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On certain extension theorems in the mixed Borel setting

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To our friend John Horváth on the occasion of his 80th birthday

Abstract

Given two sequences $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ of positive numbers, we give necessary and sufficient conditions under which the inclusions

$$\Lambda_{\{\boldsymbol{M}_1\}} \subset \left\{ \left(f^{(j)}(0) \right)_{j \in \mathbb{N}_0} \colon f \in \mathcal{D}_{\{\boldsymbol{M}_2\}}\left([-1,1]\right) \right\},$$
$$\Lambda_{(\boldsymbol{M}_1)} \subset \left\{ \left(f^{(j)}(0) \right)_{j \in \mathbb{N}_0} \colon f \in \mathcal{D}_{(\boldsymbol{M}_2)}\left([-1,1]\right) \right\}$$

hold, by means of explicit constructions. This answers a question raised by Chaumat and Chollet (Math. Ann. 298 (1994) 7–40). We also consider the case when $[-1,1]$ is replaced by $[-1,1]^m$ as well as the possibility to get ultraholomorphic extensions.

Keywords: Mixed Borel theorem; Ultradifferentiable functions; Beurling type; Roumieu type

1. Introduction

The sequences $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ and the numbers $\lambda_{p,s}$ for $p, s \in \mathbb{N}$. Throughout this paper $\boldsymbol{m}_1 = (m_{1,p})_{p \in \mathbb{N}_0}$ and $\boldsymbol{m}_2 = (m_{2,p})_{p \in \mathbb{N}_0}$ designate sequences of real numbers submitted to the following requirements:

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(a) $m_{1,0} = m_{2,0} = 1$;
(b) $m_{1,p} \leqslant m_{1,p+1}$ and $m_{2,p} \leqslant m_{2,p+1}$ for every $p \in \mathbb{N}_0$;
(c) $m_{1,p} \leqslant m_{2,p}$ for every $p \in \mathbb{N}_0$;
(d) $\lim_{p\to\infty} p/\sqrt[p]{m_{1,0}\dots m_{1,p}} = 0$;
(e) $\sum_{p=0}^{\infty} 1/m_{2,p} < \infty$.

As usual we then set $M_{1,p} = m_{1,0}\dots m_{1,p}$ and $M_{2,p} = m_{2,0}\dots m_{2,p}$ for every $p \in \mathbb{N}_0$ and get the sequences $\boldsymbol{M}_1 = (M_{1,p})_{p\in\mathbb{N}_0}$ and $\boldsymbol{M}_2 = (M_{2,p})_{p\in\mathbb{N}_0}$.

Definition. For every $p, s \in \mathbb{N}$, we set

$$\lambda_{p,s} := \sup_{0\leqslant j<p} \left(\frac{M_{1,p}}{s^p M_{2,j}}\right)^{\frac{1}{p-j}}$$

and say that *the condition* $(*)$ *holds for* $s \in \mathbb{N}$ if

$$\sup_{p\in\mathbb{N}} \frac{\lambda_{p,s}}{p} \sum_{k=p}^{\infty} \frac{1}{m_{2,k}} < \infty.$$

Now we set up notations used throughout the case of the interval $[-1, 1]$. In the fifth paragraph, we adapt them and treat the $[-1, 1]^m$ setting. In the last one we indicate how to get ultraholomorphic extensions of the jets.

The spaces $\Lambda_{\{\boldsymbol{M}_1\},r}$ *and* $\Lambda_{\{\boldsymbol{M}_1\}}$. Given $r > 0$, $\Lambda_{\{\boldsymbol{M}_1\},r}$ is the following Banach space: its elements are the sequences $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}_0}$ of $\mathbb{C}$ such that

$$|\boldsymbol{a}|_r := \sup_{p\in\mathbb{N}_0} \frac{|a_p|}{r^p M_{1,p}} < \infty$$

and it is endowed with the norm $|\cdot|_r$. We then introduce the Hausdorff (LB)-space $\Lambda_{\{\boldsymbol{M}_1\}}$ as the inductive limit of these Banach spaces.

The spaces $\mathcal{D}_{\{\boldsymbol{M}_2\},r}([-1, 1])$ *and* $\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1, 1])$. Given $r > 0$, $\mathcal{D}_{\{\boldsymbol{M}_2\},r}([-1, 1])$ is the following Banach space: its elements are the complex-valued $\mathcal{E}^\infty$-functions f on $\mathbb{R}$ with support contained in $[-1, 1]$ and such that

$$|f|_r := \sup_{p\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} \frac{|f^{(p)}(x)|}{r^p M_{2,p}} < \infty$$

and it is endowed with the norm $|\cdot|_r$. We then introduce the Hausdorff (LB)-space $\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1, 1])$ as the inductive limit of these Banach spaces.

The space $\Lambda_{(\boldsymbol{M}_1)}$. The Fréchet space $\Lambda_{(\boldsymbol{M}_1)}$ is the vector space of the sequences $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}_0}$ of complex numbers such that

$$\|\boldsymbol{a}\|_r := \sup_{p\in\mathbb{N}_0} \frac{r^p|a_p|}{M_{1,p}} < \infty, \quad \forall r \in \mathbb{N},$$

endowed with the systems of norms $\{\|\cdot\|_r\colon r \in \mathbb{N}\}$.

The space $\mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])$. The Fréchet space $\mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])$ is the vector space of the complex-valued $\mathcal{E}^\infty$-functions f on $\mathbb{R}$ with support contained in $[-1,1]$ and such that

$$\|f\|_r := \sup_{p\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} \frac{r^p|f^{(p)}(x)|}{M_{2,p}} < \infty, \quad \forall r \in \mathbb{N},$$

endowed with the systems of norms $\{\|\cdot\|_r\colon r\in\mathbb{N}\}$.

Main result. The main aim of this paper is to prove the following result, an immediate consequence of Theorems 3.2, 3.5, 4.2 and 4.4.

Theorem 1.1. *The following assertions are equivalent*:

(a) *the condition* $(*)$ *holds for some* $s\in\mathbb{N}$;
(b) $\Lambda_{\{\boldsymbol{M}_1\}} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0}\colon f\in\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])\}$;
(c) $\Lambda_{(\boldsymbol{M}_1)} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0}\colon f\in\mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])\}$.

Motivation. Our interest in this subject comes in particular from the study of [1,2,4,5]. In [5], the case $\boldsymbol{M}_1=\boldsymbol{M}_2$ is thoroughly investigated. The slightly more general case when the sequences $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are replaced by weights is considered in [1] (cf. Theorems 3.6 and 3.7). Théorème 30 of [2] provides in particular the equivalence of the assertions (b) and (c) here above under stronger conditions on the sequences $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$. Commentaires 32 of [2] give a detailed discussion of the literature and ask for explicit constructions as well as for smoother conditions.

In [4], one finds results similar to ours. The method, based on the use of the Fourier transform, is completely different and permits to consider the Whitney case (i.e., to consider jets on a closed subset of $\mathbb{R}^n$ and not only sequences at the origin). However, the conditions imposed on the sequences are stronger—$\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ must satisfy the following condition: there is $C>0$ such that $p!\leqslant C^pM_p$ and $M_{p+1}\leqslant C^pM_p$ for every $p\in\mathbb{N}$. The first part of this condition is equivalent to the boundedness of the sequence $(p/M_p^{1/p})_{p\in\mathbb{N}}$; the second part (known as stability under differential operators) is not required in our development. It is easy to describe sequences $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ verifying the conditions (a)–(e) and not the condition of stability under differential operators: one has just to consider the sequences $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ defined by $m_{1,0}=m_{2,0}=1$ and $m_{1,p}=m_{2,p}=p^p$ for every $p\in\mathbb{N}$. So our results extend those of [5] contrary to [4]. Finally let us mention that Langenbruch has proved that the condition 2.14 of [4] implies the condition $(*)$ (private communication).

It is a pleasure to thank M. Langenbruch for very fruitful discussions.

2. Some information about the sequences $\boldsymbol{m}$ and $\boldsymbol{M}$

Let us gather properties and remarks about sequences $\boldsymbol{m}$ and $\boldsymbol{M}$.

(a) *The inequality* $\lambda_{p,s}\leqslant m_{1,p}\leqslant m_{2,p}$ *holds for every* $p, s\in\mathbb{N}$ since

$$(s^{-p}M_{1,p}/M_{2,j})^{\frac{1}{p-j}} \leqslant (M_{1,p}/M_{1,j})^{\frac{1}{p-j}} = (m_{1,j+1}\dots m_{1,p})^{\frac{1}{p-j}} \leqslant m_{1,p}$$

hold for every $p, s \in \mathbb{N}$ and $j \in \{0, \dots, p-1\}$. Therefore *the condition* $(**)$

$$\sup_{p\in\mathbb{N}} \frac{m_{1,p}}{p} \sum_{k=p}^{\infty} \frac{1}{m_{2,k}} < \infty$$

(to be compared with the condition (γ_1) of [5]) *implies that the condition* $(*)$ *holds for every* $s \in \mathbb{N}$.

(b) Let $\boldsymbol{m} = (m_p)_{p\in\mathbb{N}_0}$ be an increasing sequence of real numbers such that $m_0 = 1$ and consider the sequence $\boldsymbol{M} = (M_p)_{p\in\mathbb{N}_0}$.

(b.1) If we have $\sum_{p=0}^{\infty} 1/m_p < \infty$, it is well known that $p/m_p \to 0$, hence $p/M_p^{1/p} \to 0$. In this case we may set $m_{1,p} = m_{2,p} := m_p$ for every $p \in \mathbb{N}_0$ and consider the sequences $\boldsymbol{M}_1 = \boldsymbol{M}_2 := \boldsymbol{M}$.

(b.2) The following two conditions are regularly considered:

(A) there is $A > 1$ such that $M_p \leqslant A^p M_j M_{p-j}$ for every $p \in \mathbb{N}$ and $j \in \{0, \dots, p\}$;
(B) there is $B > 1$ such that $m_p \leqslant B M_p^{1/p}$ for every $p \in \mathbb{N}$.

Chollet and Thilliez have made us aware that in fact *the condition* (A) *implies the condition* (B) *with* $B = A^2$ since

$$m_p^p \leqslant m_{p+1} \dots m_{2p} = M_{2p}/M_p \leqslant A^{2p} M_p, \quad \forall p \in \mathbb{N}.$$

(c) *If* $\boldsymbol{M}_1$ *verifies the condition* (B), *then the following conditions are equivalent*:

(i) *the condition* $(*)$ *holds for every* $s \in \mathbb{N}$;
(ii) *the condition* $(*)$ *holds for some* $s \in \mathbb{N}$;
(iii) *the condition* $(**)$ *holds.*

(i) ⇒ (ii) is trivial; (iii) ⇒ (i) is known by (a) and (ii) ⇒ (iii) holds since

$$m_{1,p} \leqslant B M_{1,p}^{1/p} \leqslant Bs(s^{-p} M_{1,p}/M_{2,0})^{1/p} \leqslant Bs\lambda_{p,s}, \quad \forall p, s \in \mathbb{N}.$$

(d) Let $\boldsymbol{M} = (M_p)_{p\in\mathbb{N}}$ and $\boldsymbol{M}' = (M'_p)_{p\in\mathbb{N}}$ be two sequences of $\mathbb{R}$.

(d.1) If the sequence $\boldsymbol{M}$ has *moderate growth* (cf. [2, “suite à croissance modérée”]), the sequence $(p!M_p)_{p\in\mathbb{N}_0}$ verifies the conditions imposed on the sequence $\boldsymbol{M}_1$ and may be baptized $\boldsymbol{M}_1$. Let us note that in this case the sequence $\boldsymbol{M}_1$ verifies the condition (A).

(d.2) If the sequence $\boldsymbol{M}'$ is *non-quasi-analytic* (in the sense of [2]), the sequence $(p!M'_p)_{p\in\mathbb{N}_0}$ verifies the conditions (a), (b) and (e) imposed on the sequence $\boldsymbol{M}_2$. If moreover it is *associated* to $\boldsymbol{M}$ (cf. [2]), there is $A_1 \geqslant 1$ such that $m_p \leqslant A_1 m'_p$ for every $p \in \mathbb{N}$ so, up to the factor A_1, it verifies the condition (c) and may be baptized $\boldsymbol{M}_2$.

(d.3) In the context of such sequences $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ let us note that the condition $(**)$ also reads

$$\sup_{p\in\mathbb{N}} \frac{M_p}{M_{p-1}} \sum_{k=p}^{\infty} \frac{M'_{k-1}}{kM'_k} < \infty.$$

Therefore the main Theorem 1.1 leads to the following corollary to be compared with the results of [2].

Corollary 2.1. *If the sequence $\boldsymbol{M}_1$ verifies the condition* (B)*, then the following conditions are equivalent*:

(a) *the condition* $(**)$ *holds*;
(b) $\Lambda_{\{\boldsymbol{M}_1\}} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0}\colon\ f \in \mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])\}$;
(c) $\Lambda_{(\boldsymbol{M}_1)} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0}\colon\ f \in \mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])\}$.

3. Extension theorems in $\mathbb{R}$: Roumieu type case

The following result is an easy consequence of the property 1.3.5 of [3].

Theorem 3.1. *If $\boldsymbol{M} = (M_p)_{p\in\mathbb{N}_0}$ is a sequence of positive numbers such that $M_0 = 1$ and $a := \sum_{p=1}^{\infty} M_{p-1}/M_p < \infty$, then there is $f \in \mathcal{D}([-a,a])$ such that $0 \leqslant f \leqslant 1$, $f^{(j)}(0) = \delta_{j,0}$ and $|f^{(j)}| \leqslant 2^j M_j$ for every $j \in \mathbb{N}_0$.*

Here we follow the ideas of Petzsche [5, Theorem 2.2].

Theorem 3.2. *If the condition $(*)$ holds for some $s \in \mathbb{N}$, then there is $d \in \mathbb{N}$ such that, for every $r \in \mathbb{N}$, there is a continuous linear extension map from $\Lambda_{\{\boldsymbol{M}_1\},r}$ into $\mathcal{D}_{\{\boldsymbol{M}_2\},dr}([-1,1])$, hence*

$$\Lambda_{\{\boldsymbol{M}_1\}} \subset \bigl\{\bigl(f^{(j)}(0)\bigr)_{j\in\mathbb{N}_0}\colon\ f \in \mathcal{D}_{\{\boldsymbol{M}_2\}}\bigl([-1,1]\bigr)\bigr\}.$$

Proof. Let us fix $h > 0$. The main tool of this proof is to consider for every $p \in \mathbb{N}_0$ the sequence

$$\boldsymbol{m} := 1, \underbrace{h\lambda_{p,s}, \ldots, h\lambda_{p,s}}_{2p}, hm_{2,2p+1}, hm_{2,2p+2}, \ldots.$$

The case $p = 0$ is particular. As we have $B := h\sum_{k=1}^{\infty} 1/hm_{2,k} < \infty$, Theorem 3.1 provides $\rho_{h,0} \in \mathcal{D}([-B/h, B/h])$ such that $0 \leqslant \rho_{h,0} \leqslant 1$, $\rho_{h,0}^{(j)}(0) = \delta_{j,0}$ for every $j \in \mathbb{N}_0$ and $|\rho_{h,0}^{(j)}(x)| \leqslant 2^j h^j M_{2,j}$ for every $j \in \mathbb{N}$ and $x \in \mathbb{R}$.

For $p \in \mathbb{N}$, we proceed as follows. As the condition $(*)$ holds for s, there is a constant $A > 1$ such that for every $p \in \mathbb{N}$, we have

$$\frac{2p}{h\lambda_{p,s}} + \sum_{k=2p+1}^{\infty} \frac{1}{hm_{2,k}} = \frac{2p}{h\lambda_{p,s}}\left(1 + \frac{\lambda_{p,s}}{2p}\sum_{k=2p+1}^{\infty}\frac{1}{m_{2,k}}\right) \leqslant \frac{2p}{h\lambda_{p,s}}A.$$

So for every $p \in \mathbb{N}$, Theorem 3.1 provides an element $\rho_{h,p}$ of the space $\mathcal{D}([-2Ap/(h\lambda_{p,s}), 2Ap/(h\lambda_{p,s})])$ verifying $0 \leqslant \rho_{h,p} \leqslant 1$, $\rho_{h,p}^{(j)}(0) = \delta_{j,0}$ for every $j \in \mathbb{N}_0$ and

$$\bigl|\rho_{h,p}^{(j)}(x)\bigr| \leqslant \begin{cases} 2^j h^j \lambda_{p,s}^j & \text{if } 1 \leqslant j \leqslant 2p, \\ 2^j h^j \lambda_{p,s}^{2p} \frac{M_{2,j}}{M_{2,2p}} & \text{if } j > 2p. \end{cases}$$

Now for every $p \in \mathbb{N}_0$, we introduce $\chi_{h,p}(x) := \rho_{h,p}(x)x^p/p! \in \mathcal{E}^\infty(\mathbb{R})$. We need to evaluate the absolute value of the derivatives of these functions.

For $p = 0$, as we have $\chi_{h,0} = \rho_{h,0}$, this is already known.

For $p \in \mathbb{N}$, we are going to prove that we have

$$\left|\chi_{h,p}^{(j)}(x)\right| \leqslant \frac{M_{2,j}}{M_{1,p}}\left(\frac{2Aes}{h}\right)^p h^j\left(2+\frac{1}{2A}\right)^j \tag{1}$$

for every $j \in \mathbb{N}_0$ and $x \in \mathbb{R}$. The Leibniz formula leads immediately to

$$\left|\chi_{h,p}^{(j)}(x)\right| \leqslant \left(\frac{2Ae}{h}\right)^p \sum_{l=\max\{0,j-p\}}^{j}\binom{j}{l}\left|\rho_{h,p}^{(l)}(x)\right|\left(\frac{2A}{h}\right)^{l-j}\frac{1}{\lambda_{p,s}^{p+l-j}}$$

if we note that for such p, j and l, we have $0 \leqslant p+l-j \leqslant p$, hence $p^{p+l-j}/(p+l-j)! \leqslant p^p/p! \leqslant e^p$. To go on further we consider two cases.

Case 1: $p \in \mathbb{N}$ and $0 \leqslant j \leqslant 2p$. In this case we immediately obtain

$$\left|\chi_{h,p}^{(j)}(x)\right| \leqslant \left(\frac{2Ae}{h}\right)^p h^j\left(2+\frac{1}{2A}\right)^j \lambda_{p,s}^{j-p},$$

hence successively

$$\left|\chi_{h,p}^{(j)}(x)\right| \leqslant \frac{M_{2,j}}{M_{1,p}}\left(\frac{2Aes}{h}\right)^p h^j\left(2+\frac{1}{2A}\right)^j \quad \text{if } 0 \leqslant j < p \tag{2}$$

since $\lambda_{p,s}^{p-j} \geqslant M_{1,p}/(s^p M_{2,j})$ for $0 \leqslant j < p$;

$$\left|\chi_{h,p}^{(p)}(x)\right| \leqslant \frac{M_{2,p}}{M_{1,p}}\left(\frac{2Ae}{h}\right)^p h^p\left(2+\frac{1}{2A}\right)^p \quad \text{if } j = p \tag{3}$$

and

$$\left|\chi_{h,p}^{(j)}(x)\right| \leqslant \frac{M_{2,j}}{M_{1,p}}\left(\frac{2Ae}{h}\right)^p h^j\left(2+\frac{1}{2A}\right)^j \quad \text{if } p < j \leqslant 2p \tag{4}$$

if we note that for $p < j \leqslant 2p$, the inequality $\lambda_{p,s} \leqslant m_{2,p}$ leads to

$$\lambda_{p,s}^{j-p} \leqslant m_{2,p}^{j-p} \leqslant m_{2,p+1}\ldots m_{2,j} = M_{2,j}/M_{2,p} \leqslant M_{2,j}/M_{1,p}.$$

Case 2: $p \in \mathbb{N}$ and $2p < j$. If necessary, we decompose the sum $\sum_{l=j-p}^{j}$ into $\sum_{l=j-p}^{2p} + \sum_{l=2p+1}^{j}$ and observe that, as in the case $p < j \leqslant 2p$, the first sum is

$$\leqslant \frac{M_{2,j}}{M_{1,p}}\left(\frac{2Ae}{h}\right)^p h^j \sum_{l=j-p}^{2p}\binom{j}{l}2^l(2A)^{l-j}.$$

In all cases the sum $\sum_{l=\max\{2p+1,j-p\}}^{j}$ is

$$\leqslant \left(\frac{2Ae}{h}\right)^p h^j \sum_{l=\max\{2p+1,j-p\}}^{j}\binom{j}{l}2^l(2A)^{l-j}\lambda_{p,s}^{p+j-l}\frac{M_{2,l}}{M_{2,2p}}$$

with

$$\lambda_{p,s}^{p+j-l}\frac{M_{2,l}}{M_{2,2p}} \leqslant m_{2,p}^{p} m_{2,p}^{j-l} m_{2,2p+1}\dots m_{2,l} \leqslant m_{2,p+1}\dots m_{2,j} \leqslant \frac{M_{2,j}}{M_{1,p}}.$$

Putting these informations together, we end up with

$$\left|\chi_{h,p}^{(j)}(x)\right| \leqslant \frac{M_{2,j}}{M_{1,p}}\left(\frac{2Ae}{h}\right)^{p} h^{j}\left(2+\frac{1}{2A}\right)^{j} \quad \text{if } 2p < j. \tag{5}$$

The inequality (1) summarizes the inequalities (2)–(5).

By use of the value $j = 0$ in the definition of $\lambda_{p,s}$, we get $M_{1,p}^{1/p} \leqslant s\lambda_{p,s}$ for every $p \in \mathbb{N}$, hence $\lim_{p\to\infty} p/\lambda_{p,s} = 0$ by means of the condition (d) imposed on the sequence $\boldsymbol{m}_1$. Therefore there is $l \in \mathbb{N}$ such that $B/l < 1/2$, $2Aes/l < 1/2$ and $2Ap/(l\lambda_{p,s}) < 1$ for every $p \in \mathbb{N}$. So if we fix $h \geqslant l$, we finally arrive with

(a) the support of $\rho_{h,p}$ is contained in $[-1,1]$ for every $p \in \mathbb{N}_0$;

(b) $$\left|\chi_{h,p}^{(j)}(x)\right| \leqslant \frac{M_{2,j}}{M_{1,p}}\left(\frac{2Aes}{h}\right)^{p} h^{j}\left(2+\frac{1}{2A}\right)^{j}$$

for every $p, j \in \mathbb{N}_0$ and $x \in \mathbb{R}$;

(c) $\chi_{h,p}^{(j)}(0) = \delta_{p,j}$ for every $p, j \in \mathbb{N}_0$.

To conclude let us prove that any integer $d > l(2 + 1/(2A))$ fits our statement. Let r be any positive integer. For every $\boldsymbol{a} \in \Lambda_{\{\boldsymbol{M}_1\},r}$, we get

$$\left|a_p\chi_{rl,p}^{(j)}(x)\right| \leqslant |\boldsymbol{a}|_r r^p M_{1,p}\frac{M_{2,j}}{M_{1,p}}\frac{2^{-p}}{r^p}(rl)^j\left(2+\frac{1}{2A}\right)^{j} \leqslant 2^{-p}|\boldsymbol{a}|_r (dr)^j M_{2,j}$$

for every $p, j \in \mathbb{N}_0$ and $x \in \mathbb{R}$, hence

$$\sum_{p=0}^{\infty}\left|a_p\chi_{rl,p}^{(j)}(x)\right| \leqslant 2|\boldsymbol{a}|_r (dr)^j M_{2,j}.$$

Therefore the map $T_r : \Lambda_{\{\boldsymbol{M}_1\},r} \to \mathcal{D}_{\{\boldsymbol{M}_2\},dr}([-1,1])$ defined by $\boldsymbol{a} \mapsto \sum_{p=0}^{\infty} a_p\chi_{rl,p}$ suits our purpose. □

Proposition 3.3. *The inclusion*

$$\Lambda_{\{\boldsymbol{M}_1\}} \subset \left\{\left(f^{(j)}(0)\right)_{j\in\mathbb{N}_0}\colon\ f \in \mathcal{D}_{\{\boldsymbol{M}_2\}}\left([-1,1]\right)\right\}$$

implies that, for every $m \in \mathbb{N}$, there is a continuous linear extension map from $\Lambda_{\{\boldsymbol{M}_1\},m}$ into some $\mathcal{D}_{\{\boldsymbol{M}_2\},r}([-1,1])$.

Proof. For every $j \in \mathbb{N}_0$, $f \mapsto f^{(j)}(0)$ clearly defines a continuous linear functional $\tau^{(j)}$ on $\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])$. Therefore

$$H := \left\{f \in \mathcal{D}_{\{\boldsymbol{M}_2\}}\left([-1,1]\right)\colon\ f^{(j)}(0) = 0,\ \forall j \in \mathbb{N}_0\right\}$$

is a closed vector subspace of $\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])$. Let us denote by ψ the canonical surjection from $\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])$ onto the quotient space $\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])/H$.

By hypothesis, for every $\boldsymbol{a} \in \Lambda_{\{\boldsymbol{M}_1\}}$, there is some $f_{\boldsymbol{a}} \in \mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])$ such that $f_{\boldsymbol{a}}^{(j)}(0) = a_j$ for every $j \in \mathbb{N}_0$. So if we set $\phi(\boldsymbol{a}) := \psi(f_{\boldsymbol{a}})$, ϕ is a well defined linear map from $\Lambda_{\{\boldsymbol{M}_1\}}$ into $\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])/H$.

Let us prove that this map ϕ is continuous. As the spaces $\Lambda_{\{\boldsymbol{M}_1\}}$ and $\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])/H$ are (LB)-spaces, it suffices to prove that its graph is sequentially closed. Let $(\boldsymbol{a}_m)_{m\in\mathbb{N}}$ be a sequence of $\Lambda_{\{\boldsymbol{M}_1\}}$ converging to $\boldsymbol{a}$ and such that the sequence $(\phi(\boldsymbol{a}_m))_{m\in\mathbb{N}}$ converges to $\psi(f)$ in $\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])/H$. For every $j \in \mathbb{N}_0$, we clearly have $a_{m,j} \to a_j$. Moreover, as $\tau^{(j)}$ vanishes on H, $\tau^{(j),\sim}$ is a continuous linear functional on $\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])/H$ such that

$$a_{m,j} = \tau^{(j),\sim}\big(\phi(\boldsymbol{a}_m)\big) \to \tau^{(j),\sim}\big(\psi(f)\big) = f^{(j)}(0).$$

So we have $a_j = f^{(j)}(0)$ for every $j \in \mathbb{N}_0$, i.e., $\phi(\boldsymbol{a}) = \psi(f)$.

Now we apply the localization theorem: for every $m \in \mathbb{N}$, there is $r \in \mathbb{N}$ such that $\phi(\Lambda_{\{\boldsymbol{M}_1\},2m}) \subset \psi(\mathcal{D}_{\{\boldsymbol{M}_2\},r}([-1,1])) =: E$. Let us endow this vector space E with the Banach structure coming from its canonical identification with $\mathcal{D}_{\{\boldsymbol{M}_2\},r}([-1,1])/(H \cap \mathcal{D}_{\{\boldsymbol{M}_2\},r}([-1,1]))$. In this way, the map $\phi\colon \Lambda_{\{\boldsymbol{M}_1\},2m} \to (E, \|\cdot\|)$ is a continuous linear map in between two Banach spaces and there is $C > 0$ such that $\|\phi(\boldsymbol{a})\| \leqslant C|\boldsymbol{a}|_{2m}$ for every $\boldsymbol{a} \in \Lambda_{\{\boldsymbol{M}_1\},2m}$.

Now for every $p \in \mathbb{N}_0$, let $\boldsymbol{e}_p$ be the sequence $(\delta_{p,j})_{j\in\mathbb{N}_0}$. Of course $\boldsymbol{e}_p$ belongs to $\Lambda_{\{\boldsymbol{M}_1\},2m}$ with $|\boldsymbol{e}_p|_{2m} = (2m)^{-p} M_{1,p}^{-1}$ and there is $\chi_p \in \mathcal{D}_{\{\boldsymbol{M}_2\},r}([-1,1])$ such that $\psi(\chi_p) = \phi(\boldsymbol{e}_p)$ with $|\chi_p|_r \leqslant 2\|\phi(\boldsymbol{e}_p)\|$.

Putting these last informations together leads to the following situation. For every $\boldsymbol{a} \in \Lambda_{\{\boldsymbol{M}_1\},m}$ and $p \in \mathbb{N}_0$, we get

$$|a_p \boldsymbol{e}_p|_{2m} \leqslant |\boldsymbol{a}|_m m^p M_{1,p} (2m)^{-p} M_{1,p}^{-1} = 2^{-p} |\boldsymbol{a}|_m,$$

hence

$$|a_p \chi_p|_r \leqslant 2|a_p| \big\|\phi(\boldsymbol{e}_p)\big\| \leqslant 2C|a_p \boldsymbol{e}_p|_{2m} \leqslant 2C2^{-p}|\boldsymbol{a}|_m.$$

Therefore the series $T\boldsymbol{a} := \sum_{p=0}^{\infty} a_p \chi_p$ defines a linear extension map from $\Lambda_{\{\boldsymbol{M}_1\},m}$ into $\mathcal{D}_{\{\boldsymbol{M}_2\},r}([-1,1])$ which is continuous since

$$|T\boldsymbol{a}|_r \leqslant \sum_{p=0}^{\infty} |a_p \chi_p|_r \leqslant 4C|\boldsymbol{a}|_m, \quad \forall \boldsymbol{a} \in \Lambda_{\{\boldsymbol{M}_1\},m}. \qquad \square$$

For the sake of completeness, let us mention the following result.

Lemma 3.4 [3, Lemma 1.3.6]. *Let $m \in \mathbb{N}$. If $a_1, \ldots, a_m$ are positive decreasing numbers with $T \leqslant a_1 + \cdots + a_m$, then for every $f \in \mathcal{E}^m(]-\infty, T])$ vanishing on $]-\infty, 0]$, one has*

$$\big|f(x)\big| \leqslant \sum_{j\in J} 2^{2j} \sup_{y<x} a_1 \ldots a_j \big|f^{(j)}(y)\big|, \quad \forall x \leqslant T,$$

where $J := \{j\colon 1 \leqslant j \leqslant m$ and $a_{j+1} < a_j$ or $j = m\}$.

Here we follow again the ideas of Petzsche as it has been done in [2, Proposition 24].

Theorem 3.5. *If $\Lambda_{\{\boldsymbol{M}_1\}} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0}\colon f \in \mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])\}$, then the condition $(*)$ holds for some $s \in \mathbb{N}$.*

Proof. The preceding proposition provides the existence of a positive integer s and of a continuous linear extension map T from $\Lambda_{\{\boldsymbol{M}_1\},1}$ into $\mathcal{D}_{\{\boldsymbol{M}_2\},s}([-1,1])$. Let us choose $C > \max\{1, \|T\|_\}$ and select $h > 0$ such that $0 < 4hs^2 < 1/2$ and $\sum_{l=0}^{\infty} h/m_{2,l} < 1$.

For every $p \in \mathbb{N}$, we consider the sequence

$$\underbrace{\frac{m_{2,2p}}{h}, \dots, \frac{m_{2,2p}}{h}}_{p}, \frac{m_{2,2p+1}}{h}, \frac{m_{2,2p+2}}{h}, \dots$$

and introduce the following notations: $\boldsymbol{e}_p := (\delta_{p,j})_{j\in\mathbb{N}_0} \in \Lambda_{\{\boldsymbol{M}_1\},1}$, $\chi_p := T\boldsymbol{e}_p$ and $\rho_{p,j}$ is defined by $\rho_{p,j}(x) := 0$ if $x \leqslant 0$ and $\rho_{p,j}(x) := \chi_p^{(j)}(x) - x^{p-j}/(p-j)!$ if $x > 0$ for every $j \in \{0,\dots,p-1\}$. Finally we choose $z \in]0,1[$ such that

$$z < \frac{ph}{m_{2,2p}} + \sum_{l=2p+1}^{\infty} \frac{h}{m_{2,l}}.$$

So everything is set up in order to apply Lemma 3.4: we get

$$\left|\rho_{p,j}(z)\right| \leqslant \sum_{k=p}^{\infty} \frac{(4h)^k}{m_{2,2p}^p m_{2,2p+1} \dots m_{2,p+k}} \left\|\rho_{p,j}^{(k)}\right\|_{[0,z]}$$

with successively for $k \geqslant p$

$$\begin{aligned}\left\|\rho_{p,j}^{(k)}\right\|_{[0,z]} &\leqslant \left\|\chi_p^{(j+k)}\right\|_{[0,z]} + 1 \leqslant |\chi_p|_s s^{j+k} M_{2,j+k} + 1 \\ &\leqslant C|\boldsymbol{e}_p|_1 s^{2k} M_{2,j+k} + 1 \leqslant \frac{2C}{M_{1,p}} s^{2k} M_{2,j+k}\end{aligned}$$

as well as

$$M_{2,j+k} \leqslant M_{2,j} m_{2,p+1} \dots m_{2,p+k} \leqslant M_{2,j} m_{2,2p}^p m_{2,2p+1} \dots m_{2,p+k},$$

hence (by use of the inequalities $0 < 4hs^2 < 1/2$)

$$\left|\rho_{p,j}(z)\right| \leqslant \frac{2C}{M_{1,p}} M_{2,j} \sum_{k=p}^{\infty} (4hs^2)^k \leqslant \frac{2C}{M_{1,p}} M_{2,j} 2^{-p+1}.$$

We now consider the special case $z = \sum_{l=2p+1}^{\infty} h/m_{2,l}$. Given $p \in \mathbb{N}$ and $j \in \{0,\dots,p-1\}$, two possibilities may occur: either

$$\chi_p^{(j)}(z) \leqslant \frac{1}{2} \frac{z^{p-j}}{(p-j)!}$$

which leads to

$$\frac{1}{2} \frac{z^{p-j}}{(p-j)!} \leqslant \frac{z^{p-j}}{(p-j)!} - \chi_p^{(j)}(z) = \left|\rho_{p,j}(z)\right| \leqslant 2C \frac{s^p M_{2,j}}{M_{1,p}},$$

or

$$\chi_p^{(j)}(z) > \frac{1}{2}\frac{z^{p-j}}{(p-j)!}$$

which leads to

$$\frac{1}{2}\frac{z^{p-j}}{(p-j)!} < \chi_p^{(j)}(z) \leqslant |\chi_p|_s s^j M_{2,j} \leqslant C\frac{s^p M_{2,j}}{M_{1,p}}.$$

So in both cases we get

$$\sum_{l=2p+1}^{\infty} \frac{h}{m_{2,l}} = z \leqslant (4C)^{\frac{1}{p-j}}(p-j)!^{\frac{1}{p-j}}\left(\frac{s^p M_{2,j}}{M_{1,p}}\right)^{\frac{1}{p-j}} \leqslant 4Cp\left(\frac{s^p M_{2,j}}{M_{1,p}}\right)^{\frac{1}{p-j}}$$

for every $p \in \mathbb{N}$ and $j \in \{0, \ldots, p-1\}$, hence

$$\frac{\lambda_{p,s}}{p}\sum_{l=2p+1}^{\infty}\frac{1}{m_{2,l}} \leqslant \frac{4C}{h}, \quad \forall p \in \mathbb{N},$$

and finally we arrive at

$$\frac{\lambda_{p,s}}{p}\sum_{l=p}^{\infty}\frac{1}{m_{2,l}} \leqslant \frac{\lambda_{p,s}}{p}\sum_{l=p}^{2p}\frac{1}{m_{2,l}} + \frac{4C}{h} \leqslant \frac{m_{1,p}}{p}\frac{p+1}{m_{2,p}} + \frac{4C}{h} \leqslant 2 + \frac{4C}{h}, \quad \forall p \in \mathbb{N},$$

which concludes the proof. □

Remark. Theorems 3.2 and 3.5 lead easily to the following result due to Petzsche.

Theorem 3.6 [5, Theorem 3.6]. *Let $(m_p)_{p\in\mathbb{N}_0}$ be an increasing sequence of real numbers such that $m_0 = 1$ and $\sum_{p=0}^{\infty} 1/m_p < \infty$. Then the continuous linear restriction map $R : \mathcal{D}_{\{M\}}([-1,1]) \to \Lambda_{\{M\}}$ defined by $f \mapsto (f^{(j)}(0))_{j\in\mathbb{N}_0}$ is surjective if and only if the condition (γ_1) of* [5] *holds.*

Proof. The information (b.1) of the second paragraph tells us that setting $m_{1,p} = m_{2,p} := m_p$ for every $p \in \mathbb{N}_0$ leads to admissible sequences $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$.

If R is surjective, Theorem 3.5 says that the condition $(*)$ holds for some $s \in \mathbb{N}$. If we remark that for every $p \in \mathbb{N}$

$$\lambda_{2p,s} \geqslant (s^{-2p} M_{1,2p}/M_{2,p})^{1/p} = s^{-2}(m_{1,p+1}\ldots m_{1,2p})^{1/p} \geqslant s^{-2} m_{1,p} = s^{-2} m_{2,p},$$

the conclusion is a direct consequence of the fact that, for every $p \in \mathbb{N}$,

$$\frac{m_{1,p}}{p}\sum_{k=p}^{\infty}\frac{1}{m_{2,k}} \leqslant \frac{m_{1,p}}{p}\sum_{k=p}^{2p-1}\frac{1}{m_{2,k}} + s^2\frac{\lambda_{2p,s}}{p}\sum_{k=2p}^{\infty}\frac{1}{m_{2,k}} \leqslant 1 + 2Cs^2.$$

As $\lambda_{p,s} \leqslant m_{1,p}$ for every $p, s \in \mathbb{N}$, the other direction is an immediate consequence of Theorem 3.2. □

4. Extension theorems in $\mathbb{R}$: Beurling type case

For the sake of completeness, let us mention the following result.

Lemma 4.1 [2, Lemme 16]. *Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence of non-negative numbers such that $\sum_{k=1}^{\infty}\alpha_k<\infty$. Let, moreover, $(\beta_k)_{k\in\mathbb{N}}$ and $(\gamma_k)_{k\in\mathbb{N}}$ be sequences of positive numbers such that $\beta_k\to 0$ and $\gamma_k\downarrow 0$. Then there is a sequence $(\lambda_k)_{k\in\mathbb{N}}$ such that $\lambda_k\uparrow\infty$, $\lambda_k\gamma_k\downarrow$, $\lambda_k\beta_k\to 0$ and $\sum_{k=p}^{\infty}\lambda_k\alpha_k\leqslant 8\lambda_p\sum_{k=p}^{\infty}\alpha_k$ for every $p\in\mathbb{N}$.*

Theorem 4.2. *If the condition $(*)$ holds for some $s\in\mathbb{N}$, then*

$$\Lambda_{(\boldsymbol{M}_1)}\subset\left\{\left(f^{(j)}(0)\right)_{j\in\mathbb{N}_0}\colon f\in\mathcal{D}_{(\boldsymbol{M}_2)}\left([-1,1]\right)\right\}.$$

Proof. Let $\boldsymbol{a}$ be any non-zero element of $\Lambda_{(\boldsymbol{M}_1)}$.

For every $r\in\mathbb{N}$, there is $C_r>0$ such that $|a_p|\leqslant C_rM_{1,p}r^{-p}$ for every $p\in\mathbb{N}_0$, hence $(|a_p|/M_{1,p})^{1/p}\leqslant C_r^{1/p}/r$, for every $p\in\mathbb{N}_0$ and $r\in\mathbb{N}$, which implies $(|a_p|/M_{1,p})^{1/p}\to 0$.

Let us set $\epsilon_p:=\sup_{k\geqslant p}(|a_p|/M_{1,p})^{1/p}$ for every $p\in\mathbb{N}$. Of course $(\epsilon_p)_{p\in\mathbb{N}}$ is a decreasing sequence of non-negative numbers such that $\epsilon_p\to 0$ and $|a_p|\leqslant\epsilon_1\ldots\epsilon_pM_{1,p}$ for every $p\in\mathbb{N}$. Let us also set $\alpha_k:=0$, $\beta_k:=\max\{\epsilon_k,k/M_{1,k}^{1/k}\}$ and $\gamma_k:=1/m_{1,k}$ for every $k\in\mathbb{N}$. The preceding lemma provides then a sequence $(\theta_k)_{k\in\mathbb{N}}$ of positive numbers such that $\theta_k\uparrow\infty$, $\theta_k\gamma_k\downarrow$ and $\theta_k\beta_k\to 0$ and we may very well impose the condition $\theta_1=1$. In fact, we have $\theta_k\gamma_k\downarrow 0$ since

$$\theta_k\gamma_k=\frac{\theta_k}{m_{1,k}}\leqslant\frac{\theta_k}{M_{1,k}^{1/k}}\leqslant\theta_k\frac{k}{M_{1,k}^{1/k}}\leqslant\theta_k\beta_k,\quad\forall k\in\mathbb{N}.$$

Now we apply the preceding lemma to the following situation: $\alpha'_k=\gamma'_k:=1/m_{2,k}$ and $\beta'_k:=\max\{1/\theta_{[k/2]}^{1/2},1/m_{2,k}\}$ for every $k\in\mathbb{N}$, where $[k/2]$ denotes the integer part of $k/2$ and $\theta_0:=1$. So we get a sequence $(\theta'_k)_{k\in\mathbb{N}}$ of positive numbers such that $\theta'_k\uparrow\infty$, $\theta'_k\gamma'_k\downarrow$, $\theta'_k\beta'_k\to 0$ and $\sum_{k=p}^{\infty}\theta'_k/m_{2,k}\leqslant 8\theta'_p\sum_{k=p}^{\infty}1/m_{2,k}$ for every $p\in\mathbb{N}_0$, and we may impose $\theta'_1=1$. As $\theta'_k\beta'_k\to 0$, we have $\lim_k\theta'_k/\theta_{[k/2]}^{1/2}=0$ and $\lim_k\theta'_k/m_{2,k}=0$, hence $\theta'_k\gamma'_k\downarrow 0$ and therefore get the existence of a constant $A>1$ such that

$$\theta'_k\leqslant A\theta_{[k/2]}^{1/2}\leqslant A\theta_{[k/2]}\leqslant A\theta_k,\quad\forall k\in\mathbb{N}.\tag{6}$$

Now we introduce the sequences $\boldsymbol{m}'_1$ and $\boldsymbol{m}'_2$ by setting $m'_{1,0}=m'_{2,0}:=1$, $m'_{1,k}:=m_{1,k}/\theta_k$ and $m'_{2,k}:=Am_{2,k}/\theta'_k$ for every $k\in\mathbb{N}$. It is clear that $\boldsymbol{m}'_1$ and $\boldsymbol{m}'_2$ are increasing sequences of positive numbers such that $m'_{1,p}\leqslant m'_{2,p}$ for every $p\in\mathbb{N}_0$, $\sum_{p=0}^{\infty}1/m'_{2,p}<\infty$ and also $p/(M'_{1,p})^{1/p}\to 0$ since

$$\frac{p}{(M'_{1,p})^{1/p}}=\frac{p}{\left(\frac{1}{\theta_1}\cdots\frac{1}{\theta_p}M_{1,p}\right)^{1/p}}\leqslant\frac{p\theta_p}{M_{1,p}^{1/p}}\leqslant\theta_p\beta_p,\quad\forall p\in\mathbb{N}.$$

As the condition $(*)$ holds for $s \in \mathbb{N}$, there is a constant $C > 0$ such that

$$\sum_{k=p}^{\infty} \frac{1}{m_{2,k}} < Cp\left(\frac{s^p M_{2,j}}{M_{1,p}}\right)^{\frac{1}{p-j}}, \quad \forall p \in \mathbb{N},\ \forall j \in \{0,\dots,p-1\}.$$

This leads to

$$\sum_{k=p}^{\infty} \frac{1}{m'_{2,k}} = \frac{1}{A}\sum_{k=p}^{\infty} \frac{\theta'_k}{m_{2,k}} \leqslant 8\theta'_p \sum_{k=p}^{\infty} \frac{1}{m_{2,k}} \leqslant 8p\theta'_p C\left(\frac{s^p m_{2,1}\dots m_{2,j}}{m_{1,1}\dots m_{1,p}}\right)^{\frac{1}{p-j}}$$
$$\leqslant 8p\theta'_p C\left(\frac{s^p \theta'_1\dots\theta'_j M'_{2,j}}{A^j\theta_1\dots\theta_p M'_{1,p}}\right)^{\frac{1}{p-j}} \leqslant 8p\theta'_p C\left(\frac{s^p M'_{2,j}}{\theta_{j+1}\dots\theta_p M'_{1,p}}\right)^{\frac{1}{p-j}}$$

for every $p \in \mathbb{N}$ and $j \in \{0,\dots,p-1\}$. Let us remark that, on one hand, for $j \in \{0,\dots,[p/2]-1\}$, we have

$$(\theta_{j+1}\dots\theta_p)^{\frac{1}{p-j}} \geqslant (\theta_{[p/2]}\dots\theta_p)^{\frac{1}{p-j}} \geqslant (\theta_{[p/2]})^{\frac{p-[p/2]}{p-j}} \geqslant (\theta_{[p/2]})^{\frac{p-[p/2]}{p}} \geqslant \sqrt{\theta_{[p/2]}}$$

and, on the other hand, for $j \in \{[p/2],\dots,p-1\}$,

$$(\theta_{j+1}\dots\theta_p)^{\frac{1}{p-j}} \geqslant \theta_{j+1} \geqslant \theta_{[p/2]}.$$

Therefore we finally get

$$\sum_{k=p}^{\infty} \frac{1}{m'_{2,k}} \leqslant 8ACp\left(\frac{s^p M'_{2,j}}{M'_{1,p}}\right)^{\frac{1}{p-j}}, \quad \forall p \in \mathbb{N},\ \forall j \in \{0,\dots,p-1\},$$

by use of the inequality (6), hence

$$\sup_{p\in\mathbb{N}} \frac{\lambda'_{p,s}}{p} \sum_{k=p}^{\infty} \frac{1}{m'_{2,k}} \leqslant 8AC$$

and we may apply Theorem 3.2 in the $'$-situation: we get

$$\Lambda_{\{\boldsymbol{M}'_1\}} \subset \left\{\left(f^{(j)}(0)\right)_{j\in\mathbb{N}_0}:\ f \in \mathcal{D}_{\{\boldsymbol{M}'_2\}}\big([-1,1]\big)\right\}.$$

Let us consider again the element $\boldsymbol{a}$ of $\Lambda_{(\boldsymbol{M}_1)}$ we started with. As we have

$$|a_p| \leqslant \epsilon_1\dots\epsilon_p M_{1,p} \leqslant \epsilon_1\theta_1\dots\epsilon_p\theta_p(\theta_1\dots\theta_p)^{-1}M_{1,p} \leqslant \epsilon_1\theta_1\dots\epsilon_p\theta_p M'_{1,p}$$

and $\epsilon_p\theta_p \leqslant \theta_p\beta_p$ for every $p \in \mathbb{N}$ with $\theta_p\beta_p \to 0$, there is an integer $p_1 \in \mathbb{N}$ such that $|a_p| \leqslant M'_{1,p}$ for every $p \geqslant p_1$; this implies $\boldsymbol{a} \in \Lambda_{\{\boldsymbol{M}'_1\}}$, hence the existence of $f \in \mathcal{D}_{\{\boldsymbol{M}'_2\}}([-1,1])$ such that $f^{(p)}(0) = a_p$ for every $p \in \mathbb{N}_0$.

Let us investigate this function f. There is $r \in \mathbb{N}$ such that $f \in \mathcal{D}_{\{\boldsymbol{M}'_2\},r}([-1,1])$, hence $K > 0$ such that $|f^{(p)}(x)| \leqslant Kr^p M'_{2,p}$ for every $p \in \mathbb{N}_0$ and $x \in \mathbb{R}$. For every $q \in \mathbb{N}$, as $\theta'_p \uparrow \infty$, there is $p_2 \in \mathbb{N}$ such that $\theta'_p > qrA$ for every $p \geqslant p_2$. Therefore for every $p > p_2$, we get

$$M_{2,p} = m_{2,1}\dots m_{2,p} = A^{-p}\theta'_1\dots\theta'_p m'_{2,1}\dots m'_{2,p}$$
$$\geqslant A^{-p}\theta'_{p_2+1}\dots\theta'_p M'_{2,p} \geqslant A^{-p_2}(qr)^{p-p_2}M'_{2,p}.$$

This leads to $|f^{(p)}(x)| \leqslant K(qrA)^{p_2}q^{-p}M_{2,p}$ for every $p > p_2$, hence $f \in \mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])$ and we conclude. $\square$

Notations. For every $m \in \mathbb{N}$, let us designate by E_m the normed space $(\mathcal{D}_{(\boldsymbol{M}_2)}([-1,1]), \|\cdot\|_m)$ and by F_m its completion.

Definition. Let $m \in \mathbb{N}$. For every $p \in \mathbb{N}_0$, the functional $\tau^{(p)}$ defined on E_m by $\tau^{(p)}(f) = f^{(p)}(0)$ is linear and continuous. Therefore it has a unique continuous linear extension on F_m, that we continue to denote by τ_p. In this way, it makes sense to say that a map $T : \Lambda_{(\boldsymbol{M}_1)} \to F_m$ is an *extension map* if $\tau^{(p)}(T\boldsymbol{a}) = a_p$ for every $\boldsymbol{a} \in \Lambda_{(\boldsymbol{M}_1)}$ and $p \in \mathbb{N}_0$.

Proposition 4.3. *The inclusion*

$$\Lambda_{(\boldsymbol{M}_1)} \subset \left\{ \left(f^{(j)}(0)\right)_{j\in\mathbb{N}_0} \colon f \in \mathcal{D}_{(\boldsymbol{M}_2)}\left([-1,1]\right) \right\}$$

implies that, for every $m \in \mathbb{N}$, there is a continuous linear extension map $T_m : \Lambda_{(\boldsymbol{M}_1)} \to F_m$ such that $T_m\boldsymbol{e}_p \in \mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])$ for every $p \in \mathbb{N}_0$.

Proof. Let m be any element of $\mathbb{N}$. Of course

$$H := \left\{ f \in \mathcal{D}_{(\boldsymbol{M}_2)}\left([-1,1]\right) \colon f^{(j)}(0) = 0,\ \forall j \in \mathbb{N}_0 \right\}$$

is a closed vector subspace of $\mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])$ and of E_m. We designate by ψ the canonical surjection from $\mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])$ onto the quotient space $\mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])/H$. For every $\boldsymbol{a} \in \Lambda_{(\boldsymbol{M}_1)}$, there is by hypothesis an element f of $\mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])$ such that $f^{(p)}(0) = a_p$ for every $p \in \mathbb{N}_0$. Proceeding as in the proof of Proposition 3.3, it is a direct matter that setting $\phi(\boldsymbol{a}) = \psi(f)$ defines ϕ as a continuous linear map from $\Lambda_{(\boldsymbol{M}_1)}$ into $\mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])/H$, hence from $\Lambda_{(\boldsymbol{M}_1)}$ into E_m/H. So there are $r \in \mathbb{N}$ and $A > 0$ such that $\|\phi(\boldsymbol{a})\|_m^{\sim} \leqslant A\|\boldsymbol{a}\|_r$ holds for every $\boldsymbol{a} \in \Lambda_{(\boldsymbol{M}_1)}$.

For every $p \in \mathbb{N}_0$, let χ_p be an element of E_m such that $\psi(\chi_p) = \phi(\boldsymbol{e}_p)$ and $\|\chi_p\|_m \leqslant 2\|\phi(\boldsymbol{e}_p)\|_m^{\sim}$. For every $\boldsymbol{a} \in \Lambda_{(\boldsymbol{M}_1)}$, we then get

$$\left\| \sum_{p=0}^{\infty} a_p\chi_p \right\|_m^{\wedge} \leqslant \sum_{p=0}^{\infty} |a_p|\|\chi_p\|_m \leqslant 2\sum_{p=0}^{\infty} \|\boldsymbol{a}\|_{2r}\frac{M_{1,p}}{(2r)^p}\left\|\phi(\boldsymbol{e}_p)\right\|_m^{\sim}$$

$$\leqslant 2A\|\boldsymbol{a}\|_{2r}\sum_{p=0}^{\infty}\frac{M_{1,p}}{(2r)^p}\|\boldsymbol{e}_p\|_r = 4A\|\boldsymbol{a}\|_{2r}.$$

Therefore the map $T : \Lambda_{(\boldsymbol{M}_1)} \to F_m$ defined by $\boldsymbol{a} \mapsto \sum_{p=0}^{\infty} a_p\chi_p$ suits our purpose. $\square$

Theorem 4.4. *If $\Lambda_{(\boldsymbol{M}_1)} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0} \colon f \in \mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])\}$, then the condition $(*)$ holds for some $s \in \mathbb{N}_0$.*

Proof. The preceding proposition provides a continuous linear extension map $T : \Lambda_{(\boldsymbol{M}_1)} \to F_1$ such that $\chi_p := T\boldsymbol{e}_p$ belongs to $\mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])$ for every $p \in \mathbb{N}_0$. The continuity of

T affords the existence of $s \in \mathbb{N}$ and $C > 1$ such that $\|T\boldsymbol{a}\|_1^{\widehat{}} \leqslant C\|\boldsymbol{a}\|_s$ for every $\boldsymbol{a} \in \Lambda_{(\boldsymbol{M}_1)}$, hence

$$\|\chi_p\|_1 = \|\chi_p\|_1^{\widehat{}} \leqslant C\|\boldsymbol{e}_p\|_s = C\frac{s^p}{M_{1,p}}, \quad \forall p \in \mathbb{N}_0.$$

Now we choose $h > 0$ such that $0 < 4hs < 1/2$ and $\sum_{l=0}^{\infty} h/m_{2,l} < 1$ and consider the increasing sequence

$$\underbrace{\frac{m_{2,2p}}{h}, \ldots, \frac{m_{2,2p}}{h}}_{p}, \frac{m_{2,2p+1}}{h}, \frac{m_{2,2p+2}}{h}, \ldots .$$

We also introduce the functions $\rho_{p,j}$ on $\mathbb{R}$ by $\rho_{p,j}(x) := 0$ if $x \leqslant 0$ and

$$\rho_{p,j}(x) := \chi_p^{(j)}(x) - \frac{x^{p-j}}{(p-j)!} \quad \text{if } x > 0$$

for every $p \in \mathbb{N}$ and $j \in \{0, \ldots, p-1\}$. Then for every $z \in]0,1[$ such that $z < ph/m_{2,2p} + \sum_{l=2p+1}^{\infty} h/m_{2,l}$, Lemma 3.4 leads to

$$\left|\rho_{p,j}(z)\right| \leqslant \sum_{k=p}^{\infty} \frac{(4h)^k}{m_{2,2p}^p m_{2,2p+1} \ldots m_{2,p+k}} \left\|\rho_{p,j}^{(k)}\right\|_{[0,z]}$$

with successively for $k \geqslant p$

$$\begin{aligned} \left\|\rho_{p,j}^{(k)}\right\|_{[0,z]} &\leqslant \left\|\chi_p^{(j+k)}\right\|_{[0,z]} + 1 \leqslant \|\chi_p\|_1 M_{2,j+k} + 1 \\ &\leqslant C\frac{s^p}{M_{1,p}} M_{2,j+k} + 1 \leqslant \frac{2C}{M_{1,p}} s^p M_{2,j+k}. \end{aligned}$$

Proceeding then as in the proof of Theorem 3.5, we obtain

$$\left|\rho_{p,j}(z)\right| \leqslant \frac{2C}{M_{1,p}} M_{2,j} 2^{-p+1},$$

and the more precise value $z = \sum_{l=2p+1}^{\infty} h/m_{2,l}$ leads to

$$\frac{\lambda_{p,s}}{p} \sum_{l=p}^{\infty} \frac{1}{m_{2,l}} \leqslant 2 + \frac{4C}{h}, \quad \forall p \in \mathbb{N}. \qquad \square$$

5. Extension theorems in $\mathbb{R}^m$

Notations. Unless otherwise stated, throughout this paragraph we consider an integer $m \geqslant 2$. Given a multi-index $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{N}_0^m$, $|\nu|$ is equal to $\nu_1 + \cdots + \nu_m$ and we introduce the numbers

$$M_{1,\nu} := M_{1,\nu_1} \ldots M_{1,\nu_m} \quad \text{and} \quad M_{2,\nu} := M_{2,\nu_1} \ldots M_{2,\nu_m}.$$

5.1. Case of the $\{\boldsymbol{M}\}$ spaces

Definitions. Given $r \in \mathbb{N}$, $\Lambda^{(m)}_{\{\boldsymbol{M}_1],r}$ is the following Banach space: its elements are the complex-valued multi-sequences $\boldsymbol{a} = (a_\nu)_{\nu \in \mathbb{N}_0^m}$ such that

$$|\boldsymbol{a}|_r := \sup_{\nu \in \mathbb{N}_0^m} \frac{|a_\nu|}{r^{|\nu|} M_{1,\nu}} < \infty$$

and it is endowed with the norm $|\cdot|_r$. We then introduce the Hausdorff (LB)-space $\Lambda^{(m)}_{\{\boldsymbol{M}_1]}$ as the inductive limit of these Banach spaces.

Given $r \in \mathbb{N}$, $\mathcal{D}_{\{\boldsymbol{M}_2],r}([-1,1]^m)$ is the following Banach space: its elements are the complex-valued functions $f \in \mathcal{E}^\infty(\mathbb{R}^m)$ such that

$$|f|_r := \sup_{\nu \in \mathbb{N}_0^m} \frac{\|\mathrm{D}^\nu f\|_{\mathbb{R}^m}}{r^{|\nu|} M_{2,\nu}} < \infty$$

and it is endowed with the norm $|\cdot|_r$. We then introduce the Hausdorff (LB)-space $\mathcal{D}_{\{\boldsymbol{M}_2]}([-1,1]^m)$ as the inductive limit of these Banach spaces.

Theorem 5.1. *If the condition $(*)$ holds for some $s \in \mathbb{N}$, then there is $d \in \mathbb{N}$ such that, for every $r \in \mathbb{N}$, there is a continuous linear extension map from $\Lambda^{(m)}_{\{\boldsymbol{M}_1],r}$ into $\mathcal{D}_{\{\boldsymbol{M}_2],dr}([-1,1]^m)$.*

Proof. We start as in the proof of Theorem 3.2 until we consider an integer $d > l(2 + 1/(2A))$, i.e., shortly after the inequality (5). Then we proceed as follows.

For every $h \geqslant l$, $\pi = (\pi_1, \ldots, \pi_m) \in \mathbb{N}_0^m$ and $(x_1, \ldots, x_m) \in \mathbb{R}^m$, let us then set $\chi_{h,\pi}(x) = \chi_{h,\pi_1}(x_1) \ldots \chi_{h,\pi_m}(x_m)$; of course these functions $\chi_{h,\pi}$ belong to $\mathcal{D}([-1,1]^m)$. Moreover, for every $\nu \in \mathbb{N}_0^m$, we have $\mathrm{D}^\nu \chi_{h,\pi}(0) = \delta_{\nu,\pi}$ and

$$\left|\mathrm{D}^\nu \chi_{h,\pi}(x)\right| \leqslant \frac{M_{2,\nu}}{M_{1,\pi}} (2Aes/h)^{|\pi|} h^{|\nu|} \bigl(2 + 1/(2A)\bigr)^{|\nu|}, \quad \forall x \in \mathbb{R}^m.$$

Let r be any positive integer. To every $\boldsymbol{a} \in \Lambda^{(m)}_{\{\boldsymbol{M}_1],r}$, we associate the series $T_r\boldsymbol{a} := \sum_{\pi \in \mathbb{N}_0^m} a_\pi \chi_{lr,\pi}$. As for every $\pi, \nu \in \mathbb{N}_0^m$ and $x \in \mathbb{R}^m$, we have

$$\begin{aligned} \left|a_\pi \mathrm{D}^\nu \chi_{lr,\pi}(x)\right| &\leqslant |\boldsymbol{a}|_r M_{2,\nu} (2Aes/l)^{|\pi|} (lr)^{|\nu|} \bigl(2 + 1/(2A)\bigr)^{|\nu|} \\ &\leqslant |\boldsymbol{a}|_r 2^{-|\pi|} (dr)^{|\nu|} M_{2,\nu}, \end{aligned}$$

this series converges in the space $\mathcal{D}_{\{\boldsymbol{M}_2],dr}([-1,1]^m)$ and T_r appears as a continuous linear extension map from $\Lambda^{(m)}_{\{\boldsymbol{M}_1],r}$ into $\mathcal{D}_{\{\boldsymbol{M}_2],dr}([-1,1]^m)$. □

Definitions. Given $r \in \mathbb{N}$, $\Lambda^{(m)}_{\{\boldsymbol{M}_1\},r}$ is the following Banach space: its elements are the complex-valued multi-sequences $\boldsymbol{a} = (a_\nu)_{\nu \in \mathbb{N}_0^m}$ such that

$$|\boldsymbol{a}|'_r := \sup_{\nu \in \mathbb{N}_0^m} \frac{|a_\nu|}{r^{|\nu|} M_{1,|\nu|}} < \infty$$

and it is endowed with the norm $|\cdot|'_r$. We then introduce the Hausdorff (LB)-space $\Lambda^{(m)}_{\{\boldsymbol{M}_1\}}$ as the inductive limit of these Banach spaces.

In a similar way, we introduce the Banach spaces $\mathcal{D}_{\{\boldsymbol{M}_2\},r}([-1,1]^m)$ for every $r \in \mathbb{N}$ and the Hausdorff (LB)-space $\mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1]^m)$.

Theorem 5.2. *The following assertions are equivalent*:

(a) *the condition* $(*)$ *holds for some* $s \in \mathbb{N}$;
(b) $\Lambda^{(m)}_{\{\boldsymbol{M}_1]} \subset \{(f^{(\nu)})_{\nu\in\mathbb{N}_0^m}\colon\ f \in \mathcal{D}_{\{\boldsymbol{M}_2]}([-1,1]^m)\}$;
(c) $\Lambda^{(m)}_{\{\boldsymbol{M}_1]} \subset \{(f^{(\nu)})_{\nu\in\mathbb{N}_0^m}\colon\ f \in \mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1]^m)\}$.

Proof. (a) $\Rightarrow$ (b) is known by Theorem 5.1 and (b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a) To every $\boldsymbol{a} \in \Lambda_{\{\boldsymbol{M}_1\}}$, let us associate the multi-sequence $\boldsymbol{b} = (b_\nu)_{\nu\in\mathbb{N}_0^m}$ defined by $b_\nu = a_{\nu_1}$ if $\nu_2 = \cdots = \nu_m = 0$ and $b_\nu = 0$ otherwise. As $\boldsymbol{a}$ belongs to $\Lambda_{\{\boldsymbol{M}_1\}}$, there is $r \in \mathbb{N}$ such that $\boldsymbol{a} \in \Lambda_{\{\boldsymbol{M}_1\},r}$, hence

$$|\boldsymbol{b}|_r = \sup_{\nu\in\mathbb{N}_0^m} \frac{|b_\nu|}{r^{|\nu|} M_{1,\nu}} = \sup_{p\in\mathbb{N}_0} \frac{|a_p|}{r^p M_{1,p}} < \infty,$$

i.e., $\boldsymbol{b}$ belongs to $\Lambda^{(m)}_{\{\boldsymbol{M}_1],r}$. Therefore there is $f \in \mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1]^m)$ verifying $\mathrm{D}^\nu f(0) = b_\nu$ for every $\nu \in \mathbb{N}_0^m$ and for which there are $s \in \mathbb{N}$ and $C > 0$ such that $|\mathrm{D}^\nu f(x)| \leqslant Cs^{|\nu|} M_{2,|\nu|}$ for every $\nu \in \mathbb{N}_0^m$ and $x \in \mathbb{R}^m$. Now we define the function g on $\mathbb{R}$ by $g(x) := f(x,0,\ldots,0)$ for every $x \in \mathbb{R}$. Of course g belongs to $\mathcal{E}^\infty(\mathbb{R})$ and, for every $p \in \mathbb{N}_0^m$, setting $\nu = (p,0,\ldots,0)$ leads to

$$\left|g^{(p)}(x)\right| = \left|\mathrm{D}^\nu f(x,0,\ldots,0)\right| \leqslant Cs^{|\nu|} M_{2,|\nu|} = Cs^p M_{2,p}, \quad \forall x \in \mathbb{R},$$

i.e., $g \in \mathcal{D}_{\{\boldsymbol{M}_2\},s}([-1,1])$. As we clearly have $g^{(p)}(0) = a_p$ for every $p \in \mathbb{N}_0$, we have arrived at $\Lambda_{\{\boldsymbol{M}_1\}} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0}\colon\ f \in \mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1])\}$ and we conclude by Theorem 1.1. □

5.2. Case of the $(\boldsymbol{M}]$ *spaces*

Definitions. The Fréchet space $\Lambda^{(m)}_{(\boldsymbol{M}_1]}$ is the vector space of the complex-valued multi-sequences $\boldsymbol{a} = (a_\nu)_{\nu\in\mathbb{N}_0^m}$ such that

$$\|\boldsymbol{a}\|_r := \sup_{\nu\in\mathbb{N}_0^m} \frac{r^{|\nu|}|a_\nu|}{M_{1,\nu}} < \infty, \quad \forall r \in \mathbb{N},$$

and it is endowed with the countable system of norms $\{\|\cdot\|_r\colon\ r \in \mathbb{N}\}$.

The Fréchet space $\mathcal{D}_{(\boldsymbol{M}_2]}([-1,1]^m)$ is the vector space of the complex-valued $\mathcal{E}^\infty$-functions on $\mathbb{R}^m$ with support contained in $[-1,1]^m$ verifying

$$\|f\|_r := \sup_{\nu\in\mathbb{N}_0^m} \frac{r^{|\nu|}\|\mathrm{D}^\nu f\|_{\mathbb{R}^m}}{M_{2,\nu}} < \infty, \quad \forall r \in \mathbb{N},$$

and it is endowed with the countable system of norms $\{\|\cdot\|_r\colon\ r \in \mathbb{N}\}$.

The Fréchet space $\Lambda^{(m)}_{(\boldsymbol{M}_1)}$ is the vector space of the complex-valued multi-sequences $\boldsymbol{a} = (a_\nu)_{\nu\in\mathbb{N}_0^m}$ such that

$$\|\boldsymbol{a}\|'_r := \sup_{\nu\in\mathbb{N}_0^m} \frac{r^{|\nu|}|a_\nu|}{M_{1,|\nu|}} < \infty, \quad \forall r \in \mathbb{N},$$

endowed with the countable system of norms $\{\|\cdot\|'_r\colon r \in \mathbb{N}\}$.

In a similar way we introduce the Fréchet space $\mathcal{D}_{(\boldsymbol{M}_2)}([-1,1]^m)$.

Theorem 5.3. *The following assertions are equivalent*:

(a) *the condition* $(*)$ *holds for some* $s \in \mathbb{N}$;
(b) $\Lambda^{(m)}_{(\boldsymbol{M}_1]} \subset \{(f^{(\nu)}(0))_{\nu\in\mathbb{N}_0^m}\colon f \in \mathcal{D}_{(\boldsymbol{M}_2]}([-1,1]^m)\}$;
(c) $\Lambda^{(m)}_{(\boldsymbol{M}_1]} \subset \{(f^{(\nu)}(0))_{\nu\in\mathbb{N}_0^m}\colon f \in \mathcal{D}_{(\boldsymbol{M}_2)}([-1,1]^m)\}$.

Proof. (a) ⇒ (b) Let $\boldsymbol{a}$ be any non-zero element of $\Lambda^{(m)}_{(\boldsymbol{M}_1]}$.

As we have $(|a_\nu|/M_{1,\nu})^{1/|\nu|} \leqslant \|\boldsymbol{a}\|_r^{1/|\nu|}/r$ for every $r \in \mathbb{N}$ and $\nu \in \mathbb{N}_0^m$, we get $\lim_{|\nu|\to\infty}(|a_\nu|/M_{1,\nu})^{1/|\nu|} = 0$. So setting $\epsilon_p^m := \sup_{|\nu|\geqslant p}(|a_\nu|/M_{1,\nu})^{1/|\nu|}$ for every $p \in \mathbb{N}$ leads to a sequence $(\epsilon_p)_{p\in\mathbb{N}}$ of non-negative numbers, decreasing to 0 and such that $|a_\nu| \leqslant \epsilon_1^m \ldots \epsilon_{|\nu|}^m M_{1,\nu}$ for every $\nu \in \mathbb{N}_0^m$ and $|\nu| \geqslant 1$.

With this sequence $(\epsilon_p)_{p\in\mathbb{N}}$ in mind, we can reproduce the argument of the proof of Theorem 4.2 and get sequences $(\theta_k)_{k\in\mathbb{N}}$, $(\theta'_k)_{k\in\mathbb{N}}$, $(M'_{1,k})_{k\in\mathbb{N}}$ and $(M'_{2,k})_{k\in\mathbb{N}}$. Let us moreover set $\theta_0 = \theta'_0 := 1$.

Then for every $\nu \in \mathbb{N}_0^m$, we have $M_{1,\nu_j} = \theta_0 \ldots \theta_{\nu_j} M'_{1,\nu_j}$ for every $j \in \{1,\ldots,m\}$, hence $M_{1,\nu} = M'_{1,\nu}\prod_{j=1}^m(\theta_0 \ldots \theta_{\nu_j})$ and in the same way $M_{2,\nu} = A^{-|\nu|}M'_{2,\nu}\prod_{j=1}^m(\theta'_0 \ldots \theta'_{\nu_j})$. This leads to

$$\begin{aligned}|a_\nu| &\leqslant (\theta_1\epsilon_1 \ldots \theta_{|\nu|}\epsilon_{|\nu|})^m M_{1,\nu}(\theta_1 \ldots \theta_{|\nu|})^{-m} \\ &\leqslant (\theta_1\epsilon_1 \ldots \theta_{|\nu|}\epsilon_{|\nu|})^m M_{1,\nu}\prod_{j=1}^m(\theta_1 \ldots \theta_{\nu_j})^{-1} \leqslant (\theta_1\epsilon_1 \ldots \theta_{|\nu|}\epsilon_{|\nu|})^m M'_{1,\nu}.\end{aligned}$$

As $\theta_p\epsilon_p \to 0$, there is $p_1 \in \mathbb{N}$ such that $|a_\nu| \leqslant M'_{1,\nu}$ for every $\nu \in \mathbb{N}_0^m$ such that $|\nu| \geqslant p_1$. This implies $\boldsymbol{a} \in \Lambda^{(m)}_{\{\boldsymbol{M}'_1]}$ and, by Theorem 5.2, there is $f \in \mathcal{D}_{\{\boldsymbol{M}'_2]}([-1,1]^m)$ such that $\mathrm{D}^\nu f(0) = a_\nu$ for every $\nu \in \mathbb{N}_0^m$. In particular, there is $r \in \mathbb{N}$ such that $f \in \mathcal{D}_{\{\boldsymbol{M}'_2],r}([-1,1]^m)$.

To conclude, let us prove that f belongs also to $\mathcal{D}_{(\boldsymbol{M}_2]}([-1,1]^m)$. Indeed, for every $q \in \mathbb{N}$, as $\theta'_p \uparrow \infty$, there is $p_2 \in \mathbb{N}$ such that $\theta'_p > (rqA)^m$ for every $p \geqslant p_2$. So for every $\nu \in \mathbb{N}_0^m$ such that $[|\nu|/m] \geqslant p_2 + 1$, we successively get

$$\begin{aligned}M_{2,\nu} &\geqslant A^{-|\nu|}\theta'_1 \ldots \theta'_{[|\nu|/m]}M'_{2,\nu} \geqslant A^{-|\nu|}\theta'_{p_2+1} \ldots \theta'_{[|\nu|/m]}M'_{2,\nu} \\ &\geqslant A^{-|\nu|}(\theta'_{p_2+1})^{[|\nu|/m]-p_2}M'_{2,\nu} \geqslant A^{-m(p_2+1)}(rq)^{|\nu|-(p_2+1)m}M'_{2,\nu}.\end{aligned}$$

Therefore we have obtained

$$\|\mathrm{D}^\nu f\|_{\mathbb{R}^m} \leqslant |f|_r r^{|\nu|}M'_{2,\nu} \leqslant |f|_r(rqA)^{m(p_2+1)}q^{-|\nu|}M_{2,\nu}$$

for all such ν's, hence $f \in \mathcal{D}_{\{\boldsymbol{M}_2],q}([-1,1]^m)$.

(b) $\Rightarrow$ (c) is trivial since $\mathcal{D}_{(\boldsymbol{M}_2]}([-1,1]^m) \subset \mathcal{D}_{(\boldsymbol{M}_2)}([-1,1]^m)$.

(c) $\Rightarrow$ (a) To every $\boldsymbol{a} \in \Lambda_{(\boldsymbol{M}_1)}$, let us associate the multi-sequence $\boldsymbol{b} = (b_\nu)_{\nu\in\mathbb{N}_0^m}$ defined by $b_\nu = a_{\nu_1}$ if $\nu_2 = \cdots = \nu_m = 0$ and $b_\nu = 0$ otherwise. For every $r \in \mathbb{N}$, we then have

$$\|\boldsymbol{b}\|_r = \sup_{\nu\in\mathbb{N}_0^m} \frac{r^{|\nu|}|b_\nu|}{M_{1,\nu}} = \sup_{p\in\mathbb{N}_0} \frac{r^p|a_p|}{M_{1,p}} = \|\boldsymbol{a}\|_r,$$

i.e., $\boldsymbol{b} \in \Lambda^{(m)}_{(\boldsymbol{M}_1]}$. Therefore there is $f \in \mathcal{D}_{(\boldsymbol{M}_2)}([-1,1]^m)$ such that $\mathrm{D}^\nu f(0) = b_\nu$ for every $\nu \in \mathbb{N}_0^m$. Now we define the function g on $\mathbb{R}$ by $g(x) := f(x,0,\ldots,0)$ for every $x \in \mathbb{R}$. Of course g belongs to $\mathcal{E}(\mathbb{R}^m)$ and, for every $p \in \mathbb{N}_0$, setting $\nu = (p,0,\ldots,0) \in \mathbb{N}_0^m$ leads to $|g^{(p)}(x)| = |\mathrm{D}^\nu f(x,0,\ldots,0)| \leqslant \|f\|'_r r^{-p} M_{2,p}$ for every $r \in \mathbb{N}$ and $p \in \mathbb{N}_0$, i.e., $g \in \mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])$. As we clearly have $g^{(p)}(0) = a_p$ for every $p \in \mathbb{N}_0$, we have obtained the inclusion $\Lambda_{(\boldsymbol{M}_1)} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0}\colon f \in \mathcal{D}_{(\boldsymbol{M}_2)}([-1,1])\}$ and we conclude at once by Theorem 1.1. $\square$

5.3. A consequence of condition (A)

Proposition 5.4. *If the sequence $\boldsymbol{M}_1$ verifies condition* (A)*, then the inequalities $M_{1,\nu} \leqslant M_{1,|\nu|} \leqslant A^{m|\nu|} M_{1,\nu}$ hold for every $\nu \in \mathbb{N}_0^m$, hence the equalities $\Lambda^{(m)}_{\{\boldsymbol{M}_1]} = \Lambda^{(m)}_{\{\boldsymbol{M}_1\}}$ and $\Lambda^{(m)}_{(\boldsymbol{M}_1]} = \Lambda^{(m)}_{(\boldsymbol{M}_1)}$ hold for these locally convex spaces.*

Proof. For every $\nu \in \mathbb{N}_0^m$, the inequality $M_{1,\nu} \leqslant M_{1,|\nu|}$ is clear and from $M_{1,|\nu|} \leqslant A^{|\nu|} M_{1,\nu_1+\cdots+\nu_{m-1}} M_{1,\nu_m}$, we deduce $M_{1,|\nu|} \leqslant A^{m|\nu|} M_{1,\nu_1} \ldots M_{1,\nu_m} = A^{m|\nu|} M_{1,\nu}$. $\square$

So if we recall that condition (A) implies condition (B) which in turn leads to the part (c) of Section 2, Theorems 5.2 and 5.3 provide the following result to be compared with those of [2].

Theorem 5.5. *If the sequence $\boldsymbol{M}_1$ verifies condition* (A)*, then the following conditions are equivalent*:

(a) *the condition* $(**)$ *holds*;
(b) $\Lambda^{(m)}_{\{\boldsymbol{M}_1\}} \subset \{(f^{(\nu)}(0))_{\nu\in\mathbb{N}_0^m}\colon f \in \mathcal{D}_{\{\boldsymbol{M}_2]}([-1,1]^m)\}$;
(c) $\Lambda^{(m)}_{\{\boldsymbol{M}_1\}} \subset \{(f^{(\nu)}(0))_{\nu\in\mathbb{N}_0^m}\colon f \in \mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1]^m)\}$;
(d) $\Lambda^{(m)}_{(\boldsymbol{M}_1)} \subset \{(f^{(\nu)}(0))_{\nu\in\mathbb{N}_0^m}\colon f \in \mathcal{D}_{(\boldsymbol{M}_2]}([-1,1]^m)\}$;
(e) $\Lambda^{(m)}_{(\boldsymbol{M}_1)} \subset \{(f^{(\nu)}(0))_{\nu\in\mathbb{N}_0^m}\colon f \in \mathcal{D}_{(\boldsymbol{M}_2)}([-1,1]^m)\}$.

6. Ultraholomorphic extension

It is possible to replace the $\mathcal{D}$ spaces by spaces of complex-valued $\mathcal{E}^\infty$-functions on $\mathbb{R}^m$ with ultraholomorphic extension on some open neighbourhood of $\mathbb{R}^m \setminus \{0\}$ in $\mathbb{C}^m$. In particular, such functions are real-analytic on $\mathbb{R}^m \setminus \{0\}$. The key results to get such properties are Theorems 4.3 and 6.2 of [6]: they lead immediately to results such as the following ones.

Theorem 6.1. (a) *The following assertions are equivalent*:

(a) *the condition* $(*)$ *holds for some* $s \in \mathbb{N}$;
(b) $\Lambda_{\{\boldsymbol{M}_1\}} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0}\colon f \in \mathcal{F}_\infty\{\boldsymbol{M}_2, D_{\mathbb{R}\setminus\{0\}}\}\}$;
(c) $\Lambda_{(\boldsymbol{M}_1)} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0}\colon f \in \mathcal{F}_\infty(\boldsymbol{M}_2, D_{\mathbb{R}\setminus\{0\}})\}$.

(b) *The following assertions are equivalent*:

(a) *the condition* $(*)$ *holds for some* $s \in \mathbb{N}$;
(b) $\Lambda^{(m)}_{\{\boldsymbol{M}_1]} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0}\colon f \in \mathcal{F}_\infty\{\boldsymbol{M}_2, D_{\mathbb{R}^m\setminus\{0\}}\}\}$;
(c) $\Lambda^{(m)}_{(\boldsymbol{M}_1]} \subset \{(f^{(j)}(0))_{j\in\mathbb{N}_0}\colon f \in \mathcal{F}_\infty(\boldsymbol{M}_2, D_{\mathbb{R}^m\setminus\{0\}})\}$.

Let us explain this in the Beurling case; the Roumieu case can be treated in a very analogous way.

Given a sequence such as $\boldsymbol{M}_2$, one finds in [6] a construction associating to every proper open subset Ω of $\mathbb{R}^m$, an open subset D_Ω of $\mathbb{C}^m$ verifying in particular the following properties: $D_\Omega \cap \mathbb{R}^m = \Omega$ and $(u + iv \in D_\Omega \Rightarrow u \in \Omega$ and $|v| < d(u, \partial\Omega))$. The Fréchet space $\mathrm{C}(\boldsymbol{M}_2, \Omega)$ is the vector space of the complex-valued $\mathcal{E}^\infty$-functions f on Ω such that

$$\|f\|_r := \sup_{\alpha\in\mathbb{N}_0^m} \frac{2^{(r+1)|\alpha|}\|\mathrm{D}^\alpha f\|_\Omega}{M_{2,|\alpha|}} < \infty, \quad \forall r \in \mathbb{N},$$

endowed with the system of norms $\{\|\cdot\|_r\colon r \in \mathbb{N}\}$. Given a proper open subset U of $\mathbb{C}^m$, the Fréchet space $\mathcal{H}_\infty(\boldsymbol{M}_2, U)$ is the vector space of the holomorphic functions g on U such that

$$\|g\|_r := \sup_{\alpha\in\mathbb{N}_0^m} \frac{2^{(r+1)|\alpha|}\|\mathrm{D}^\alpha g\|_\Omega}{M_{2,|\alpha|}} < \infty, \quad \forall r \in \mathbb{N},$$

endowed with the system of norms $\{|\cdot|_r\colon r \in \mathbb{N}\}$.

Then one can establish the following result.

Theorem 6.2 [6, Theorem 4.3]. *For every proper open subset Ω of $\mathbb{R}^m$, there is a continuous linear map T_Ω from $\mathrm{C}(\boldsymbol{M}_2, \Omega)$ into $\mathcal{H}_\infty(\boldsymbol{M}_2, D_\Omega)$ such that for every $f \in \mathrm{C}(\boldsymbol{M}_2, \Omega)$, $\varepsilon > 0$ and $s \in \mathbb{N}$, there is a compact subset K of Ω such that $|\mathrm{D}^\alpha(T_\Omega f)(u + iv) - \mathrm{D}^\alpha f(u)| \leqslant \varepsilon$ for every $u + iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$ verifying $u \in \Omega \setminus K$ and $|\alpha| \leqslant s$.*

Given a proper open subset U of $\mathbb{C}^m$, we designate by $\mathcal{F}(U)$ the vector space of the functions f defined on $\mathbb{R}^m \cup U$ verifying $f|_{\mathbb{R}^m} \in \mathcal{E}^\infty(\mathbb{R}^m)$, $f|_U \in \mathcal{H}(U)$ and such that

$\lim_{z\to x} \mathrm{D}^\alpha(f|_U)(z) = \mathrm{D}^\alpha(f|_{\mathbb{R}^m})(x)$ for every $\alpha \in \mathbb{N}_0^n$ and $x \in \partial_{\mathbb{R}^m}(\mathbb{R}^m \cap U)$. The Fréchet space $\mathcal{F}_\infty(\boldsymbol{M}_2, U)$ is then the vector space of the elements f of $\mathcal{F}(U)$ such that

$$\|f\|_r := \sup_{\alpha\in\mathbb{N}_0^m} \frac{2^{(r+1)|\alpha|}\|\mathrm{D}^\alpha f\|_{\mathbb{R}^m\cup U}}{M_{|\alpha|}} < \infty, \quad \forall r \in \mathbb{N},$$

endowed with the fundamental system of norms $\{\|\cdot\|_r\colon r \in \mathbb{N}\}$.

The use of these informations leads directly to the equivalence (a) $\Leftrightarrow$ (c) of the announced results: if $f \in \mathcal{D}_{\{\boldsymbol{M}_2\}}([-1,1]^m)$ extends $\boldsymbol{a}$, g defined on $\mathbb{R}^m \cup D_{\mathbb{R}^m\setminus\{0\}}$ by $g(0) = a_0$ and $g(z) := T_{\mathbb{R}^m\setminus\{0\}}(f|_{\mathbb{R}^m\setminus\{0\}})(z)$ otherwise belongs to $\mathcal{F}_\infty(\boldsymbol{M}_2, D_{\mathbb{R}^m\setminus\{0\}})$ and extends $\boldsymbol{a}$ too.

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