(x) = 1 - bx^\beta \) with \( b > 0 \) and \( \beta > 1 \). In this case, from the theory of convex analysis [29], we have the following simple result.

**Lemma 2.1.** If \( f(x) \) is continuous, strictly increasing and linear over a convex set \( \Omega \) in \( R^n \), then its inverse \( f^{-1} \) is linear.

**Theorem 2.1.** Let \( A : U \to R^n \) be a monotone mapping with respect to \( h \), where \( h : U \to U \) is a mapping. Suppose that \( A(x) \) and \( g_i(x) \) are linear for \( i = 1, 2, \ldots, m \) and for all \( y \in U \), \( \mu_{\Omega_y}(x) \) is continuous, strictly decreasing and linear. Then we can find a solution to the system of fuzzy variational inequalities (1.1) by solving the following programming problem:

\[
\begin{align*}
\max & \quad \alpha \\
\text{s.t.} & \quad \mu_{S_1}^{-1}(\alpha) - A(x)h(x) \geq 0, \\
& \quad \mu_{S_2}^{-1}(\alpha) + A(x)h(x) \geq 0, \\
& \quad -\mu_{S_j}^{-1}(\alpha) + d_{j-2}A(x) \geq 0, \quad j = 3, 4, \ldots, m + 2, \\
& \quad g_i(x) \geq 0, \quad i = 1, 2, \ldots, m, \\
& \quad 0 \leq \alpha \leq 1, \quad x \in R^n. 
\end{align*}
\]

(2.1)

**Proof.** Since \( A \) is a monotone mapping with respect to \( h \), i.e.,

\[ \langle A(x) - A(y), h(x) - h(y) \rangle \geq 0, \quad \forall x, y \in U, \]

the existence of solutions to (1.1) can be guaranteed, and it follows from (1.6) and Lemma 2.1 that our result can be found directly. □

Notice that problem (2.1) is a convex programming problem with variables \( x_1, x_2, \ldots, x_n \), which are confined by the first four sets of constraints, and \( \alpha \), which lies in between 0 and 1. Various methods can be applied to solve general convex programming problems (see, for example, [30]). Considering the structure of (2.1), we are interested in developing an efficient algorithm based on the framework of “method of centres”. This approach can be traced back to Huard’s work [31]. The basic concepts are easy to understand and very adaptive to new developments. To describe the approach, we denote the feasible domain of (2.1) by a set \( V \) and define some terminologies. A general assumption for this approach is that \( V \) is bounded and the interior of \( V \) is non-empty.

**Definition 2.1.** Given any point \( (x, \alpha) \) in the convex domain \( V \), we define the “distance \( L \) of \( (x, \alpha) \) to the boundary of \( V \)” by a continuous function

\[
L((x, \alpha), V) = \min_{i=1,2,\ldots,m, j=3,4,\ldots,m+2} \{ \mu_{S_1}^{-1}(\alpha) - A(x)h(x), \mu_{S_2}^{-1}(\alpha) + A(x)h(x), \\
-\mu_{S_j}^{-1}(\alpha) + d_{j-2}A(x), g_i(x), \alpha, 1 - \alpha \}.
\]

**Definition 2.2.** Given a convex domain \( V \) and a distance function \( L((x, \alpha), V) \) defined on the domain, we call a point \( (\tilde{x}, \tilde{\alpha}) \in V \) the “centre of \( V \)”, if it maximizes the distance function \( L((x, \alpha), V) \), i.e.,

\[
(\tilde{x}, \tilde{\alpha}); L((\tilde{x}, \tilde{\alpha}), V) = \max\{d((x, \alpha), V) | (x, \alpha) \in V \}.
\]
The basic idea of the “method of centres” could be described as an iterative method in terms of the transitions from a current iterate \((x^k, \alpha^k)\) to a new iterate \((x^{k+1}, \alpha^{k+1})\). Let \((x^k, \alpha^k)\) be a point of \(V\), we consider the convex domain \(W_k = V \cap \{(x, \alpha) | x \geq \alpha^k\}\). Then the new iterate \((x^{k+1}, \alpha^{k+1})\) is a solution of the centre of \(W_k\) and defined as

\[(x^{k+1}, \alpha^{k+1}) \in L((x^{k+1}, \alpha^{k+1}), W_k) = \max\{d((x, \alpha), W_k) | (x, \alpha) \in W_k\},\]

where

\[L((x, \alpha), W_k) = \min_{i=1, 2, \ldots, m+j} \min_{j=3, 4, \ldots, m+2} \{\alpha - \alpha^k, \mu_{\tilde{S}_1}(\alpha) - A(x)h(x), \mu_{\tilde{S}_2}(\alpha) + A(x)h(x),
- \mu_{\tilde{S}_j}(\alpha) + d_{j-2}^\prime A(x), g_i(x), \alpha, 1 - \alpha\}\]

is the distance function defined on the convex domain \(W_k\). We start working again with \((x^{k+1}, \alpha^{k+1})\) instead of \((x^k, \alpha^k)\). A sequence of points, \((x^k, \alpha^k)\), is thus obtained with the following properties (see [31]):

(a) The sequence of domains \(W_k\) satisfies

\[W_k' \subset W_k \subset V, \quad \forall k' > k.\]

Since \(V\) has a non-empty interior, all the domains \(W_k\) also have a non-empty interior, except the last one in cases where the sequence becomes finite.

(b) The value of \(\alpha\) is strictly increasing in every iteration.

(c) The sequence converges to an optimal solution of problem (2.1).

(d) The case of a finite sequence could occur only when the optimal solution belongs to the interior of the domain \(V\).

For the above framework, the major computational work lies in the determination of the centres required, i.e., at the \(k\)th iteration, we need to resolve the following nonlinear programming problem:

\[
\max_{x, \alpha} \min_{i=1, 2, \ldots, m+j} \min_{j=3, 4, \ldots, m+2} \{\alpha - \alpha^k, \mu_{\tilde{S}_1}(\alpha) - A(x)h(x), \mu_{\tilde{S}_2}(\alpha) + A(x)h(x),
- \mu_{\tilde{S}_j}(\alpha) + d_{j-2}^\prime A(x), g_i(x), \alpha, 1 - \alpha\},
\]

which is equivalent to the following “Min–max problem”:

\[-\min_{x, \alpha} L'(x, \alpha), W_k) = -\min_{x, \alpha} \max_{i=1, 2, \ldots, m+j} \min_{j=3, 4, \ldots, m+2} \{\alpha - \alpha^k, A(x)h(x) - \mu_{\tilde{S}_1}(\alpha),
- A(x)h(x) - \mu_{\tilde{S}_2}(\alpha),
- d_{j-2}^\prime A(x) + \mu_{\tilde{S}_j}(\alpha),
- g_i(x), -\alpha, \alpha - 1\}. \tag{2.2}\]

Again, there are many different algorithms for solving the above problem [32]; notice that \(A(x)h(x) - \mu_{\tilde{S}_1}(\alpha), - A(x)h(x) - \mu_{\tilde{S}_2}(\alpha), - d_{j-2}^\prime A(x) - \mu_{\tilde{S}_j}(\alpha), j = 3, 4, \ldots, m + 2, \) and \(- g_i(x)\) for \(i = 1, 2, \ldots, m\), are convex. However, they could be non-differentiable in general practice. To overcome this potential problem, we adopt the newly proposed “entropic regularization procedure” [33]. This procedure guarantees that, for an arbitrarily small \(\epsilon > 0\), an \(\epsilon\)-optimal solution of the “min–max” problem (2.2) can be obtained by the following unconstrained smooth convex program:

\[-\min_{x, \alpha} L_p(x, \alpha), W_k) = -\min_{x, \alpha} \frac{1}{p} \ln \left\{ \exp[p(\alpha^k - \alpha)] + \exp[p(A(x)h(x) - \mu_{\tilde{S}_1}(\alpha))] + \exp[p(-A(x)h(x) - \mu_{\tilde{S}_2}(\alpha))] + \sum_{j=3}^{m+2} \exp[p(-d_{j-2}^\prime A(x) + \mu_{\tilde{S}_j}(\alpha))] + \sum_{i=1}^{m} \exp[p(-g_i(x))] + \exp[p(-\alpha)] + \exp[p(\alpha - 1)] \right\}
\]

with a sufficiently large \(p\). In other words, \(\min_{x, \alpha} L_p(x, \alpha), W_k)\) provides a centre of \(W_k\), as \(p \to \infty\). It should be noted that in practice an accurate approximation can be obtained by using a moderately large \(p\). Also because of the special “log-exponential” form of \(L_p(x, \alpha), W_k)\), most overflow problems in computation can be avoided. Moreover, since it is an unconstrained, smooth, and convex optimization problem, the commonly used solution methods, such as the quasi-Newton line search of the MATLAB software, can be readily applied.
If \( h = I \), then from Theorem 2.1, we have the following result.

**Corollary 2.1.** Let \( A : U \rightarrow R^n \) be a monotone mapping. Suppose that \( A(x) \), \( g_i(x)(i = 1, 2, \ldots, m) \) and \( \mu_{\Omega}(x) \) are the same as in Theorem 2.1. Then we can find a solution to the system of fuzzy variational inequalities (1.2) by solving the following programming problem:

\[
\begin{align*}
\max & \quad \alpha \\
\text{s.t.} & \quad \mu_{S_1}^{-1}(\alpha) - A(x) \geq 0, \\
& \quad \mu_{S_2}^{-1}(\alpha) + A(x) \geq 0, \\
& \quad -\mu_{S_j}(\alpha) + d_jA(x) \geq 0, \quad j = 3, 4, \ldots, m + 2, \\
& \quad g_i(x) \geq 0, \quad i = 1, 2, \ldots, m, \\
& \quad 0 \leq \alpha \leq 1, \quad x \in R^n.
\end{align*}
\]

3. An algorithm with a numerical example

Based on the concepts discussed in the previous section, here we propose a “method of centres with entropic regularization techniques” for finding a solution to the system of fuzzy inequalities (1.1). The inputs of the proposed algorithm include the initial iterate \((x^0, \alpha^0)\) which is an interior point of \( V \) defined by (2.1), a sufficiently small constant \( \epsilon > 0 \), and an upper bound \( Q \) which is the maximum number of unconstrained minimizations to be performed.

**Algorithm 3.1.** Step 1. Set \( m = 0 \).

Step 2. Starting from \((x^m, \alpha^m)\), apply a standard quasi-Newton line search of the MATLAB software to solve the unconstrained smooth convex program (2.2) with a sufficiently large \( p \). Denote its solution by \((x^{m+1}, \alpha^{m+1})\).

Step 3. If \( m > 1 \) and \( \| (x^{m+1}, \alpha^{m+1}) - (x^m, \alpha^m) \|_2 \leq \epsilon \), then the algorithm terminates with \((x^{m+1}, \alpha^{m+1})\) as the solution. If \( m > Q \), then the algorithm terminates with a failure.

Step 4. \( m \leftarrow m + 1 \) and go to Step 2.

**Numerical Example 3.1.** To illustrate the validity of our approach, we consider the fuzzy variational inequalities (1.1) with

\[
g_1(x) = x_1 \geq 0, \quad g_2(x) = x_2 \geq 0, \quad g_3(x) = x_3 \geq 0, \quad g_4(x) = x_4 \geq 0,
\]

\[
h(x) = \begin{pmatrix}
2x_1 \\
x_2 + 1 \\
3x_3 - 2 \\
-x_4 + 1
\end{pmatrix}, \quad A(x) = \begin{pmatrix}
-4x_1 - 5x_2 - 9x_3 - 11x_4 + 111.57 \\
x_1 + x_2 + x_3 + x_4 - 15 \\
7x_1 + 5x_2 + 3x_3 + 2x_4 - 80 \\
3x_1 + 5x_2 + 10x_3 + 15x_4 - 100
\end{pmatrix}
\]

and

\[
d_1 = (1, 0, 0, 0), \quad d_2 = (0, 1, 0, 0), \quad d_3 = (0, 0, 1, 0), \quad d_4 = (0, 0, 0, 1),
\]

which is equivalent to the following fuzzy generalized complementarity problem:

Find \( x \in R^4 \) such that

\[
g_1(x) = x_1 \geq 0, \quad g_2(x) = x_2 \geq 0, \quad g_3(x) = x_3 \geq 0, \quad g_4(x) = x_4 \geq 0,
\]

\[
f_1(x) = \langle A(x), h(x) \rangle = -8x_1^2 + x_2^2 + 9x_3^2 - 15x_4^2 - 9x_1x_2 + 3x_1x_3 - 25x_1x_4 + 16x_2x_3
\]

\[
-4x_2x_4 - 4x_3x_4 + 213.14x_1 - 19x_2 - 235x_3 + 112x_4 + 45 \geq 0,
\]

\[
f_2(x) = \langle -A(x), h(x) \rangle = 8x_1^2 - x_2^2 - 9x_3^2 + 15x_4^2 + 9x_1x_2 - 3x_1x_3 + 25x_1x_4 - 16x_2x_3
\]

\[
+ 4x_2x_4 + 4x_3x_4 - 213.14x_1 + 19x_2 + 235x_3 - 112x_4 - 45 \geq 0,
\]

\[
f_3(x) = d_1^tA(x) = 11x_1 + 11x_2 + 14x_3 + 18x_4 - 195 \geq 0,
\]

\[
f_4(x) = d_2^tA(x) = -8x_1 - 5x_2 - 2x_3 + 2x_4 + 91.57 \geq 0,
\]
Applying Bellman and Zadeh’s method of fuzzy decision making with the membership function \( \tilde{\mu}_j(x) \), \( j = 1, 2, \ldots, 6 \), being specified as

\[
\begin{align*}
\mu_{\tilde{\xi}_1}(x) &= \begin{cases} 
1, & \text{if } f_1(x) > 0, \\
1 - \frac{f_1(x)}{5}, & \text{if } -5 < f_1(x) \leq 0, \\
0, & \text{if } f_1(x) \leq -5,
\end{cases} \\
\mu_{\tilde{\xi}_2}(x) &= \begin{cases} 
1, & \text{if } f_2(x) > 0, \\
1 - \frac{f_2(x)}{17}, & \text{if } -17 < f_2(x) \leq 0, \\
0, & \text{if } f_2(x) \leq -17,
\end{cases} \\
\mu_{\tilde{\xi}_3}(x) &= \begin{cases} 
1, & \text{if } f_3(x) > 0, \\
1 - f_3(x), & \text{if } -1 < f_3(x) \leq 0, \\
0, & \text{if } f_3(x) \leq -1,
\end{cases} \\
\mu_{\tilde{\xi}_4}(x) &= \begin{cases} 
1, & \text{if } f_4(x) > 0, \\
1 - \frac{f_4(x)}{3}, & \text{if } -3 < f_4(x) \leq 0, \\
0, & \text{if } f_4(x) \leq -3,
\end{cases} \\
\mu_{\tilde{\xi}_5}(x) &= \begin{cases} 
1, & \text{if } f_5(x) > 0, \\
1 - \frac{f_5(x)}{8}, & \text{if } -8 < f_5(x) \leq 0, \\
0, & \text{if } f_5(x) \leq -8,
\end{cases} \\
\mu_{\tilde{\xi}_6}(x) &= \begin{cases} 
1, & \text{if } f_6(x) > 0, \\
1 - \frac{f_6(x)}{10}, & \text{if } -10 < f_6(x) \leq 0, \\
0, & \text{if } f_6(x) \leq -10.
\end{cases}
\end{align*}
\]

Applying Bellman and Zadeh’s method of fuzzy decision making [34], the maximizing solution \( x^* \) of this problem, is given by solving the following nonlinear convex programming problem:

\[
\begin{align*}
\max & \quad \alpha \\
\text{s.t.} & \quad 5(1 - \alpha) + 8x_1^2 - x_2^2 - 9x_3^2 + 15x_4^2 + 9x_1x_2 - 3x_1x_3 + 25x_1x_4 - 16x_2x_3 \\
& \quad + 4x_2x_4 + 4x_3x_4 - 213.14x_1 + 19x_2 + 235x_3 - 112x_4 - 45 \geq 0, \\
& \quad 17(1 - \alpha) - 8x_1^2 + x_2^2 + 9x_3^2 - 15x_4^2 - 9x_1x_2 + 3x_1x_3 - 25x_1x_4 + 16x_2x_3 \\
& \quad - 4x_2x_4 - 4x_3x_4 + 213.14x_1 - 19x_2 - 235x_3 + 112x_4 + 45 \geq 0, \\
& \quad -1(1 - \alpha) + 11x_1 + 11x_2 + 14x_3 + 18x_4 - 195 \geq 0, \\
& \quad -3(1 - \alpha) - 8x_1 - 5x_2 - 2x_3 + 2x_4 + 91.57 \geq 0, \\
& \quad -8(1 - \alpha) - x_2 - 2x_3 - 5x_4 + 3.47 \geq 0, \\
& \quad -10(1 - \alpha) - 12x_1 - 11x_2 - 13x_3 - 14x_4 + 206.57 \geq 0, \\
& \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, 0 \leq \alpha \leq 1.
\end{align*}
\]

Using the proposed Algorithm 3.1 to solve this problem (3.1), at the kth iteration, we consider the following nonlinear programming problem:

\[
\begin{align*}
\max & \quad \min_{\alpha} \{ \alpha - \alpha^k, \alpha, 1 - \alpha, 5(1 - \alpha) + 8x_1^2 - x_2^2 - 9x_3^2 + 15x_4^2 + 9x_1x_2 - 3x_1x_3 \\
& \quad + 25x_1x_4 - 16x_2x_3 + 4x_2x_4 + 4x_3x_4 - 213.14x_1 + 19x_2 + 235x_3 - 112x_4 - 45, \\
& \quad 17(1 - \alpha) - 8x_1^2 + x_2^2 + 9x_3^2 - 15x_4^2 - 9x_1x_2 + 3x_1x_3 - 25x_1x_4 + 16x_2x_3 \\
& \quad - 4x_2x_4 - 4x_3x_4 + 213.14x_1 - 19x_2 - 235x_3 + 112x_4 + 45, \\
& \quad -1(1 - \alpha) + 11x_1 + 11x_2 + 14x_3 + 18x_4 - 195, \\
& \quad -3(1 - \alpha) - 8x_1 - 5x_2 - 2x_3 + 2x_4 + 91.57, \\
& \quad -8(1 - \alpha) - x_2 - 2x_3 - 5x_4 + 3.47, \\
& \quad -10(1 - \alpha) - 12x_1 - 11x_2 - 13x_3 - 14x_4 + 206.57, x_1, x_2, x_3, x_4 \}. 
\end{align*}
\]
This problem is equivalent to the following “min–max” problem:

\[
\begin{align*}
\min_{\alpha, x} & \quad \{\alpha^k - \alpha, -\alpha, \alpha - 1, \\
& -8x_1^2 + x_2^2 + 9x_3^2 - 15x_4^2 - 9x_1x_2 + 3x_1x_3 - 25x_1x_4 + 16x_2x_3 - 4x_2x_4 \\
& - 4x_3x_4 + 213.14x_1 - 19x_2 - 235x_3 + 112x_4 + 45 - 5(1 - \alpha), \\
& 8x_1^2 - x_2^2 - 9x_3^2 + 15x_4^2 + 9x_1x_2 - 3x_1x_3 + 25x_1x_4 - 16x_2x_3 + 4x_2x_4 \\
& + 4x_3x_4 - 213.14x_1 + 19x_2 + 235x_3 - 112x_4 - 45 - 17(1 - \alpha), \\
& -11x_1 - 11x_2 - 14x_3 - 18x_4 + 195 + (1 - \alpha), 8x_1 + 5x_2 + 2x_3 - 2x_4 - 91.57 + 3(1 - \alpha), \\
& x_2 + 2x_3 + 5x_4 - 3.47 + 8(1 - \alpha), 12x_1 + 11x_2 + 13x_3 + 14x_4 - 206.57 + 10(1 - \alpha), \\
& -x_1, -x_2, -x_3, -x_4 \}. 
\end{align*}
\]

An \(\epsilon\)-optimal solution of the “min–max” problem (3.2) can be obtained by solving an unconstrained and smooth nonlinear programming problem:

\[
\begin{align*}
& \min_{\alpha, x} \quad \frac{1}{p} \ln[p(\alpha^k - \alpha) + \exp[p(-\alpha)] + \exp[p(\alpha - 1)] \\
& + \exp[p(-8x_1^2 + x_2^2 + 9x_3^2 - 15x_4^2 - 9x_1x_2 + 3x_1x_3 - 25x_1x_4 + 16x_2x_3 \\
& - 4x_2x_4 - 4x_3x_4 + 213.14x_1 - 19x_2 - 235x_3 + 112x_4 + 45 - 5(1 - \alpha))] \\
& + \exp[p(8x_1^2 - x_2^2 - 9x_3^2 + 15x_4^2 + 9x_1x_2 - 3x_1x_3 + 25x_1x_4 - 16x_2x_3 \\
& + 4x_2x_4 + 4x_3x_4 - 213.14x_1 + 19x_2 + 235x_3 - 112x_4 - 45 - 17(1 - \alpha))] \\
& + \exp[p(-11x_1 - 11x_2 - 14x_3 - 18x_4 + 195 + (1 - \alpha))] \\
& + \exp[p(8x_1 + 5x_2 + 2x_3 - 2x_4 - 91.57 + 3(1 - \alpha))] \\
& + \exp[p(x_2 + 2x_3 + 5x_4 - 3.47 + 8(1 - \alpha))] \\
& + \exp[p((12x_1 + 11x_2 + 13x_3 + 14x_4 - 206.57 + 10(1 - \alpha))] \\
& + \exp[p(-x_1)] + \exp[p(-x_2)] + \exp[p(-x_3)] + \exp[p(-x_4)] \\
& \text{with } p \text{ being sufficiently large.}
\end{align*}
\]

Setting \(\epsilon = 10^{-5}\) and fixed \(p = 1000\) for each iteration, we solve problem (3.3) by the commonly used the quasi-Newton line search of the MATLAB software and obtain \(x^* = (6.0294, 0.6062, 6.7981, 1.7246)\) with membership degree \(\alpha^* = 0.8915\). Computational results for this problem are listed in Table 1.

Acknowledgments
The authors are grateful to the referees for their valuable comments and suggestions.

References