Periodic solutions of the nonlinear telegraph equations with bounded nonlinearities

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Abstract

In this article, using the Leray–Schauder degree theory, we discuss existence, nonexistence and multiplicity for the periodic solutions of the nonlinear telegraph equation

\[ u_{tt} - u_{xx} + cu_t + \Phi(u) = f(t, x) + s, \]

where \( c > 0, \Phi \in C(\mathbb{R}), f \in C(T^2) \) and \( s \) is a parameter.

Keywords: Telegraph equation; Periodic solutions; Bounded nonlinearities; Leray–Schauder degree

1. Introduction

In this article we study existence, nonexistence and multiplicity for the periodic solutions of the nonlinear telegraph equation

\[ u_{tt} - u_{xx} + cu_t + \Phi(u) = f(t, x) + s, \quad (1) \]

where \( c > 0, \Phi \) is continuous on \( \mathbb{R} \), \( f \) is continuous on \( \mathbb{R}^2 \) and \( 2\pi \)-periodic in \( t \) and \( x \) and \( s \) is a real parameter.

The existence of a periodic solution of nonlinear problem (1) with bounded nonlinearity \( \Phi \) has been studied by many authors, see [5–7] and references therein. In [5] the nonlinearity \( \Phi \) has different limits at \( \pm \infty \) (Landesman–Lazer type nonlinearity) and in [6,7] \( \Phi \) is \( 2\pi \)-periodic (forced pendulum type nonlinearity).

Let \( T^2 \) be the torus defined by

\[ T^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}). \]

A point of \( T^2 \) is denoted by \((\hat{t}, \hat{x})\), where \((t, x)\) is a point of \( \mathbb{R}^2 \) and \( \hat{t} = t + 2\pi\mathbb{Z}, \hat{x} = x + 2\pi\mathbb{Z} \). Doubly periodic functions will be identified to functions defined on the torus. In particular, the notations
stand for the spaces of doubly periodic functions with the indicated degree of regularity. The space of distributions on $T^2$ is denoted by $D'(T^2)$. The maximum norm in $C(T^2)$ is denoted by $\| \cdot \|_{\infty}$. $B_r$ denotes the open ball of center 0 and radius $r$ in $C(T^2)$. By a periodic solution of (1) we understand a function $u \in C(T^2)$ satisfying

$$\int_{T^2} u_{tt} - u_{xx} - cu_t + \int_{T^2} \Phi(u) \varphi = \int_{T^2} (f(t, x) + s) \varphi, \quad \forall \varphi \in D(T^2),$$

in other words

$$u_{tt} - u_{xx} + cu_t + \Phi(u) = f(t, x) + s \quad \text{in} \quad D'(T^2).$$

In what follows we assume that the following properties are satisfied.

(H1) $\int_{T^2} f = 0$.

(H2) There exists $l \in \mathbb{R}$ such that $\Phi(u) > l$ for all $u \in \mathbb{R}$ and $\lim_{u \to \pm \infty} \Phi(u) = l$.

(H3) $\Phi(u_2) - \Phi(u_1) \leq \nu(c)(u_2 - u_1)$ for every $u_1, u_2$ with $u_1 \leq u_2$. The constant $\nu(c) \in ]c^2/4, (c^2 + 1)/4]$ will be specified later.

Our main result is the following one.

**Theorem 1.** If conditions (H1)–(H3) hold, then there exists $s^* \in [l, \sup_{R} \Phi]$ such that problem (1) has zero, at least one or at least two periodic solutions according to $s \notin [l, s^*], s = s^* \text{ or } s \in [l, s^*]$.

This type of nonlinearity $\Phi$ has been considered for the first time in [8] and the corresponding result for second-order differential equations has been proved in [4] following a different approach based upon a result from [3].

The methodology used in this paper has been introduced in [2] in order to extend in various directions the result in [4]. Our main tool is the Leray–Schauder degree together with some results concerning the linear problem associated to (1) proved in [6].

2. Proof of the main result

Let $\tilde{C}(T^2)$ be the closed subspace of $C(T^2)$ defined by

$$\tilde{C}(T^2) = \left\{ \tilde{u} \in C(T^2) : \int_{T^2} \tilde{u} = 0 \right\}.$$ 

Using Proposition 4.4 in [6], we define the compact linear operator

$$R_0 : \tilde{C}(T^2) \to \tilde{C}(T^2), \quad R_0(\tilde{u}) = \tilde{u},$$

where $\tilde{u}$ is the unique solution of the linear problem

$$u_{tt} - u_{xx} + cu_t = \tilde{h}(t, x) \quad \text{in} \quad D'(T^2), \quad \int_{T^2} u = 0.$$ 

Using (H1) we can define the nonlinear compact operator

$$\mathcal{M} : \mathbb{R} \times \tilde{C}(T^2) \to \tilde{C}(T^2)$$

by the formula

$$\mathcal{M}(\tilde{u}, \tilde{u}) = R_0 \left( f - \Phi(\tilde{u} + \tilde{u}) + \frac{1}{4\pi^2} \int_{T^2} \Phi(\tilde{u} + \tilde{u}) \right).$$

The idea of the following lemma comes from [1].
Lemma 1. The set \( S \) of solutions \((\bar{u}, \bar{u}) \in \mathbb{R} \times \bar{C}(\mathbb{T}^2)\) of problem
\[
\bar{u} = \mathcal{M}(\bar{u}, \bar{u})
\]
contains a subset \( C \) whose projection on \( \mathbb{R} \) is \( \mathbb{R} \). Moreover,
\[
\exists \rho_1 > 0: \quad \|\bar{u}\|_\infty \leq \rho_1, \quad \forall (\bar{u}, \bar{u}) \in S.
\]

Proof. We have that
\[
\|\mathcal{M}(\bar{u}, \bar{u})\|_\infty \leq \|R_0\| \left(\|f\|_\infty + 2 \sup_{\mathbb{R}} |\Phi| \right) =: \rho_1 \quad \text{for all } (\bar{u}, \bar{u}) \in \mathbb{R} \times \bar{C}(\mathbb{T}^2)
\]
Let us fix \( \bar{u} \in \mathbb{R} \). The nonlinear compact operator
\[
\mathcal{M}(\bar{u}, \cdot) : \bar{C}(\mathbb{T}^2) \to \bar{C}(\mathbb{T}^2)
\]
has the range contained in \( B_{\rho_1} \). Then the conclusion follows from Schauder fixed point theorem. \( \Box \)

Let us decompose any \( u \in C(\mathbb{T}^2) \) in the form
\[
 u = \tilde{u} + \bar{u} \quad \text{where } \tilde{u} = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} u \quad \text{and } \bar{u} \in \bar{C}(\mathbb{T}^2).
\]
Consider the continuous function
\[
\gamma : \mathbb{R} \times \bar{C}(\mathbb{T}^2) \to \mathbb{R}, \quad \gamma(\bar{u}, \bar{u}) = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \Phi(\bar{u} + \bar{u}).
\]
If \( u = \tilde{u} + \bar{u} \) is a solution of (1), then \( s = \gamma(\tilde{u}, \bar{u}) \) and \( \bar{u} = \mathcal{M}(\bar{u}, \bar{u}) \). Reciprocally, if \((\bar{u}, \bar{u}) \in \mathbb{R} \times \bar{C}(\mathbb{T}^2)\) such that \( \bar{u} = \mathcal{M}(\bar{u}, \bar{u}) \), then \( u = \tilde{u} + \bar{u} \) is a solution of (1) with \( s = \gamma(\tilde{u}, \bar{u}) \). Let
\[
 S_j = \{ s \in \mathbb{R} : \text{(1) has at least } j \text{ solutions} \} \quad (j \geq 1)
\]
Lemma 2. We have that \( S_1 \neq \emptyset, S_1 \subset [l, \sup_{\mathbb{R}} \Phi] \) and \( s^* := \sup S_1 \in S_1 \).

Proof. Let \( u \) be a possible solution of (1). Then
\[
 s = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \Phi(u)
\]
and using (H2), we deduce that \( s \in [l, \sup_{\mathbb{R}} \Phi] \). This means that \( S_1 \subset [l, \sup_{\mathbb{R}} \Phi] \). On the other hand, let \((\bar{u}, \bar{u}) \in C \), where \( C \) is given in Lemma 1. We have that \( u = \tilde{u} + \bar{u} \) is a solution of (1) with \( s = \gamma(\tilde{u}, \bar{u}) \), hence \( S_1 \neq \emptyset \). Now, let \( \{s_n\} \subset S_1 \) be a sequence which converges to \( s^* \). Let \( u_n \) be a solution of (1) with \( s = s_n \). It follows that
\[
 s_n = \gamma(\tilde{u}_n, \bar{u}_n) \quad \text{and } \bar{u}_n = \mathcal{M}(\bar{u}_n, \bar{u}_n).
\]
Using Lemma 1, it follows that
\[
\|\bar{u}_n\|_\infty \leq \rho_1.
\]
Hence, if up to a subsequence \( \bar{u}_n \to \pm \infty \), then using (H2) it follows that \( \gamma(\tilde{u}_n, \bar{u}_n) \to l \), which means that \( s^* \neq l \), contradiction. We have proved that \( \{(\bar{u}_n, \bar{u}_n)\} \) is a bounded sequence in \( \mathbb{R} \times \bar{C}(\mathbb{T}^2) \). Because \( \mathcal{M} \) is completely continuous, we can assume, passing to a subsequence, that
\[
\mathcal{M}(\bar{u}_n, \bar{u}_n) \to \bar{u} \quad \text{and } \bar{u}_n \to \bar{u}.
\]
We deduce that
\[
\bar{u} = \mathcal{M}(\bar{u}, \bar{u}), \quad \gamma(\bar{u}, \bar{u}) = s^*
\]
and \( u = \tilde{u} + \bar{u} \) is a solution of (1) with \( s = s^* \). \( \Box \)
Lemma 3. Let $l < s_1 < s^*$. Then, there is $\rho_2 > 0$ such that any possible solution $u$ of (1) with $s \in [s_1, s^*]$ belongs to $B_{\rho_2}$.

Proof. Let $s \in [s_1, s^*]$ and $u$ be a possible solution of (1). It follows that

$$s = \gamma(\bar{u}, \tilde{u}) \quad \text{and} \quad \tilde{u} = M(\bar{u}, \bar{u}).$$

From Lemma 1, we deduce that

$$\|\bar{u}\|_{\infty} \leq \rho_1.$$

Because $l < s_1 \leq \gamma(\bar{u}, \tilde{u})$, it follows from (H2) that there exists $\rho'_1 > 0$ such that

$$|\bar{u}| < \rho'_1.$$  

The conclusion follows taking $\rho_2 = \rho_1 + \rho'_1$. □

Lemma 4. We have that $|l, s^*| \subset S_2$.

Proof. Let $\nu(c)$ be the constant given in [6, Theorem 2.3]. This constant is such that for any $h \in C(T^2)$, the linear problem

$$u_{tt} - u_{xx} + cu_t + v(c)u = h(t, x) \quad \text{in } D'(T^2)$$

has a unique solution

$$u = R_{v(c)}(h).$$

A strong maximum principle holds, in the sense that, $h \geq 0$ on $\mathbb{R}^2$ and $\int_{T^2} h > 0$, then $R_{v(c)}(h) > 0$ on $\mathbb{R}^2$.

Moreover, the linear operator

$$R_{v(c)} : C(T^2) \rightarrow C(T^2)$$

is compact (see [6, Proposition 2.1]). For any $(s, u) \in \mathbb{R} \times C(T^2)$ let us define

$$G(s, u) = R_{v(c)}(f + s - \Phi(u) + v(c)u).$$

It is clear that

$$G : \mathbb{R} \times C(T^2) \rightarrow C(T^2)$$

is a nonlinear compact operator and $u$ is a solution of (1) iff $u = G(s, u)$.

Let $s_1 \in |l, s^*|$ be fixed and let $u^*$ be a solution of (1) with $s = s^*$ given by Lemma 2. Using Lemma 1 and (H2), it follows that there exists $(\bar{u}_*, \tilde{u}_*) \in C$ such that

$$u_* = \bar{u}_* + \tilde{u}_* < u^* \quad \text{on } \mathbb{R}^2 \quad \text{and} \quad \gamma(\bar{u}_*, \tilde{u}_*) < s_1.$$  

Consider the open bounded convex set in $C(T^2)$ defined by

$$\Omega = \{ u \in C(T^2) : u_* < u < u^* \}.$$  

We will show that

$$G(s_1, \Omega) \subset \Omega.$$  

Let $u \in \Omega$ and $v = G(s_1, u)$. If $w = u^* - v$, then $w = R_{v(c)}(h)$ where

$$h = s^* - s_1 + v(c)(u^* - u) - (\Phi(u^*) - \Phi(u)).$$

Using (H3) we deduce that $h > 0$ on $\mathbb{R}^2$ which implies that $w > 0$. Analogously we can prove that $u_* < v$ on $\mathbb{R}^2$. We have proved that $G(s_1, \Omega) \subset \Omega$. This implies that

$$d_{LS}[I - G(s_1, \cdot), \Omega, 0] = 1.$$  

(2)
where $d_{LS}$ denotes the Leray–Schauder degree. Hence the existence property of the Leray–Schauder degree implies that $G(s_1, \cdot)$ has a fixed point in $\Omega$ which is also a solution of (1) with $s = s_1$. On the other hand, using Lemma 2, it follows that if $s_2 > s^*$ is fixed, then $u \neq G(s_2, u)$ for all $u \in C(\mathbb{T}^2)$. Hence
\[ d_{LS}[I - G(s_2, \cdot), B_{\rho_2}, 0] = 0, \]
where $\rho_2$ is given in Lemma 3. Using Lemmas 2, 3 and the invariance property of the Leray–Schauder degree, we deduce that
\[ d_{LS}[I - G(s_1, \cdot), B_{\rho_2}, 0] = d_{LS}[I - G(s_2, \cdot), B_{\rho_2}, 0] = 0, \]
which together with the excision property of the Leray–Schauder degree give
\[ d_{LS}[I - G(s_1, \cdot), B_{\rho}, 0] = 0, \]
for $\rho > 0$ sufficiently large satisfying $\overline{\Omega} \subset B_{\rho}$. Using (2), (3) and the additivity property of the Leray–Schauder degree we deduce that
\[ d_{LS}[I - G(s_1, \cdot), B_{\rho} \setminus \overline{\Omega}, 0] = -1, \]
and the existence property of the Leray–Schauder degree implies that $G(s_1, \cdot)$ has a second fixed point in $B_{\rho} \setminus \overline{\Omega}$ which is also a solution of (1) with $s = s_1$. □

**End of the proof of Theorem 1.** Now, our main result follows from Lemmas 2 and 4.

**Example 1.** Let $c > 0$ be such that
\[ \sqrt{2} \exp\left(-\frac{1}{2}\right) \leq \frac{c^2}{4}. \]
Using Theorem 1 it follows that there exists $s^*$ in $]0, 1]$ such that problem
\[ u_{tt} - u_{xx} + cu_t + \exp(-u^2) = \sin(t + x) + s \]
has zero, at least one or at least two periodic solutions according to $s \not\in ]0, s^*[, s = s^*$ or $s \in ]0, s^*[.$

**References**


