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# On solvability of functional equations and system of functional equations arising in dynamic programming

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**Abstract**

The purpose of this paper is to study solvability of two classes of functional equations and a class of system of functional equations arising in dynamic programming of multistage decision processes. By using fixed point theorems, a few existence and uniqueness theorems of solutions and iterative approximation for solving these classes of functional equations are established. Under certain conditions, some existence theorems of coincidence solutions for the class of system of functional equations are shown. Some examples are given to demonstrate the advantage of our results than existing ones in the literature.


*Keywords:* Dynamic programming; Functional equation; System of functional equations; Coincidence solutions; Fixed point theorems; Iterative approximation; Nonexpansive mapping

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## 1. Introduction

This paper deals with solvability of functional equations and system of functional equations arising in dynamic programming of multistage decision processes as follows:

$$f(x) = \operatorname*{opt}_{y\in D}\bigl\{u(x,y) + \operatorname{opt}\bigl\{p_i(x,y) + A_i\bigl(x,y,f\bigl(a_i(x,y)\bigr)\bigr)\colon\ i=1,2\bigr\}\bigr\},\quad \forall x\in S, \tag{1.1}$$

$$f(x) = \operatorname*{opt}_{y\in D}\bigl\{u(x,y) + \operatorname{opt}\bigl\{p_i(x,y) + q_i(x,y)f\bigl(a_i(x,y)\bigr)\colon\ i=1,2\bigr\}\bigr\},\quad \forall x\in S, \tag{1.2}$$

and

$$\begin{aligned} f(x) &= \operatorname*{opt}_{y\in D}\bigl\{u_1(x,y) + \operatorname{opt}\bigl\{A_1\bigl(x,y,g\bigl(a_1(x,y)\bigr)\bigr), B_1\bigl(x,y,g\bigl(b_1(x,y)\bigr)\bigr)\bigr\}\bigr\},\quad \forall x\in S,\\ g(x) &= \operatorname*{opt}_{y\in D}\bigl\{u_2(x,y) + \operatorname{opt}\bigl\{A_2\bigl(x,y,f\bigl(a_2(x,y)\bigr)\bigr), B_2\bigl(x,y,f\bigl(b_2(x,y)\bigr)\bigr)\bigr\}\bigr\},\quad \forall x\in S, \end{aligned} \tag{1.3}$$

where opt denotes the sup or inf, $x$ and $y$ stand for the state and decision vectors, respectively, $a_1$, $a_2$, $b_1$, and $b_2$ represent the transformations of the process, $f(x)$ and $g(x)$ denote the optimal return functions with initial state $x$.

It is clear that (1.1)–(1.3) include many functional equations and systems of functional equations as special cases, respectively. For instance, the following functional equations

$$f(x) = \inf_{y\in D}\max\bigl\{p(x,y), q(x,y)f\bigl(a(x,y)\bigr)\bigr\},\quad \forall x\in S, \tag{1.4}$$

$$f(x) = \inf_{y\in D}\max\bigl\{p(x,y), f\bigl(a(x,y)\bigr)\bigr\},\quad \forall x\in S, \tag{1.5}$$

$$f(x) = \sup_{y\in D} H\bigl(x,y,f\bigl(a(x,y)\bigr)\bigr),\quad \forall x\in S, \tag{1.6}$$

$$f(x) = \max_{y\in S(x)}\bigl\{p(x,y) + h(x,y)f\bigl(a(x,y)\bigr)\bigr\}, \tag{1.7}$$

$$f(x) = \sup_{y\in D}\bigl\{p(x,y) + f\bigl(a(x,y)\bigr)\bigr\},\quad \forall x\in S, \tag{1.8}$$

$$f(x) = \sup_{y\in D}\bigl\{p(x,y) + A\bigl(x,y,f\bigl(a(x,y)\bigr)\bigr)\bigr\},\quad \forall x\in S, \tag{1.9}$$

$$f(x) = \inf_{y\in D} A(x,y,f),\quad \forall x\in S, \tag{1.10}$$

studied by Bellman [3,4], Bellman and Lee [5], Bellman and Roosta [6], Bhakta and Choudhury [7], Bhakta and Mitra [8], Liu [12], Liu and Ume [13], and others [1,2], respectively; the systems of functional equations

$$\begin{cases} f(x) = \sup_{y\in D}\{u(x,y) + G(x,y,g(a(x,y)))\},\quad \forall x\in S,\\ g(x) = \sup_{y\in D}\{u(x,y) + F(x,y,f(a(x,y)))\},\quad \forall x\in S, \end{cases} \tag{1.11}$$

$$\begin{cases} f(x) = \inf_{y \in D}\{v(x,y) + G(x,y,g(a(x,y)))\}, & \forall x \in S, \\ g(x) = \sup_{y \in D}\{u(x,y) + F(x,y,f(b(x,y)))\}, & \forall x \in S, \end{cases} \tag{1.12}$$

investigated by Chang [10], Chang and Ma [11], and Liu [12], respectively. By using fixed point theorems, we establish the existence and uniqueness of solutions and iterative approximation for the functional equations (1.1) and (1.2). Under suitable conditions, we investigate the properties of solutions for the functional equation (1.2) and construct an iterative algorithm for solving this class of functional equation. We also obtain the sufficient conditions which guarantee the existence of coincidence solutions for the system of functional equations (1.3). The results presented in this paper generalize, improve, and unify a number of results due to Bellman [3], Bhakta and Choudhury [7], Bhakta and Mitra [8], Chang [10], Liu [12], and Liu and Ume [13]. Several examples which dwell upon the importance of our results are also included.

## 2. Preliminaries

In this section, we introduce notations, definitions, and a result that will be used in the remainder of the paper.

Let $R = (-\infty, +\infty)$, $R^+ = [0, +\infty)$, $R^- = (-\infty, 0]$, and $I$ denote the identity mapping. For any $t \in R$, $[t]$ stands for the largest integer not exceeding $t$. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ be real Banach spaces, let $S \subseteq X$ be the state space, let $D \subseteq Y$ be the decision space. Let $B(S)$ be the set of real-valued bounded functions on $S$, and $BB(S)$ denote the set of all real-valued functions on $S$ that are bounded on bounded subsets of $S$. According to ordinary addition of functions and scalar multiplication and endowing a norm $\|f\|_1 = \sup_{x \in S} |f(x)|$ for $f \in B(S)$, then $(B(S), \|\cdot\|_1)$ is a Banach space. For any $k \geqslant 1$ and $f, g \in BB(S)$, let

$$d_k(f,g) = \sup\bigl\{\bigl|f(x) - g(x)\bigr|:\ x \in \bar{B}(0,k)\bigr\},$$

$$d(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(f,g)}{1 + d_k(f,g)},$$

where $\bar{B}(0,k) = \{x:\ x \in S \text{ and } \|x\| \leqslant k\}$. Then $\{d_k\}_{k \geqslant 1}$ is a countable family of pseudometrics on $BB(S)$. A sequence $\{x_n\}_{n \geqslant 1}$ in $BB(S)$ is said to *converge* to a point $x$ in $BB(S)$ if for any $k \geqslant 1$, $d_k(x_n, x) \to 0$ as $n \to \infty$, and to be a *Cauchy sequence* if for any $k \geqslant 1$, $d_k(x_n, x_m) \to 0$ as $n, m \to \infty$. It is clear that $(BB(S), d)$ is a complete metric space. A metric space $(M, \rho)$ is said to be *metrically convex* if for each $x, y \in M$, there is a $z \neq x, y$ for which $\rho(x,y) = \rho(x,z) + \rho(z,y)$. Clearly any Banach space is metrically convex. Define

$$\Phi_1 = \left\{(\varphi, \psi):\ \varphi \text{ and } \psi : R^+ \to R^+ \text{are nondecreasing and } \sum_{n=0}^{\infty} \psi\bigl(\varphi^n(t)\bigr) < \infty \text{ for all } t > 0\right\},$$

$$\Phi_2 = \bigl\{(\varphi, \psi):\ (\varphi, \psi) \in \Phi_1 \text{ and } \psi(t) > 0 \text{ for all } t > 0\bigr\},$$

$$\Phi_3 = \bigl\{\varphi:\ \varphi : R^+ \to R^+ \text{ satisfies } \varphi(t) < t \text{ for all } t > 0\bigr\}.$$

**Lemma 2.1** [9]. *Suppose that $(M, \rho)$ is a completely metrically convex metric space and that $f : M \to M$ satisfies*

$$\rho\big(f(x), f(y)\big) \leqslant \varphi\big(\rho(x, y)\big), \quad \forall x, y \in M,$$

*where $\varphi : \bar{P} \to R^+$ satisfies $\varphi(t) < t$ for all $t \in \bar{P} - \{0\}$, $P = \{d(x, y)\colon x, y \in M\}$ and $\bar{P}$ denotes the closure of $P$. Then $f$ has a unique fixed point $u \in M$ and $\lim_{n\to\infty} f^n(x) = u$ for each $x \in M$.*

$\bar{f}$ and $\bar{g}$ are called *coincidence solutions of the system* of functional equations (1.3) if

$$\bar{f}(x) = \operatorname*{opt}_{y\in D}\big\{u_1(x, y) + \operatorname{opt}\big\{A_1\big(x, y, \bar{g}\big(a_1(x, y)\big)\big), B_1\big(x, y, \bar{g}\big(b_1(x, y)\big)\big)\big\}\big\},$$
$$\forall x \in S,$$
$$\bar{g}(x) = \operatorname*{opt}_{y\in D}\big\{u_2(x, y) + \operatorname{opt}\big\{A_2\big(x, y, \bar{f}\big(a_2(x, y)\big)\big), B_2\big(x, y, \bar{f}\big(b_2(x, y)\big)\big)\big\}\big\},$$
$$\forall x \in S.$$

## 3. Existence of coincidence solutions

**Theorem 3.1.** *Let $a_i, b_i : S \times D \to S$, $u_i : S \times D \to R$, and $A_i, B_i : S \times D \times R \to R$ be mappings for $i = 1, 2$, and let $(\varphi, \psi)$ be in $\Phi_1$ satisfying the following conditions*:

(C1) $\max\{|u_i(x, y)|\colon i = 1, 2\} \leqslant \psi(\|x\|)$ *for all* $(x, y) \in S \times D$;
(C2) $\max\{\|a_i(x, y)\|, \|b_i(x, y)\|\colon i = 1, 2\} \leqslant \varphi(\|x\|)$ *for all* $(x, y) \in S \times D$;
(C3) $0 \leqslant \operatorname{opt}\{A_2(x, y, z), B_2(x, y, z)\}$ *and* $\max\{|A_i(x, y, z)|, |B_i(x, y, z)|\colon i = 1, 2\} \leqslant |z|$ *for all* $(x, y, z) \in S \times D \times R$;
(C4) *given $(x, y) \in S \times D$, $A_i(x, y, \cdot)$ and $B_i(x, y, \cdot)$ are both left continuous and nondecreasing with respect to the third argument on $R$ for $i = 1, 2$.*

*Then the system of functional equations*

$$f(x) = \sup_{y\in D}\big\{u_1(x, y) + \operatorname{opt}\big\{A_1\big(x, y, g\big(a_1(x, y)\big)\big), B_1\big(x, y, g\big(b_1(x, y)\big)\big)\big\}\big\},$$
$$\forall x \in S,$$
$$g(x) = \sup_{y\in D}\big\{u_2(x, y) + \operatorname{opt}\big\{A_2\big(x, y, f\big(a_2(x, y)\big)\big), B_2\big(x, y, f\big(b_2(x, y)\big)\big)\big\}\big\},$$
$$\forall x \in S, \tag{3.1}$$

*possesses coincidence solutions in $BB(S)$.*

**Proof.** For any $x \in S$, put

$$g_0(x) = \sup_{y\in D} u_2(x, y),$$

$$g_{2n}(x) = \sup_{y \in D}\bigl\{u_2(x,y) + \operatorname{opt}\bigl\{A_2\bigl(x,y,f_{2n-1}\bigl(a_2(x,y)\bigr)\bigr),\ B_2\bigl(x,y,f_{2n-1}\bigl(b_2(x,y)\bigr)\bigr)\bigr\}\bigr\}, \quad n \geqslant 1,$$
$$f_{2n+1}(x) = \sup_{y \in D}\bigl\{u_1(x,y) + \operatorname{opt}\bigl\{A_1\bigl(x,y,g_{2n}\bigl(a_1(x,y)\bigr)\bigr),\ B_1\bigl(x,y,g_{2n}\bigl(b_1(x,y)\bigr)\bigr)\bigr\}\bigr\}, \quad n \geqslant 0.$$

In terms of (C3) and (C4) we know that for each $x \in S$,

$$g_0(x) \leqslant g_2(x) \leqslant \cdots \leqslant g_{2n}(x) \leqslant g_{2n+2}(x) \leqslant \cdots,$$
$$f_1(x) \leqslant f_3(x) \leqslant \cdots \leqslant f_{2n-1}(x) \leqslant f_{2n+1}(x) \leqslant \cdots. \tag{3.2}$$

Let $x$ be in $S$ and $k = [\|x\|] + 1$. By virtue of (C1)–(C3), we derive that

$$\bigl|g_0(x)\bigr| \leqslant \sup_{y \in D}\bigl|u_2(x,y)\bigr| \leqslant \psi\bigl(\|x\|\bigr)$$

and

$$\begin{aligned}\bigl|f_1(x)\bigr| &\leqslant \sup_{y \in D}\bigl\{\bigl|u_1(x,y)\bigr| + \max\bigl\{\bigl|A_1\bigl(x,y,g_0\bigl(a_1(x,y)\bigr)\bigr)\bigr|,\ \bigl|B_1\bigl(x,y,g_0\bigl(b_1(x,y)\bigr)\bigr)\bigr|\bigr\}\bigr\}\\
&\leqslant \sup_{y \in D}\bigl\{\psi\bigl(\|x\|\bigr) + \max\bigl\{\bigl|g_0\bigl(a_1(x,y)\bigr)\bigr|, \bigl|g_0\bigl(b_1(x,y)\bigr)\bigr|\bigr\}\bigr\}\\
&\leqslant \sup_{y \in D}\bigl\{\psi\bigl(\|x\|\bigr) + \max\bigl\{\psi\bigl(\bigl\|a_1(x,y)\bigr\|\bigr), \psi\bigl(\bigl\|b_1(x,y)\bigr\|\bigr)\bigr\}\bigr\}\\
&\leqslant \sum_{i=0}^{1}\psi\bigl(\varphi^i\bigl(\|x\|\bigr)\bigr).\end{aligned}$$

Similarly we conclude that for each $n \geqslant 0$,

$$\bigl|g_{2n}(x)\bigr| \leqslant \sum_{i=0}^{2n}\psi\bigl(\varphi^i\bigl(\|x\|\bigr)\bigr) \leqslant \sum_{i=0}^{\infty}\psi\bigl(\varphi^i(k)\bigr)$$

and

$$\bigl|f_{2n+1}(x)\bigr| \leqslant \sum_{i=0}^{2n+1}\psi\bigl(\varphi^i\bigl(\|x\|\bigr)\bigr) \leqslant \sum_{i=0}^{\infty}\psi\bigl(\varphi^i(k)\bigr),$$

which imply that $g_{2n}$ and $f_{2n+1}$ are bounded on $\bar{B}(0,k)$ for each $n \geqslant 0$. From (3.2) we conclude that there exist $f, g \in BB(S)$ such that

$$\lim_{n\to\infty} g_{2n}(x) = g(x), \qquad \lim_{n\to\infty} f_{2n+1}(x) = f(x), \quad x \in \bar{B}(0,k) \tag{3.3}$$

and

$$\max\bigl\{\bigl|g(x)\bigr|, \bigl|f(x)\bigr|\bigr\} \leqslant \sum_{i=0}^{\infty}\psi\bigl(\varphi^i(k)\bigr), \quad x \in \bar{B}(0,k).$$

Set

$$\begin{aligned}
&M(x)=\sup_{y\in D}\bigl\{u_1(x,y)+\operatorname{opt}\bigl\{A_1\bigl(x,y,g\bigl(a_1(x,y)\bigr)\bigr),B_1\bigl(x,y,g\bigl(b_1(x,y)\bigr)\bigr)\bigr\}\bigr\},\\
&\quad x\in S,\\
&N(x)=\sup_{y\in D}\bigl\{u_2(x,y)+\operatorname{opt}\bigl\{A_2\bigl(x,y,f\bigl(a_2(x,y)\bigr)\bigr),B_2\bigl(x,y,f\bigl(b_2(x,y)\bigr)\bigr)\bigr\}\bigr\},\\
&\quad x\in S.
\end{aligned}\tag{3.4}$$

It follows from (3.2) and (3.4) that for any $(x,y)\in S\times D$ and $n\geqslant 0$,

$$\begin{aligned}
&u_1(x,y)+\operatorname{opt}\bigl\{A_1\bigl(x,y,g_{2n}\bigl(a_1(x,y)\bigr)\bigr),B_1\bigl(x,y,g_{2n}\bigl(b_1(x,y)\bigr)\bigr)\bigr\}\\
&\quad\leqslant f_{2n+1}(x)\leqslant M(x),\\
&u_2(x,y)+\operatorname{opt}\bigl\{A_2\bigl(x,y,f_{2n-1}\bigl(a_2(x,y)\bigr)\bigr),B_2\bigl(x,y,f_{2n-1}\bigl(b_2(x,y)\bigr)\bigr)\bigr\}\\
&\quad\leqslant g_{2n}(x)\leqslant N(x).
\end{aligned}\tag{3.5}$$

Letting $n\to\infty$ in (3.5), by (C4), (3.2), and (3.3) we see that for any $(x,y)\in S\times D$,

$$\begin{aligned}
&u_1(x,y)+\operatorname{opt}\bigl\{A_1\bigl(x,y,g\bigl(a_1(x,y)\bigr)\bigr),B_1\bigl(x,y,g\bigl(b_1(x,y)\bigr)\bigr)\bigr\}\leqslant f(x)\leqslant M(x),\\
&u_2(x,y)+\operatorname{opt}\bigl\{A_2\bigl(x,y,f\bigl(a_2(x,y)\bigr)\bigr),B_2\bigl(x,y,f\bigl(b_2(x,y)\bigr)\bigr)\bigr\}\leqslant g(x)\leqslant N(x),
\end{aligned}$$

which yield that

$$M(x)\leqslant f(x)\leqslant M(x),\qquad N(x)\leqslant g(x)\leqslant N(x),\quad x\in S.$$

That is, $M(x)=f(x)$ and $N(x)=g(x)$ for $x\in S$. Therefore, $f$ and $g$ are coincidence solutions of the system of functional equations (3.1). This completes the proof. □

By similar arguments as in the proof of Theorem 3.1, we have the following results and their proofs are thus omitted.

**Theorem 3.2.** *Let $a_i,b_i:S\times D\to S$, $u_i:S\times D\to R$, and $A_i,B_i:S\times D\times R\to R$ be mappings for $i=1,2$, and let $(\varphi,\psi)$ be in $\Phi_1$ satisfying conditions* (C1)–(C3) *and*

(C5) *given $(x,y)\in S\times D$, $A_1(x,y,\cdot)$ and $B_1(x,y,\cdot)$ are left continuous and nondecreasing with respect to the third argument on $R$, and $A_2(x,y,\cdot)$ and $B_2(x,y,\cdot)$ are right continuous and nondecreasing with respect to the third argument on $R$.*

*Then the system of functional equations*

$$\begin{aligned}
&f(x)=\inf_{y\in D}\bigl\{u_1(x,y)+\operatorname{opt}\bigl\{A_1\bigl(x,y,g\bigl(a_1(x,y)\bigr)\bigr),B_1\bigl(x,y,g\bigl(b_1(x,y)\bigr)\bigr)\bigr\}\bigr\},\\
&\quad\forall x\in S,\\
&g(x)=\sup_{y\in D}\bigl\{u_2(x,y)+\operatorname{opt}\bigl\{A_2\bigl(x,y,f\bigl(a_2(x,y)\bigr)\bigr),B_2\bigl(x,y,f\bigl(b_2(x,y)\bigr)\bigr)\bigr\}\bigr\},\\
&\quad\forall x\in S,
\end{aligned}\tag{3.6}$$

*possesses coincidence solutions in $BB(S)$.*

**Theorem 3.3.** *Let $a_i, b_i : S \times D \to S$, $u_i : S \times D \to R$, and $A_i, B_i : S \times D \times R \to R$ be mappings for $i = 1, 2$, and let $(\varphi, \psi)$ be in $\Phi_1$ satisfying conditions* (C1), (C2) *and*

(C6) $\operatorname{opt}\{A_2(x, y, z), B_2(x, y, z)\} \leqslant 0$ *and* $\max\{|A_i(x, y, z)|, |B_i(x, y, z)|\colon i = 1, 2\} \leqslant |z|$ *for all* $(x, y, z) \in S \times D \times R$;
(C7) *given* $(x, y) \in S \times D$, $A_i(x, y, \cdot)$ *and* $B_i(x, y, \cdot)$ *are right continuous and nondecreasing with respect to the third argument on* $R$ *for* $i = 1, 2$.

*Then the system of functional equations*

$$\begin{aligned} f(x) &= \inf_{y \in D}\bigl\{u_1(x, y) + \operatorname{opt}\bigl\{A_1\bigl(x, y, g\bigl(a_1(x, y)\bigr)\bigr), B_1\bigl(x, y, g\bigl(b_1(x, y)\bigr)\bigr)\bigr\}\bigr\}, \\ &\quad \forall x \in S, \\ g(x) &= \inf_{y \in D}\bigl\{u_2(x, y) + \operatorname{opt}\bigl\{A_2\bigl(x, y, f\bigl(a_2(x, y)\bigr)\bigr), B_2\bigl(x, y, f\bigl(b_2(x, y)\bigr)\bigr)\bigr\}\bigr\}, \\ &\quad \forall x \in S. \end{aligned} \tag{3.7}$$

**Remark 3.1.**

(a) In case $\psi = I$, $A_i = B_i$, and $a_i = b_i$ for $i = 1, 2$, then Theorem 3.1 reduces Theorem 4.2 of Liu [12], which, in turn, generalizes Theorem 2.3 of Bhakta and Mitra [8] and Theorem 4 of Chang [10].
(b) If $\psi = I$, $A_i = B_i$, and $a_i = b_i$ for $i = 1, 2$, then Theorem 3.2 reduces to Theorem 4.1 of Liu [12].

The following example demonstrates that Theorems 3.1 and 3.2 extend properly the results due to Bhakta and Mitra [8], Chang [10], and Liu [12].

**Example 3.1.** Let $X = Y = R$ and $S = D = R^+$. Define $a_i, b_i : S \times D \to S$, $u_i$: $S \times D \to R$, $A_i, B_i : S \times D \times R \to R$ for $i = 1, 2$, and $\varphi, \psi : R^+ \to R^+$ by

$$a_1(x, y) = \begin{cases} \frac{x \sin(1 - x - y^2)}{2 + x + y^x} & \text{for } x^2 + y^2 < 1, \\ \frac{x}{1 + x^2 + y^2} & \text{for } x^2 + y^2 \geqslant 1, \end{cases}$$

$$b_1(x, y) = \begin{cases} \frac{\sin x}{2 + \ln(1 + xy)} & \text{for } x^2 + y^2 < 1, \\ \frac{x \cos(x - y)}{1 + x^2 + y^2} & \text{for } x^2 + y^2 \geqslant 1, \end{cases}$$

$$a_2(x, y) = \frac{x}{3 + \sin(x^2 y - 2)}, \qquad b_2(x, y) = \frac{x}{2 + xy^2},$$

$$u_1(x, y) = \frac{x^2}{1 + xy}, \qquad u_2(x, y) = -\frac{x^3}{1 + x + y},$$

$$A_1(x, y, z) = \begin{cases} \frac{z}{1 + x^2 + y} & \text{for } z \leqslant 0, \\ \frac{z|\sin(x^2 - 2y^2)|}{z + \ln(1 + xy)} & \text{for } z > 0, \end{cases}$$

$$B_1(x,y,z)=\begin{cases}\frac{z}{2+\sin(x-y^2)} & \text{for } z\leqslant 0,\\ \frac{z|\cos(x^2-2y^2)|}{z+2^{xy}} & \text{for } z>0,\end{cases}$$
$$A_2(x,y,z)=\begin{cases}0 & \text{for } z\leqslant 0,\\ \frac{z}{1+xy+z} & \text{for } z>0,\end{cases}\qquad B_2(x,y,z)=\begin{cases}0 & \text{for } z\leqslant 0,\\ \frac{z}{2^{x+y}+z} & \text{for } z>0,\end{cases}$$

where $(x,y)\in S\times D$, and

$$\psi(t)=t^2,\qquad \varphi(t)=\frac{1}{2}t\quad \text{for } t\in R^+.$$

It is easy to verify that the assumptions of Theorems 3.1 and 3.2 are satisfied. It follows from Theorems 3.1 and 3.2 that the functional equations (3.1) and (3.6) possess coincidence solutions in $BB(S)$, respectively. However, [8, Theorem 2.3], [10, Theorem 4.1], and [12, Theorems 4.1 and 4.2] are not applicable for the functional equations (3.1) and (3.6) because

$$\max\{|u_1(x,y)|,|u_2(x,y)|\}\leqslant\|x\|$$

does not hold for $(x,y)=(3,1/3)\in S\times D$.

## 4. Existence and uniqueness of solutions

**Theorem 4.1.** *Let $a_i: S\times D\to S$, $u, p_i: S\times D\to R$, and $A_i: S\times D\times R\to R$ be mappings for $i=1,2$, and let $\varphi$ be in $\Phi_3$ satisfying*

(D1) *$u$, $p_i$, and $A_i$ are bounded for $i=1,2$;*
(D2)

$$\max\{|A_i(x,y,s(a_i(x,y)))-A_i(x,y,t(a_i(x,y)))|:\ i=1,2\}\leqslant\varphi(\|s-t\|_1),$$
$$\forall(x,y,s,t)\in S\times D\times B(S)\times B(S).$$

*Then the functional equation* (1.1) *possesses a unique solution $w\in B(S)$ and $\{H^n z\}_{n\geqslant 1}$ converges to $w$ for each $z\in B(S)$, where*

$$Hz(x)=\operatorname*{opt}_{y\in D} C(x,y,z),\quad \forall(x,z)\in S\times B(S),$$
$$C(x,y,z)=u(x,y)+\operatorname{opt}\{p_i(x,y)+A_i(x,y,z(a_i(x,y))):\ i=1,2\},$$
$$\forall(x,y,z)\in S\times D\times B(S).$$

**Proof.** It follows from (D1) that $H$ is a mapping from $B(S)$ into itself. It is easy to see that $H$ has a unique fixed point $w\in B(S)$ if and only if the functional equation (1.1) possesses a unique solution $w\in B(S)$. Given $\varepsilon>0$, $x\in S$, and $s,t\in B(S)$, if $\operatorname{opt}_{y\in D}=\sup_{y\in D}$, then there exist $y,h\in D$ such that

$$Hs(x)<C(x,y,s)+\varepsilon,\qquad Ht(x)<C(x,h,t)+\varepsilon,$$
$$Hs(x)\geqslant C(x,h,s),\qquad Ht(x)\geqslant C(x,y,t),$$

which imply that

$$\begin{aligned}\big|Hs(x)-Ht(x)\big| &< \max\big\{\big|C(x,y,s)-C(x,y,t)\big|,\big|C(x,h,s)-C(x,h,t)\big|\big\}+\varepsilon\\ &= \max\big\{\big|\mathrm{opt}\big\{p_i(x,y)+A_i\big(x,y,s\big(a_i(x,y)\big)\big)\colon\ i=1,2\big\}\\ &\qquad -\mathrm{opt}\big\{p_i(x,y)+A_i\big(x,y,t\big(a_i(x,y)\big)\big)\colon\ i=1,2\big\}\big|,\\ &\qquad \big|\mathrm{opt}\big\{p_i(x,h)+A_i\big(x,h,s\big(a_i(x,h)\big)\big)\colon\ i=1,2\big\}\\ &\qquad -\mathrm{opt}\big\{p_i(x,h)+A_i\big(x,h,t\big(a_i(x,h)\big)\big)\colon\ i=1,2\big\}\big|\big\}+\varepsilon\\ &\leqslant \max\big\{\big|A_i\big(x,y,s\big(a_i(x,y)\big)\big)-A_i\big(x,y,t\big(a_i(x,y)\big)\big)\big|,\\ &\qquad \big|A_i\big(x,h,s\big(a_i(x,h)\big)\big)-A_i\big(x,h,t\big(a_i(x,h)\big)\big)\big|\colon\ i=1,2\big\}\\ &\quad +\varepsilon\leqslant\varphi\big(\|s-t\|_1\big)+\varepsilon,\end{aligned}$$

which gives that

$$\|Hs-Ht\|_1\leqslant\varphi\big(\|s-t\|_1\big)+\varepsilon; \tag{4.1}$$

if $\mathrm{opt}_{y\in D}=\inf_{y\in D}$, similarly we conclude that (4.1) holds also. Letting $\varepsilon\to 0$ in (4.1), we have

$$\|Hs-Ht\|_1\leqslant\varphi\big(\|s-t\|_1\big).$$

It follows from Lemma 2.1 that $H$ has a unique fixed point $w\in B(S)$ and $\{H^n z\}_{n\geqslant 1}$ converges to $w$ for each $z\in B(S)$. This completes the proof. □

**Remark 4.1.** If $a_1=a_2$, $A_1=A_2$, and $p_1=p_2=0$, then Theorem 4.1 reduces to a result which generalizes Theorem 2.1 of Bhakta and Mitra [8]. The example below shows that Theorem 4.1 is indeed an extension of the result due to Bhakta and Mitra [8].

**Example 4.1.** Let $X=Y=R$, $S=R^-$, and $D=R^+$. Define $\varphi: R^+\to R^+$ by $\varphi(t)=\frac{1}{2}t$ for all $t\geqslant 0$. Then Theorem 4.1 ensures that the following functional equation

$$\begin{aligned}f(x)=\mathop{\mathrm{opt}}_{y\in D}\Bigg\{&2\sin\big(x-y^2\big)+\mathrm{opt}\Bigg\{\cos(x-2y+1)\\ &+\frac{1}{1+|x|+yx^2}\cdot\frac{1}{1+|f(3\sin(x^2-y^2))|},\\ &\frac{1}{1+\ln(1+x^2y)}+\frac{x}{1+x^2+y}\cdot\sin\big(f(2xy)\big)\Bigg\}\Bigg\},\quad \forall x\in S,\end{aligned} \tag{4.2}$$

possesses a unique solution in $B(S)$. But [8, Theorem 2.1] is not valid for the functional equation (4.2).

**Theorem 4.2.** *Let $a_i: S\times D\to S$, $u$, $p_i$, and $q_i: S\times D\to R$ be mappings for $i=1,2$. Suppose that the following conditions hold*:

(D3) *$u$, $p_1$, and $p_2$ are bounded on $\bar{B}(0,k)\times D$ for each $k\geqslant 1$*;
(D4) $\max\{\|a_i(x,y)\|\colon\ i=1,2\}\leqslant\|x\|$ *for all $(x,y)\in S\times D$*;
(D5) *there exists a constant $r$ such that*

$$\max\{|q_i(x,y)|:\ i=1,2\} \leqslant r < 1, \quad \forall (x,y) \in S \times D.$$

*Then the functional equation* (1.2) *possesses a unique solution* $w \in BB(S)$ *and* $\{H^n z\}_{n\geqslant 1}$ *converges to* $w$ *for any* $z \in BB(S)$, *where* $H$ *is defined by*

$$Hz(x) = \operatorname*{opt}_{y\in D} C(x,y,z), \quad \forall (x,z) \in S \times BB(S), \tag{4.3}$$

$$C(x,y,z) = u(x,y) + \operatorname{opt}\{p_i(x,y) + q_i(x,y)z(a_i(x,y)):\ i=1,2\},$$
$$\forall (x,y,z) \in S \times D \times BB(S). \tag{4.4}$$

**Proof.** In terms of (D3) and (D4), we deduce that for each $k \geqslant 1$ and $g \in BB(S)$, there exist $\alpha(k) > 0$ and $\beta(k,g) > 0$ such that

$$\max\{|u(x,y)|, |p_i(x,y)|:\ i=1,2\} \leqslant \alpha(k), \quad \forall (x,y) \in \bar{B}(0,k) \times D$$

and

$$\max\{|g(a_i(x,y))|:\ i=1,2\} \leqslant \beta(k,g), \quad \forall (x,y) \in \bar{B}(0,k) \times D.$$

It follows from (D5) and the above inequalities for each $k \geqslant 1$, $z \in BB(S)$, and $(x,y) \in \bar{B}(0,k) \times D$,

$$\begin{aligned} |C(x,y,z)| &\leqslant |u(x,y)| + \max\{|p_i(x,y)| + |q_i(x,y)||z(a_i(x,y))|:\ i=1,2\} \\ &\leqslant 2\alpha(k) + \beta(k,z), \end{aligned}$$

which means that $H$ is a mapping from $BB(S)$ into itself. Given $\varepsilon > 0$, $k \geqslant 1$, $x \in \bar{B}(0,k)$, and $g, h \in BB(S)$. Suppose that $\operatorname{opt}_{y\in D} = \inf_{y\in D}$. Then there exist $s, t \in D$ such that

$$\begin{aligned} &Hg(x) > C(x,s,g) - \varepsilon, \qquad Hh(x) > C(x,t,h) - \varepsilon, \\ &Hg(x) \leqslant C(x,t,g), \qquad Hh(x) \leqslant C(x,s,h). \end{aligned} \tag{4.5}$$

On account of (4.3)–(4.5), (D4) and (D5), we know that

$$\begin{aligned} &|Hg(x) - Hh(x)| \\ &\quad < \max\{|C(x,s,g) - C(x,s,h)|, |C(x,t,g) - C(x,t,h)|\} + \varepsilon \\ &\quad \leqslant \max\{|\operatorname{opt}\{p_i(x,s) + q_i(x,s)g(a_i(x,s)):\ i=1,2\} \\ &\qquad\qquad - \operatorname{opt}\{p_i(x,s) + q_i(x,s)h(a_i(x,s)):\ i=1,2\}|, \\ &\qquad\qquad |\operatorname{opt}\{p_i(x,t) + q_i(x,t)g(a_i(x,t)):\ i=1,2\} \\ &\qquad\qquad - \operatorname{opt}\{p_i(x,t) + q_i(x,t)h(a_i(x,t)):\ i=1,2\}|\} + \varepsilon \\ &\quad \leqslant \max\{|q_i(x,s)||g(a_i(x,s)) - h(a_i(x,s))|, \\ &\qquad\qquad |q_i(x,t)||g(a_i(x,t)) - h(a_i(x,t))|:\ i=1,2\} + \varepsilon \\ &\quad \leqslant \max\{|q_i(x,s)|, |q_i(x,t)|:\ i=1,2\} \max\{|g(a_i(x,s)) - h(a_i(x,s))|, \\ &\qquad\qquad |g(a_i(x,t)) - h(a_i(x,t))|:\ i=1,2\} + \varepsilon \\ &\quad \leqslant r d_k(g,h) + \varepsilon, \end{aligned}$$

which gives that

$$d_k(Hg, Hh) \leqslant r d_k(g, h) + \varepsilon.$$

Similarly we can conclude the above inequality for $\mathrm{opt}_{y\in D} = \sup_{y\in D}$. Letting $\varepsilon \to 0$, we immediately obtain that

$$d_k(Hg, Hh) \leqslant r d_k(g, h) = \varphi\big(d_k(g, h)\big),$$

where $\varphi(t) = rt$ for each $t \geqslant 0$. Thus [7, Theorem 2.2] guarantees that $H$ has a unique fixed point $w \in BB(S)$ and $\{H^n z\}_{n\geqslant 1}$ converges to $w$ for any $z \in BB(S)$. Consequently, $w$ is also a unique solution of the functional equation (1.2). This completes the proof. □

**Remark 4.2.** In case $u = q_1 = p_2 = 0$, $\mathrm{opt}_{y\in D} = \inf_{y\in D}$, and opt = max, then Theorem 4.2 reduces to Theorem 3.4 of Bhakta and Choudhury [7] and a result of Bellman [3, p. 149]. The example below reveals that Theorem 4.2 generalizes substantially the results due to Bhakta and Choudhury [7] and Bellman [3].

**Example 4.2.** Let $X = Y = R$, $S = R^+$, and $D = R^-$. By making use of Theorem 4.2, we assert that the following functional equation:

$$f(x) = \operatorname*{opt}_{y\in D}\left\{x^2\sin(x-y) + \mathrm{opt}\left\{\frac{x^2+x}{1+y^2} + \frac{\sin(x^2-2y)}{3+x+2|y|} f\left(\frac{y\sin(2x-3y)}{1+x+y^2}\right),\right.\right.$$
$$\left.\left.\frac{\ln(1+x^2)}{1+2x|y|+y^2} + \frac{\cos(x+y^2)}{3+2x|y|} f\left(\frac{(x^2+1)\cos(x^2-y^2)}{1+3x+2|y|}\right)\right\}\right\},$$
$$\forall x \in S, \tag{4.6}$$

possesses a unique solution $w \in BB(S)$. However, the results in [3] and [7] are inapplicable for the functional equation (4.6).

**Theorem 4.3.** *Let $a_i : S \times D \to S$, $u$, $p_i$, and $q_i : S \times D \to R$ be mappings for $i = 1, 2$, and let $(\varphi, \psi)$ be in $\Phi_2$ satisfying*

(D6) $\max\{|u(x, y)|, |p_i(x, y)|: i = 1, 2\} \leqslant \psi(\|x\|)$ *for all* $(x, y) \in S \times D$;
(D7) $\max\{\|a_i(x, y)\|: i = 1, 2\} \leqslant \varphi(\|x\|)$ *for all* $(x, y) \in S \times D$;
(D8) $\max\{|q_i(x, y)|: i = 1, 2\} \leqslant 1$ *for all* $(x, y) \in S \times D$.

*Then the functional equation* (1.2) *possesses a solution $w \in BB(S)$ that satisfies the following conditions*:

(D9) *the sequence $\{w_n\}_{n\geqslant 0}$ defined by*

$$w_0(x) = \operatorname*{opt}_{y\in D}\big\{u(x, y) + \mathrm{opt}\big\{p_i(x, y): i = 1, 2\big\}\big\}, \quad \forall x \in S,$$
$$w_n(x) = \operatorname*{opt}_{y\in D}\big\{u(x, y) + \mathrm{opt}\big\{p_i(x, y)$$
$$+ q_i(x, y) w_{n-1}\big(a_i(x, y)\big): i = 1, 2\big\}\big\}, \quad \forall x \in S,\ n \geqslant 1,$$

*converges to $w$*;

(D10) *if* $x_0 \in S$, $\{y_n\}_{n\geqslant 1} \subset D$, *and* $x_n \in \{a_i(x_{n-1}, y_n)\colon i = 1, 2\}$ *for each* $n \geqslant 1$, *then*

$$\lim_{n\to\infty} w(x_n) = 0;$$

(D11) $w$ *is unique with respect to condition* (D10).

**Proof.** First of all we show that

$$\varphi(t) < t, \quad \forall t > 0. \tag{4.7}$$

If not, then there exists some $t > 0$ such that $\varphi(t) \geqslant t$. From $(\varphi, \psi) \in \Phi_2$ we derive that

$$0 < \psi(t) \leqslant \psi\big(\varphi(t)\big) \leqslant \psi\big(\varphi^2(t)\big) \leqslant \cdots \leqslant \psi\big(\varphi^n(t)\big) \to 0 \quad \text{as } n \to \infty,$$

that is,

$$0 < \psi(t) \leqslant 0,$$

which is a contradiction.

Now we claim that $H$ is a nonexpansive mapping from $BB(S)$ into itself, where $H$ is defined by (4.3) and (4.4). Given $k \geqslant 1$ and $g \in BB(S)$, we know that by (4.7) and (D7),

$$\max\big\{\big\|a_i(x, y)\big\|\colon i = 1, 2\big\} \leqslant \varphi\big(\|x\|\big) \leqslant k, \quad \forall (x, y) \in \bar{B}(0, k) \times D,$$

which yields that there exists $h(k, g) > 0$ with

$$\max\big\{\big|g\big(a_i(x, y)\big)\big|\colon i = 1, 2\big\} \leqslant h(k, g), \quad \forall (x, y) \in \bar{B}(0, k) \times D.$$

In view of (4.4), (D6), (D8), and the above inequality, we see that

$$\begin{aligned}
\big|C(x, y, g)\big| &\leqslant \big|u(x, y)\big| + \max\big\{\big|p_i(x, y)\big| + \big|q_i(x, y)\big|\big|g\big(a_i(x, y)\big)\big|\colon i = 1, 2\big\} \\
&\leqslant \big|u(x, y)\big| + \max\big\{\big|p_i(x, y)\big|\colon i = 1, 2\big\} \\
&\quad + \max\big\{\big|q_i(x, y)\big|\colon i = 1, 2\big\} \max\big\{\big|g\big(a_i(x, y)\big)\big|\colon i = 1, 2\big\} \\
&\leqslant 2\psi\big(\|x\|\big) + \max\big\{\big|g\big(a_i(x, y)\big)\big|\colon i = 1, 2\big\} \\
&\leqslant 2\psi(k) + h(k, g), \quad \forall (x, y) \in \bar{B}(0, k) \times D,
\end{aligned}$$

which means that

$$\big|Hg(x)\big| \leqslant \sup_{y\in D}\big|C(x, y, g)\big| \leqslant 2\psi(k) + h(k, g), \quad \forall x \in \bar{B}(0, k).$$

Therefore $Hg$ is bounded on bounded subsets of $S$. That is, $H$ is a self mapping on $BB(S)$. Let $\varepsilon > 0$ and $(x, s, t) \in \bar{B}(0, k) \times BB(S) \times BB(S)$. Suppose that $\mathrm{opt}_{y\in D} = \inf_{y\in D}$. Then there exist $y, z \in D$ such that

$$\begin{aligned}
&Hs(x) > C(x, y, s) - \varepsilon, \qquad Ht(x) > C(x, z, t) - \varepsilon, \\
&Hs(x) \leqslant C(x, z, s), \qquad Ht(x) \leqslant C(x, y, t).
\end{aligned} \tag{4.8}$$

By virtue of (4.4), (4.7), (4.8), and (D6)–(D8), we get that

$$\begin{aligned}\left|Hs(x)-Ht(x)\right| &< \max\left\{\left|C(x,y,s)-C(x,y,t)\right|, \left|C(x,z,s)-C(x,z,t)\right|\right\}+\varepsilon \\ &\leqslant \max\Bigl\{\Bigl|\operatorname{opt}\bigl\{p_i(x,y)+q_i(x,y)s\bigl(a_i(x,y)\bigr)\colon\ i=1,2\bigr\} \\ &\qquad - \operatorname{opt}\bigl\{p_i(x,y)+q_i(x,y)t\bigl(a_i(x,y)\bigr)\colon\ i=1,2\bigr\}\Bigr|, \\ &\qquad \Bigl|\operatorname{opt}\bigl\{p_i(x,z)+q_i(x,z)s\bigl(a_i(x,z)\bigr)\colon\ i=1,2\bigr\} \\ &\qquad - \operatorname{opt}\bigl\{p_i(x,z)+q_i(x,z)t\bigl(a_i(x,z)\bigr)\colon\ i=1,2\bigr\}\Bigr|\Bigr\}+\varepsilon \\ &\leqslant \max\Bigl\{\left|q_i(x,y)\right|\left|s\bigl(a_i(x,y)\bigr)-t\bigl(a_i(x,y)\bigr)\right|, \\ &\qquad \left|q_i(x,z)\right|\left|s\bigl(a_i(x,z)\bigr)-t\bigl(a_i(x,z)\bigr)\right|\colon\ i=1,2\Bigr\}+\varepsilon \\ &\leqslant \max\bigl\{\left|q_i(x,y)\right|, \left|q_i(x,z)\right|\colon\ i=1,2\bigr\}d_k(s,t)+\varepsilon \\ &\leqslant d_k(s,t)+\varepsilon,\end{aligned}$$

which implies that

$$d_k(Hs,Ht) \leqslant d_k(s,t)+\varepsilon. \tag{4.9}$$

Similarly (4.9) holds for $\operatorname{opt}_{y\in D}=\sup_{y\in D}$. Letting $\varepsilon\to 0$ in (4.9), we obtain that

$$d_k(Hs,Ht) \leqslant d_k(s,t),$$

which yields that

$$d(Hs,Ht)=\sum_{k=1}^{\infty}\frac{1}{2^k}\cdot\frac{d_k(Hs,Ht)}{1+d_k(Hs,Ht)} \leqslant \sum_{k=1}^{\infty}\frac{1}{2^k}\cdot\frac{d_k(s,t)}{1+d_k(s,t)}=d(s,t). \tag{4.10}$$

Now we claim that for each $x\in S$,

$$\left|w_n(x)\right| \leqslant 2\sum_{j=0}^{n}\psi\bigl(\varphi^j\bigl(\|x\|\bigr)\bigr), \quad \forall n\geqslant 0. \tag{4.11}$$

From (D6) we conclude that

$$\left|w_0(x)\right| \leqslant \sup_{y\in D}\bigl\{\left|u(x,y)\right|+\max\bigl\{\left|p_i(x,y)\right|\colon\ i=1,2\bigr\}\bigr\} \leqslant 2\psi\bigl(\|x\|\bigr),$$

that is, (4.11) holds for $n=0$. Suppose that (4.11) holds for some $n\geqslant 0$. Using (D6)–(D8), we deduce that

$$\begin{aligned}\left|w_{n+1}(x)\right| &\leqslant \sup_{y\in D}\bigl\{\left|u(x,y)\right|+\max\bigl\{\left|p_i(x,y)\right|+\left|q_i(x,y)\right|\left|w_n\bigl(a_i(x,y)\bigr)\right|\colon\ i=1,2\bigr\}\bigr\} \\ &\leqslant \sup_{y\in D}\bigl\{\psi\bigl(\|x\|\bigr)+\max\bigl\{\left|p_i(x,y)\right|\colon\ i=1,2\bigr\} \\ &\qquad +\max\bigl\{\left|q_i(x,y)\right|\colon\ i=1,2\bigr\}\max\bigl\{\left|w_n\bigl(a_i(x,y)\bigr)\right|\colon\ i=1,2\bigr\}\bigr\} \\ &\leqslant \sup_{y\in D}\left\{2\psi\bigl(\|x\|\bigr)+\max\left\{2\sum_{j=0}^{n}\psi\bigl(\varphi^j\bigl(\left\|a_i(x,y)\right\|\bigr)\bigr)\colon\ i=1,2\right\}\right\} \\ &\leqslant 2\sum_{j=0}^{n+1}\psi\bigl(\varphi^j\bigl(\|x\|\bigr)\bigr),\end{aligned}$$

which gives that (4.11) holds for all $n \geqslant 0$.

Next we assert that $\{w_n\}_{n\geqslant 0}$ is a Cauchy sequence in $(BB(S), d)$. Given $k \geqslant 1$ and $x_0 \in \bar{B}(0,k)$. Let $\varepsilon > 0$, $n \geqslant 1$, and $m \geqslant 1$. Suppose that $\mathrm{opt}_{y\in D} = \inf_{y\in D}$. From (4.3) we can select $y, z \in D$ such that

$$\begin{aligned} &w_n(x_0) > C(x_0, y, w_{n-1}) - 2^{-1}\varepsilon, \qquad w_{n+m}(x_0) > C(x_0, z, w_{n+m-1}) - 2^{-1}\varepsilon,\\ &w_n(x_0) \leqslant C(x_0, z, w_{n-1}), \qquad w_{n+m}(x_0) \leqslant C(x_0, y, w_{n+m-1}). \end{aligned} \tag{4.12}$$

In terms of (4.12), (4.4), (D6)–(D8), we determine $y_1 \in \{y, z\}$ and $x_1 \in \{a_i(x_0, y_1)\colon\ i = 1, 2\}$ such that

$$\begin{aligned} &\bigl|w_{n+m}(x_0) - w_n(x_0)\bigr|\\ &\quad < \max\Bigl\{\bigl|C\bigl(x_0, y, w_{n-1}\bigr) - C\bigl(x_0, y, w_{n+m-1}\bigr)\bigr|, \bigl|C\bigl(x_0, z, w_{n-1}\bigr)\\ &\qquad\quad - C\bigl(x_0, z, w_{n+m-1}\bigr)\bigr|\Bigr\} + 2^{-1}\varepsilon\\ &\quad \leqslant \max\Bigl\{\bigl|\mathrm{opt}\bigl\{p_i(x_0, y) + q_i(x_0, y)w_{n-1}\bigl(a_i(x_0, y)\bigr)\colon\ i = 1, 2\bigr\}\\ &\qquad\quad - \mathrm{opt}\bigl\{p_i(x_0, y) + q_i(x_0, y)w_{n+m-1}\bigl(a_i(x_0, y)\bigr)\colon\ i = 1, 2\bigr\}\bigr|,\\ &\qquad\quad \bigl|\mathrm{opt}\bigl\{p_i(x_0, z) + q_i(x_0, z)w_{n-1}\bigl(a_i(x_0, z)\bigr)\colon\ i = 1, 2\bigr\}\\ &\qquad\quad - \mathrm{opt}\bigl\{p_i(x_0, z) + q_i(x_0, z)w_{n+m-1}\bigl(a_i(x_0, z)\bigr)\colon\ i = 1, 2\bigr\}\bigr|\Bigr\} + 2^{-1}\varepsilon\\ &\quad \leqslant \max\Bigl\{\max\bigl\{\bigl|q_i(x_0, y)\bigr|\colon\ i = 1, 2\bigr\}\\ &\qquad\quad \times \max\bigl\{\bigl|w_{n-1}\bigl(a_i(x_0, y)\bigr) - w_{n+m-1}\bigl(a_i(x_0, y)\bigr)\bigr|\colon\ i = 1, 2\bigr\},\\ &\qquad\quad \max\bigl\{\bigl|q_i(x_0, z)\bigr|\colon\ i = 1, 2\bigr\}\\ &\qquad\quad \times \max\bigl\{\bigl|w_{n-1}\bigl(a_i(x_0, z)\bigr) - w_{n+m-1}\bigl(a_i(x_0, z)\bigr)\bigr|\colon\ i = 1, 2\bigr\}\Bigr\} + 2^{-1}\varepsilon\\ &\quad \leqslant \max\Bigl\{\bigl|w_{n-1}\bigl(a_i(x_0, y)\bigr) - w_{n+m-1}\bigl(a_i(x_0, y)\bigr)\bigr|,\\ &\qquad\quad \bigl|w_{n-1}\bigl(a_i(x_0, z)\bigr) - w_{n+m-1}\bigl(a_i(x_0, z)\bigr)\bigr|\colon\ i = 1, 2\Bigr\} + 2^{-1}\varepsilon\\ &\quad = \bigl|w_{n+m-1}(x_1) - w_{n-1}(x_1)\bigr| + 2^{-1}\varepsilon, \end{aligned}$$

that is

$$\bigl|w_{n+m}(x_0) - w_n(x_0)\bigr| < \bigl|w_{n+m-1}(x_1) - w_{n-1}(x_1)\bigr| + 2^{-1}\varepsilon. \tag{4.13}$$

Similarly we see that (4.13) holds for $\mathrm{opt}_{y\in D} = \sup_{y\in D}$. Proceeding in this way, we find $y_j \in D$ and $x_j \in \{a_i(x_{j-1}, y_j)\colon\ i = 1, 2\}$ for $j \in \{2, 3, \ldots, n\}$ such that

$$\begin{aligned} &\bigl|w_{n+m-1}(x_1) - w_{n-1}(x_1)\bigr| < \bigl|w_{n+m-2}(x_2) - w_{n-2}(x_2)\bigr| + 2^{-2}\varepsilon,\\ &\bigl|w_{n+m-2}(x_2) - w_{n-2}(x_2)\bigr| < \bigl|w_{n+m-3}(x_3) - w_{n-3}(x_3)\bigr| + 2^{-3}\varepsilon,\\ &\ldots\\ &\bigl|w_{m+1}(x_{n-1}) - w_1(x_{n-1})\bigr| < \bigl|w_m(x_n) - w_0(x_n)\bigr| + 2^{-n}\varepsilon. \end{aligned} \tag{4.14}$$

On account of (4.7), (4.11), (4.13), (4.14), and (D7), we have

$$\begin{aligned}
\left|w_{n+m}(x_0) - w_n(x_0)\right| &< \left|w_m(x_n) - w_0(x_n)\right| + \varepsilon \leqslant \left|w_m(x_n)\right| + \left|w_0(x_n)\right| + \varepsilon \\
&\leqslant 2\sum_{j=0}^{m} \psi\left(\varphi^j\left(\|x_n\|\right)\right) + 2\psi\left(\|x_n\|\right) + \varepsilon \\
&\leqslant 2\sum_{j=0}^{m} \psi\left(\varphi^{j+1}\left(\|x_{n-1}\|\right)\right) + 2\psi\left(\varphi\left(\|x_{n-1}\|\right)\right) + \varepsilon \\
&\leqslant \cdots \\
&\leqslant 2\sum_{j=0}^{m} \psi\left(\varphi^{j+n}\left(\|x_0\|\right)\right) + 2\psi\left(\varphi^n\left(\|x_0\|\right)\right) + \varepsilon \\
&\leqslant 2\sum_{j=n-1}^{m+n} \psi\left(\varphi^j(k)\right) + \varepsilon,
\end{aligned}$$

which means that

$$d_k(w_{n+m}, w_n) \leqslant 2\sum_{j=n-1}^{\infty} \psi\left(\varphi^j(k)\right) + \varepsilon.$$

Letting $\varepsilon \to 0$ in the above inequality, we immediately conclude that

$$d_k(w_{n+m}, w_n) \leqslant 2\sum_{j=n-1}^{\infty} \psi\left(\varphi^j(k)\right),$$

which implies that $\{w_n\}_{n\geqslant 0}$ is a Cauchy sequence in $(BB(S), d)$ because

$$\sum_{n=0}^{\infty} \psi\left(\varphi^n(t)\right) < \infty$$

for each $t > 0$. Let $\{w_n\}_{n\geqslant 0}$ converge to $w \in BB(S)$. It follows from (4.10) that

$$\begin{aligned}
d(Hw, w) &\leqslant d(Hw, Hw_n) + d(Hw_n, w) \\
&\leqslant d(w, w_n) + d(w_{n+1}, w) \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

that is, $Hw = w$. Hence the functional equation (1.2) possesses a solution $w$.

Let $\varepsilon > 0$, $x_0 \in S$, $\{y_n\}_{n\geqslant 1} \subset D$, and $x_n \in \{a_i(x_{n-1}, y_n):\ i = 1, 2\}$ for each $n \geqslant 1$. Set $k = [\|x_0\|] + 1$. It is easy to verify that there exists an positive integer $m$ satisfying

$$d_k(w, w_n) + 2\sum_{j=n}^{\infty} \psi\left(\varphi^j(k)\right) < \varepsilon, \quad \forall n > m. \tag{4.15}$$

Note that (D7) and (4.7) ensure that

$$\|x_n\| \leqslant \varphi\left(\|x_{n-1}\|\right) \leqslant \cdots \leqslant \varphi^n\left(\|x_0\|\right) \leqslant \varphi^n(k) < k, \quad \forall n \geqslant 1. \tag{4.16}$$

Thus (4.11), (4.15), and (4.16) guarantee that

$$\begin{aligned}\left|w(x_n)\right| &\leqslant \left|w(x_n)-w_n(x_n)\right|+\left|w_n(x_n)\right| \leqslant d_k(w,w_n)+2\sum_{j=0}^{n}\psi\left(\varphi^j\left(\|x_n\|\right)\right)\\ &\leqslant d_k(w,w_n)+2\sum_{j=n}^{\infty}\psi\left(\varphi^j(k)\right)<\varepsilon, \quad \forall n>m,\end{aligned}$$

which gives that $\lim_{n\to\infty} w(x_n)=0$.

Finally we show that (D11) holds. Assume that the functional equation (1.2) possesses another solution $z\in BB(S)$, which satisfies condition (D10). Let $\varepsilon>0$ and $x_0\in S$. Suppose that $\mathrm{opt}_{y\in D}=\inf_{y\in D}$. It follows from (4.3) that there exist $y,h\in D$ with

$$\begin{aligned}&w(x_0)>C(x_0,y,w)-2^{-1}\varepsilon, \qquad z(x_0)>C(x_0,h,z)-2^{-1}\varepsilon,\\ &w(x_0)\leqslant C(x_0,h,w), \qquad z(x_0)\leqslant C(x_0,y,z).\end{aligned}$$

From the above inequalities, (4.4) and (D8), we find $y_1\in\{y,h\}\subset D$ and $x_1\in\{a_i(x_0,y_1)\colon i=1,2\}$ such that

$$\begin{aligned}&\left|w(x_0)-z(x_0)\right|\\ &\quad<\max\left\{\left|C(x_0,y,w)-C(x_0,y,z)\right|,\left|C(x_0,h,w)-C(x_0,h,z)\right|\right\}+2^{-1}\varepsilon\\ &\quad\leqslant\max\Big\{\Big|\mathrm{opt}\left\{p_i(x_0,y)+q_i(x_0,y)w\left(a_i(x_0,y)\right)\colon i=1,2\right\}\\ &\qquad\qquad-\mathrm{opt}\left\{p_i(x_0,y)+q_i(x_0,y)z\left(a_i(x_0,y)\right)\colon i=1,2\right\}\Big|,\\ &\qquad\qquad\Big|\mathrm{opt}\left\{p_i(x_0,h)+q_i(x_0,h)w\left(a_i(x_0,h)\right)\colon i=1,2\right\}\\ &\qquad\qquad-\mathrm{opt}\left\{p_i(x_0,h)+q_i(x_0,h)z\left(a_i(x_0,h)\right)\colon i=1,2\right\}\Big|\Big\}+2^{-1}\varepsilon\\ &\quad\leqslant\max\Big\{\left|q_i(x_0,y)\right|\left|w\left(a_i(x_0,y)\right)-z\left(a_i(x_0,y)\right)\right|,\\ &\qquad\qquad\left|q_i(x_0,h)\right|\left|w\left(a_i(x_0,h)\right)-z\left(a_i(x_0,h)\right)\right|\colon i=1,2\Big\}+2^{-1}\varepsilon\\ &\quad\leqslant\max\left\{\left|q_i(x_0,y)\right|,\left|q_i(x_0,h)\right|\colon i=1,2\right\}\max\Big\{\left|w\left(a_i(x_0,y)\right)-z\left(a_i(x_0,y)\right)\right|,\\ &\qquad\qquad\left|w\left(a_i(x_0,h)\right)-z\left(a_i(x_0,h)\right)\right|\colon i=1,2\Big\}+2^{-1}\varepsilon\\ &\quad\leqslant\left|w(x_1)-z(x_1)\right|+2^{-1}\varepsilon. \qquad (4.17)\end{aligned}$$

Similarly (4.17) holds also for $\mathrm{opt}_{y\in D}=\sup_{y\in D}$. Proceeding in this way, we select $y_j\in D$ and $x_j\in\{a_i(x_{j-1},y_j)\colon i=1,2\}$ for $j\in\{2,3,\ldots,n\}$ such that

$$\begin{aligned}&\left|w(x_1)-z(x_1)\right|<\left|w(x_2)-z(x_2)\right|+2^{-2}\varepsilon,\\ &\left|w(x_2)-z(x_2)\right|<\left|w(x_3)-z(x_3)\right|+2^{-3}\varepsilon,\\ &\ldots\\ &\left|w(x_{n-1})-z(x_{n-1})\right|<\left|w(x_n)-z(x_n)\right|+2^{-n}\varepsilon. \qquad (4.18)\end{aligned}$$

On account of (4.17), (4.18), and (D10), we derive that

$$\left|w(x_0)-z(x_0)\right|<\left|w(x_n)-z(x_n)\right|+\varepsilon\to\varepsilon \quad \text{as } n\to\infty,$$

which yields that

$$\left|w(x_0)-z(x_0)\right|\leqslant\varepsilon.$$

Letting $\varepsilon \to 0$ in the above inequality, we deduce that $w(x_0) = z(x_0)$. This completes the proof. $\square$

**Theorem 4.4.** *Let $a_i : S \times D \to S$, $u$, $p_i$, and $q_i : S \times D \to R$ be mappings for $i = 1, 2$, and let $(\varphi, \psi)$ be in $\Phi_2$ satisfying conditions* (D6)–(D8). *Then the functional equation*

$$f(x) = \operatorname*{opt}_{y \in D}\bigl\{u(x, y) + \max\bigl\{p_i(x, y) + q_i(x, y) f\bigl(a_i(x, y)\bigr)\colon i = 1, 2\bigr\}\bigr\}, \quad \forall x \in S, \tag{4.19}$$

*possesses a solution $w \in BB(S)$ that satisfies conditions* (D10), (D11), *and*

(D12) *the sequence $\{w_n\}_{n \geqslant 0}$ defined by*

$$w_0(x) = \operatorname*{opt}_{y \in D}\bigl\{u(x, y) + \max\bigl\{p_i(x, y)\colon i = 1, 2\bigr\}\bigr\}, \quad \forall x \in S,$$

$$w_n(x) = \operatorname*{opt}_{y \in D}\bigl\{u(x, y) + \max\bigl\{p_i(x, y) + q_i(x, y) w_{n-1}\bigl(a_i(x, y)\bigr)\colon i = 1, 2\bigr\}\bigr\}, \quad \forall x \in S,$$

*converges to $w$;*

(D13) *if $q_i(x, y) = 1$ for each $(x, y) \in S \times D$ and $i = 1, 2$, then for given $\varepsilon > 0$, $i \in \{1, 2\}$, and $x_0 \in S$, there exist $\{y_n\}_{n \geqslant 1} \subset D$ and $x_n = a_i(x_{n-1}, y_n)$, $n \geqslant 1$, such that*

$$w(x_0) - \sum_{n=1}^{\infty}\bigl[u(x_{n-1}, y_n) + p_i(x_{n-1}, y_n)\bigr] \geqslant -\varepsilon.$$

*Moreover, if $u(x, y) + p_i(x, y) \geqslant 0$ for any $(x, y) \in S \times D$, then*

$$w(x) \geqslant 0, \quad \forall x \in S.$$

**Proof.** It follows from Theorem 4.3 that the functional equation (4.19) possesses a solution $w \in BB(S)$ that satisfies (D10)–(D12). Let $q_i(x, y) = 1$ for any $(x, y) \in S \times D$ and $i = 1, 2$. Given $\varepsilon > 0$, $i \in \{1, 2\}$, and $x_0 \in S$, then there exist $y_1 \in D$ and $x_1 = a_i(x_0, y_1)$ with

$$\begin{aligned} w(x_0) &> u(x_0, y_1) + \max\bigl\{p_i(x_0, y_1) + w\bigl(a_i(x_0, y_1)\bigr)\colon i = 1, 2\bigr\} - 2^{-1}\varepsilon \\ &\geqslant u(x_0, y_1) + p_i(x_0, y_1) + w(x_1) - 2^{-1}\varepsilon. \end{aligned}$$

Similarly we find $y_j \in D$ and $x_j = a_i(x_{j-1}, y_j)$ for $j \in \{2, 3, \ldots, n\}$ satisfying

$$w(x_{j-1}) > u(x_{j-1}, y_j) + p_i(x_{j-1}, y_j) + w(x_j) - 2^{-j}\varepsilon, \quad j \in \{2, 3, \ldots, n\}.$$

It follows that

$$w(x_0) - \sum_{j=1}^{n}\bigl[u(x_{j-1}, y_j) + p_i(x_{j-1}, y_j)\bigr] > w(x_n) - \bigl(1 - 2^{-n}\bigr)\varepsilon. \tag{4.20}$$

Since $\sum_{n=0}^{\infty} \psi(\varphi^n(t)) < \infty$ for each $t > 0$, it follows from (D6) that

$$\max\bigl\{\bigl|u(x_{n-1}, y_n)\bigr|, \bigl|p_i(x_{n-1}, y_n)\bigr|\bigr\} \leqslant \psi\bigl(\|x_{n-1}\|\bigr) \leqslant \psi\bigl(\varphi^{n-1}\bigl(\|x_0\|\bigr)\bigr).$$

Consequently, $\sum_{n=1}^{\infty}[|u(x_{n-1}, y_n)| + |p_i(x_{n-1}, y_n)|]$ is convergent. Letting $n \to \infty$ in (4.20), by (D10) we see that

$$w(x_0) - \sum_{j=1}^{\infty}\big[u(x_{j-1}, y_j) + p_i(x_{j-1}, y_j)\big] \geqslant -\varepsilon. \tag{4.21}$$

Suppose that $u(x, y) + p_i(x, y) \geqslant 0$ for each $(x, y) \in S \times D$. By virtue of (4.21) we know that

$$w(x_0) \geqslant -\varepsilon.$$

Letting $\varepsilon \to 0$ in the above inequality, we have

$$w(x_0) \geqslant 0.$$

This completes the proof. □

By similar arguments as in the proof of Theorem 4.4, we have the following result and its proof is omitted.

**Theorem 4.5.** *Let $a_i : S \times D \to S$, $u$, $p_i$, and $q_i : S \times D \to R$ be mappings for $i = 1, 2$, and let $(\varphi, \psi)$ be in $\Phi_2$ satisfying conditions* (D6)–(D8). *Then the functional equation*

$$f(x) = \operatorname*{opt}_{y \in D}\big\{u(x, y) + \min\big\{p_i(x, y) + q_i(x, y) f\big(a_i(x, y)\big)\colon\ i = 1, 2\big\}\big\},\quad \forall x \in S, \tag{4.22}$$

*possesses a solution $w \in BB(S)$ that satisfies conditions* (D10), (D11), *and*

(D14) *the sequence $\{w_n\}_{n \geqslant 0}$ defined by*

$$w_0(x) = \operatorname*{opt}_{y \in D}\big\{u(x, y) + \min\big\{p_i(x, y)\colon\ i = 1, 2\big\}\big\},\quad \forall x \in S,$$

$$w_n(x) = \operatorname*{opt}_{y \in D}\big\{u(x, y) + \min\big\{p_i(x, y) + q_i(x, y) w_{n-1}\big(a_i(x, y)\big)\colon\ i = 1, 2\big\}\big\},\quad \forall x \in S,$$

*converges to $w$;*

(D15) *if $q_i(x, y) = 1$ for each $(x, y) \in S \times D$ and $i = 1, 2$, then for given $\varepsilon > 0$, $i \in \{1, 2\}$, and $x_0 \in S$, there exist $\{y_n\}_{n \geqslant 1} \subset D$ and $x_n = a_i(x_{n-1}, y_n)$, $n \geqslant 1$, such that*

$$w(x_0) - \sum_{n=1}^{\infty}\big[u(x_{n-1}, y_n) + p_i(x_{n-1}, y_n)\big] \leqslant \varepsilon.$$

*Moreover, if $u(x, y) + p_i(x, y) \leqslant 0$ for any $(x, y) \in S \times D$, then*

$$w(x) \leqslant 0,\quad \forall x \in S.$$

**Remark 4.3.** Theorems 4.3–4.5 extend and improve Theorem 2.4 of Bhakta and Mitra [8], Theorem 3.5 of Bhakta and Choudhury [7], Theorem 3.5 of Liu [12], Corollaries 3.1–3.3 of Liu and Ume [13] and a result of Bellman [3, p, 149].

The following example demonstrates that Theorems 4.3–4.5 generalizes substantially the results in [3,7,8,12,13].

**Example 4.3.** Let $X = Y = D = R$ and $S = R^+$. Define $\psi, \varphi : R^+ \to R^+$ by

$$\psi(t) = t^2 \quad \text{and} \quad \varphi(t) = \frac{1}{2}t, \quad \forall t \in R^+.$$

It follows from Theorems 4.3–4.5 that the functional equations below

$$\begin{aligned} f(x) = \operatorname*{opt}_{y\in D} \biggl\{ & \frac{x^2}{1+|\sin(xy)|} \\ & + \operatorname{opt}\biggl\{ \frac{x^2\sin(xy)}{1+|\cos(x+y+1)|} + \sin\bigl(x^2-y^2\bigr) f\biggl(\frac{x^2y}{1+x^2y^2}\biggr), \\ & \quad \frac{x^3y}{1+x^2+y^2} + \cos^2(x-y+1) f\biggl(\frac{x^{3/2}y\sin(x-y)}{1+x+y^2}\biggr)\biggr\}\biggr\}, \\ & \forall x \in S, \end{aligned} \tag{4.23}$$

$$\begin{aligned} f(x) = \operatorname*{opt}_{y\in D} \biggl\{ & \frac{x^2}{1+|\sin(xy)|} + \max\biggl\{ \frac{x^2}{1+|y|} + f\biggl(\frac{x^2\sin(x^2-y^2)}{1+2x}\biggr), \\ & \frac{x^2y}{1+y^2} + f\biggl(\frac{x^{3/2}\cos(x^2-y)}{1+x}\biggr)\biggr\}\biggr\}, \quad \forall x \in S, \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} f(x) = \operatorname*{opt}_{y\in D} \biggl\{ & \frac{x^2}{1+|\sin(xy)|} \\ & + \min\biggl\{ \frac{x^2\sin(x^3y-2yx)}{1+\ln(1+xy^2)} + f\biggl(\frac{x}{2+|\cos(xy^2-x^2y)|}\biggr), \\ & \quad \frac{x^2\sin(x^2-y^2+1)}{1+y^2} + f\biggl(\frac{x^2y^2}{1+x^2+y^4}\biggr)\biggr\}\biggr\}, \quad \forall x \in S, \end{aligned} \tag{4.25}$$

possess a unique solution in $BB(S)$, respectively. However, the corresponding results in [3,7,8,12,13] are not valid for the functional equations (4.23)–(4.25) because

$$\left| \frac{x^2}{1+|\sin(xy)|} \right| \leqslant M|x|$$

does not hold for $(x_M, y_M) = (1+M, 0) \in S \times D$, where $M$ is a positive constant.

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