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Refined energy inequality with application to well-posedness for the fourth order nonlinear Schrödinger type equation on torus

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ABSTRACT

We consider the time local and global well-posedness for the fourth order nonlinear Schrödinger type equation (4NLS) on the torus. The nonlinear term of (4NLS) contains the derivatives of unknown function and this prevents us to apply the classical energy method. To overcome this difficulty, we introduce the modified energy and derive an a priori estimate for the solution to (4NLS).

1. Introduction

We consider the fourth order nonlinear Schrödinger type equation (4NLS) on the torus $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$:

$$\begin{cases} i\partial_t\psi + \partial_x^2\psi + \nu\partial_x^4\psi = \mathcal{N}(\psi, \overline{\psi}, \partial_x\psi, \partial_x\overline{\psi}, \partial_x^2\psi, \partial_x^2\overline{\psi}), \\ \psi(0,x) = \phi(x), \quad x \in \mathbf{T}, \end{cases} \tag{1}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\psi : \mathbf{R}\times\mathbf{T} \to \mathbf{C}$ is an unknown function, and $\phi : \mathbf{T} \to \mathbf{C}$ is a given function. The nonlinear term $\mathcal{N}$ is given by

$$\begin{aligned} &\mathcal{N}(\psi, \overline{\psi}, \dots, \partial_x^2\psi, \partial_x^2\overline{\psi}) \\ &\quad = \lambda_1|\psi|^2\psi + \lambda_2|\psi|^4\psi + \lambda_3(\partial_x\psi)^2\overline{\psi} + \lambda_4|\partial_x\psi|^2\psi + \lambda_5\psi^2\partial_x^2\overline{\psi} + \lambda_6|\psi|^2\partial_x^2\psi, \end{aligned}$$

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where $\nu \neq 0$ and λ_j, $j = 1, \ldots, 6$, are real constants. Eq. (1) arises in the context of a motion of vortex filament. More precisely, using the localized induction approximation, Da Rios [5] proposed some equation which approximates the three-dimensional motion of an isolated vortex filament embedded in an inviscid incompressible fluid filling an infinite region. The Da Rios equation is reduced to the cubic nonlinear Schrödinger equation

$$i\partial_t \psi + \partial_x^2 \psi = -\frac{1}{2}|\psi|^2\psi, \quad (t,x) \in \mathbf{R} \times \mathbf{T}, \tag{2}$$

via the Hasimoto transform [9]. To describe the motion of actual vortex filament more precisely, some detailed models taking into account the effect from higher order corrections of equation have been introduced by Fukumoto and Moffatt [7]. The Fukumoto–Moffatt equation is rewritten as (1) by using the Hasimoto transform. For the physical background of (1), see Fukumoto and Moffatt [7].

In this paper we consider the time local well-posedness for (1) on the Sobolev spaces $H^m(\mathbf{T})$. Our notion of well-posedness contains the existence and uniqueness of the solution and the continuity of the data-to-solution map. We also consider the persistent property of the solution, that is, the solution describes a continuous curve in $H^m(\mathbf{T})$ whenever $\phi \in H^m(\mathbf{T})$. Our motivation to consider the time local well-posedness for (1) is that we are interested in the stability of the standing wave solution $\psi(t,x) = e^{i\omega t}\varphi_\omega(x)$ to (1). When (1) is completely integrable (see the later half of this section below for the details), (1) has a sech-type standing wave solution. The orbital stability in $H^m(\mathbf{R})$ of the sech-type standing wave solution is proved in [20]. On the other hand we easily see that (1) has an exact periodic standing wave solution of the form $\psi(t,x) = \kappa e^{i\tau x + i\omega t}$ for some real constants κ, τ and ω. It is interesting that the sech-type standing wave and the periodic standing wave correspond to the tornado like curve and the helicoid curve in the motion of the vortex filament, see Kida [16].

For the first step to show the orbital stability of the sech-type and the periodic standing wave, we need to prove the global well-posedness for (1) in the Sobolev spaces on the real line $\mathbf{R}$ and on the torus $\mathbf{T}$, respectively. Concerning the local well-posedness of (1) on real line $\mathbf{R}$, in [23,11,24,12,25] it was shown that the initial value problem of (1) is locally well-posed in Sobolev space $H^s(\mathbf{R})$ with $s > 1/2$ by using the Fourier restriction method introduced by Bourgain [3] and Kenig, Ponce, and Vega [14,15]. As far as we know, there is no result on the well-posedness of (1) under the periodic boundary condition.

In this paper we focus on the well-posedness of (1) on the torus. There is a large literature on the well-posedness for the dispersive equations in the torus. See for instance [6,13,19,21] for the linear dispersive equations and [1,3,4,8,10,22,26,27] for the nonlinear dispersive equations. We summarize the well-posedness on the derivative nonlinear Schrödinger equation with the periodic boundary condition. Tsutsumi and Fukuda [26,27] proved the local and global well-posedness for the Schrödinger equation with some nonlinearity on the torus by using the classical energy method. Grünrock and Herr [8] and Herr [10] obtained sharp well-posedness results for some derivative nonlinear Schrödinger equation on the torus by using the Fourier restriction method. The well-posedness of the Schrödinger equation for more general derivative nonlinearity in the n-dimensional torus was given by Chihara [4]. We note that the classical energy method does not work in his setting. Chihara conquered this problem by using the pseudo-differential operators with non-smooth coefficients on the torus.

As pointed out by Doi [6], the existence of trapped geodesics breaks the smoothing effect for the Schrödinger equation. This fact indicates that the dispersive equations on the torus do not have fine smoothing properties such as the Kato local smoothing effect for the real line case. Therefore the proof of the well-posedness on the torus becomes increasingly harder than the real line case. To state our results, we introduce several notations. Given a function ψ on $\mathbf{T}$, we define the Fourier coefficient of ψ, by

$$\hat{\psi}(n) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \psi(x) e^{-inx}\, dx, \quad n \in \mathbf{Z}.$$

Let m be a non-negative integer. $H^m(\mathbf{T})$ denotes the all tempered distributions on $\mathbf{T}$ satisfying

$$\|\psi\|_{H_x^m} = \left(\sum_{n\in\mathbf{Z}} \langle n\rangle^{2m} \left|\hat{\psi}(n)\right|^2\right)^{1/2} < +\infty,$$

where $\langle n\rangle = \sqrt{1+n^2}$.

The main result in this paper is the following:

Theorem 1.1. *Let $m \geqslant 4$ be an integer. Then (1) is locally well-posed in the following sense: For any $\phi \in H^m(\mathbf{T})$, there exist a time $T = T(\|\phi\|_{H^m}) > 0$ and a unique solution ψ of (1) satisfying*

$$\psi \in C\big([0,T); H^m(\mathbf{T})\big).$$

Moreover, the data-to-solution map $H^m(\mathbf{T}) \to C([0,T]; H^m(\mathbf{T}))$ $(\phi \mapsto \psi(t))$ is continuous.

The difficulty to guarantee local well-posedness of (1) comes from the loss of a derivatives. More precisely, the standard energy estimate gives only the following:

$$\frac{d}{dt}\left\|\partial_x^m \psi(t)\right\|_{L_x^2}^2 = 2\left\{2\lambda_3 + \lambda_4 + 2(m-1)\lambda_6\right\} \operatorname{Im} \int_{\mathbf{T}} \overline{\psi}\partial_x\psi \cdot \partial_x^m\overline{\psi}\partial_x^{m+1}\psi\, dx - 2\lambda_5 \operatorname{Im} \int_{\mathbf{T}} \psi^2\left(\partial_x^{m+1}\overline{\psi}\right)^2 dx + \text{l.o.t.} \tag{3}$$

Since the first and second terms in the right hand side of (3) contain the $(m+1)$-derivatives of ψ, we cannot control those factors in terms of the H^m norm of ψ. Therefore this estimate does not give any information on the solution.

For the real line case, the unitary group $\{e^{it(\partial_x^2+\nu\partial_x^4)}\}_{t\in\mathbf{R}}$ generated by the linear operator $i\partial_x^2 + i\nu\partial_x^4$ gains extra smoothness in space variable, see Kenig, Ponce, and Vega [13]. Thanks to this smoothing property for $\{e^{it(\partial_x^2+\nu\partial_x^4)}\}_{t\in\mathbf{R}}$, [23,11,24,12] overcome a loss of derivatives and guarantee the well-posedness of (1) on $\mathbf{R}$. Although the corresponding unitary group on $\mathbf{T}$ has some Strichartz type estimate (see Moyua and Vega [21]), the unitary group does not have fine properties such as the local smoothing effect on $\mathbf{R}$. Therefore it is not likely that the contraction mapping principle works for (1) on $\mathbf{T}$. This situation leads us to abandon making use of the property of the unitary group $\{e^{it(\partial_x^2+\nu\partial_x^4)}\}_{t\in\mathbf{R}}$ and take different approaches to this issue.

Let us return the estimate (3). If we contrive to eliminate the worst terms, we may obtain an a priori estimate of solution. In this paper we take a hint from Kwon [17], we introduce the"modified" energy:

$$\left[E_m(\psi)\right](t) = \left\|\partial_x^m\psi(t)\right\|_{L_x^2}^2 + \left\|\psi(t)\right\|_{L_x^2}^2 + C_m\left\|\psi(t)\right\|_{L_x^2}^{4m+2} + \frac{\lambda_5}{\nu}\operatorname{Re}\int_{\mathbf{T}}\left(\partial_x^{m-1}\psi\right)^2\overline{\psi}^2\,dx + \frac{2\lambda_3+\lambda_4+2(m-1)\lambda_6}{4\nu}\int_{\mathbf{T}}\left|\partial_x^{m-1}\psi\right|^2|\psi|^2\,dx$$

where C_m is a sufficiently large constant depending only on m so that $E_m(\psi)$ is positive. Thanks to the correction terms we can eliminate the worst factors in (3) and evaluate the H^m norm of the solution ψ to (1) in terms of the H^m norm of the initial data ϕ. This is a crucial point in the proof of Theorem 1.1.

It is known that (1) is completely integrable if and only if $\lambda_1 = -1/2$, $\lambda_2 = -3\nu/8$, $\lambda_3 = -3\nu/2$, $\lambda_4 = -\nu$, $\lambda_5 = -\nu/2$ and $\lambda_6 = -2\nu$. In this case (1) has infinitely many conservation quantities (see Langer and Perline [18]). The first three conservation quantities for (1) are given by

$$I_0(\psi) = \frac{1}{2}\int_{\mathbf{T}} |\psi|^2\, dx,$$

$$I_1(\psi) = \frac{1}{2}\int_{\mathbf{T}} |\partial_x\psi|^2\, dx - \frac{1}{8}\int_{\mathbf{T}} |\psi|^4\, dx,$$

$$I_2(\psi) = \frac{1}{2}\int_{\mathbf{T}} \left|\partial_x^2\psi\right|^2 dx + \frac{3}{4}\int_{\mathbf{T}} |\psi|^2\overline{\psi}\partial_x^2\psi\, dx + \frac{1}{8}\int_{\mathbf{T}} |\psi|^2\psi\partial_x^2\overline{\psi}\, dx$$
$$+ \frac{5}{8}\int_{\mathbf{T}} (\partial_x\psi)^2\overline{\psi}^2\, dx + \frac{3}{4}\int_{\mathbf{T}} |\partial_x\psi|^2|\psi|^2\, dx + \frac{1}{16}\int_{\mathbf{T}} |\psi|^6\, dx.$$

In general, the conservation quantities for (1) are expressed as

$$I_m(\psi) = \frac{1}{2}\int_{\mathbf{T}} \left|\partial_x^m\psi\right|^2 + \int_{\mathbf{T}} Q\left(\psi, \overline{\psi}, \ldots, \partial_x^{m-1}\psi, \partial_x^{m-1}\overline{\psi}\right) dx,$$

where Q is some polynomial in $(\psi, \overline{\psi}, \ldots, \partial_x^{m-1}\psi, \partial_x^{m-1}\overline{\psi})$ satisfying the inequality $\int |Q|\, dx \leqslant C\|\psi\|_{L_x^2}^{\alpha}\|\partial_x^m\psi\|_{L_x^2}^{\beta}$ for some $\alpha > 0$ and $0 < \beta < 2$. Therefore combining the local existence Theorem 1.1 in $H^m(\mathbf{T})$, the conservation laws $I_m(\psi)(t) = I_m(\psi)(0)$ and Young's inequality, we obtain the global existence theorem for (1) in $H^m(\mathbf{T})$:

Theorem 1.2. *Assume $\lambda_1 = -1/2$, $\lambda_2 = -3\nu/8$, $\lambda_3 = -3\nu/2$, $\lambda_4 = -\nu$, $\lambda_5 = -\nu/2$ and $\lambda_6 = -2\nu$. Then* (1) *is globally well-posed in $H^m(\mathbf{T})$, with $m \geqslant 4$.*

Finally we point out that we may well be able to extend Theorem 1.1 to the case where m is not an integer combining our proof with the estimates for the fractional derivatives. In this paper we do not touch on this issue.

The plan of this paper is as follows. Section 2 is devoted to the parabolic regularization associated to (1). In Section 3, we introduce the modified energy and give an a priori estimate for the solution to (1) and prove the existence of solution to (1). In Section 4 we give the proof of uniqueness and persistent properties of solution to (1) and continuous dependence on the initial data associated to (1).

2. Parabolic regularization

In this section, we consider the parabolic regularization of (1) in $H^m(\mathbf{T})$. We first give the Gagliardo–Nirenberg inequality for the periodic functions.

Lemma 2.1. *Let l and m be integers satisfying $0 \leqslant l \leqslant m-1$ and let $2 \leqslant p \leqslant \infty$. Then there exists a constant C depending only on l, m and p such that for any $\psi \in H^m(\mathbf{T})$,*

$$\left\|\partial_x^l\psi\right\|_{L_x^p} \leqslant C \times \begin{cases} \|\psi\|_{L_x^2}^{1-\alpha}\|\partial_x^m\psi\|_{L_x^2}^{\alpha} & (1 \leqslant l \leqslant m-1), \\ \|\psi\|_{L_x^2}^{1-\alpha}\|\partial_x^m\psi\|_{L_x^2}^{\alpha} + \|\psi\|_{L_x^2} & (l = 0), \end{cases}$$

where $\alpha = (l + 1/2 - 1/p)/m$. Especially, $\|\partial_x^l\psi\|_{L_x^p} \leqslant C\|\psi\|_{L_x^2}^{1-\alpha}\|\psi\|_{H_x^m}^{\alpha}$.

Proof. See [22, Section 2]. □

Next we consider the parabolic regularization of (1). To this end, we introduce the regularizing sequence used in Bona and Smith [2]. Let $\varphi \in C^\infty(\mathbf{R})$ be such that $0 \leqslant \varphi(\xi) \leqslant 1$ for $\xi \in \mathbf{R}$, $\varphi^{(k)}(0) = 0$ for $k \in \mathbf{N}$ and $\varphi(\xi)$ tends exponentially to 0 as $|\xi| \to \infty$. We define for $\epsilon \in (0,1]$,

$$\phi_\epsilon(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbf{Z}} \varphi(\epsilon n)\hat{\phi}(n) e^{inx}.$$

Then, $\{\phi_\epsilon\}_{\epsilon>0} \in H^\infty(\mathbf{T})$ and $\|\phi - \phi_\epsilon\|_{H_x^m} \to 0$ as $\epsilon \to 0$. Furthermore, for any $l \geqslant 0$,

$$\|\phi_\epsilon\|_{H_x^{m+l}} \leqslant C\epsilon^{-l}\|\phi\|_{H_x^m},$$

$$\|\phi - \phi_\epsilon\|_{H_x^m} \leqslant C\|\phi\|_{H_x^m},$$

$$\|\phi - \phi_\epsilon\|_{H_x^{m-l}} \leqslant C\epsilon^{l}\|\phi\|_{H_x^m}.$$

We consider the regularized problem of (1):

$$\begin{cases} i\partial_t \psi_\epsilon + \partial_x^2 \psi_\epsilon + (\nu + i\epsilon)\partial_x^4 \psi_\epsilon = \mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2 \psi_\epsilon, \partial_x^2 \overline{\psi}_\epsilon\big), \\ \psi_\epsilon(0,x) = \phi_\epsilon(x), \end{cases} \tag{4}$$

where $\psi_\epsilon : \mathbf{R} \times \mathbf{T} \to \mathbf{C}$ is an unknown function, and $\phi_\epsilon : \mathbf{T} \to \mathbf{C}$ is a Bona–Smith approximation of ϕ. For (4), we have the following lemma.

Lemma 2.2. *Let $m \geqslant 3$ be an integer. For any $\phi \in H^m(\mathbf{T})$, there exist a time $T_\epsilon = T(\epsilon, \|\phi\|_{H^m}) > 0$ and a unique solution ψ_ϵ of* (4) *satisfying*

$$\psi_\epsilon \in C\big([0, T_\epsilon), H^m(\mathbf{T})\big).$$

Proof. We shall prove (4) by using the Banach fixed point theorem. Let $\{W_\epsilon(t)\}_{t \geqslant 0}$ be a contraction semi-group generated by the linear operator $i\partial_x^2 + i\nu\partial_x^4 - \epsilon\partial_x^4$:

$$\big[W_\epsilon(t)\phi\big](x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbf{Z}} \hat{\phi}(n) e^{inx + (-in^2 + i\nu n^4 - \epsilon n^4)t}.$$

Then, the initial value problem (4) is rewritten as the integral equation

$$\psi_\epsilon(t) = W_\epsilon(t)\phi_\epsilon - i \int_0^t W_\epsilon(t-\tau)\mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2 \psi_\epsilon, \partial_x^2 \overline{\psi}_\epsilon\big)(\tau)\, d\tau.$$

We put $r = \|\phi\|_{H_x^m}$. For $T > 0$, we define

$$X_T^r = \Big\{\psi \in C\big([0,T]; H^m(\mathbf{T})\big) \,\Big|\, \sup_{t \in [0,T)} \big\|\psi(t)\big\|_{H_x^m} \leqslant 2r\Big\}.$$

We shall show that the map

$$\Phi(\psi_\epsilon) = W_\epsilon(t)\phi_\epsilon - i\int_0^t W_\epsilon(t-\tau)\mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)(\tau)\,d\tau$$

is a contraction on X_T^r for choosing T suitably.

We easily see that

$$\big\|\Phi(\psi_\epsilon)(t)\big\|_{H_x^m} \leqslant \|\phi\|_{H_x^m} + \int_0^t \big\|W_\epsilon(t-\tau)\mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)(\tau)\big\|_{H_x^m}\,d\tau. \tag{5}$$

By Plancherel's identity, we obtain

$$\begin{aligned}
&\int_0^t \big\|W_\epsilon(t-\tau)\mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)(\tau)\big\|_{H_x^m}\,d\tau \\
&\quad = \int_0^t \bigg\{\sum_{n\in\mathbf{Z}} \langle n\rangle^{2m}\big|\widehat{\mathcal{N}}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)(\tau,n)e^{(-in^2+i\nu n^4-\epsilon n^4)(t-\tau)}\big|^2\bigg\}^{1/2}d\tau \\
&\quad = \int_0^t \bigg\{\sum_{n\in\mathbf{Z}} \langle n\rangle^{2m-4}\big|\widehat{\mathcal{N}}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)(\tau,n)\big|^2\langle n\rangle^4 e^{-2\epsilon n^4(t-\tau)}\bigg\}^{1/2}d\tau \\
&\quad \leqslant \int_0^t \sup_{n\in\mathbf{Z}}\big\{\langle n\rangle^2 e^{-\epsilon n^4(t-\tau)}\big\}\big\|\mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)(\tau)\big\|_{H_x^{m-2}}\,d\tau.
\end{aligned}$$

Since $\sup_{n\in\mathbf{Z}}\{\langle n\rangle^2 e^{-\epsilon n^4(t-\tau)}\} \leqslant 1 + \epsilon^{-1/2}(t-\tau)^{-1/2}$,

$$\begin{aligned}
&\int_0^t \big\|W_\epsilon(t-\tau)\mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)(\tau)\big\|_{H_x^m}\,d\tau \\
&\quad \leqslant C\int_0^t \big\{1 + \epsilon^{-1/2}(t-\tau)^{-1/2}\big\}\big\|\mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)(\tau)\big\|_{H_x^{m-2}}\,d\tau \\
&\quad \leqslant C\big(t + \epsilon^{-1/2}t^{1/2}\big)\sup_{t\in[0,T)}\big\|\mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)(t)\big\|_{H_x^{m-2}}.
\end{aligned} \tag{6}$$

Collecting (5) and (6), we have

$$\sup_{t\in[0,T)}\big\|\Phi(\psi_\epsilon)(t)\big\|_{H_x^m} \leqslant \|\phi\|_{H_x^m} + C\big(T + \epsilon^{-1/2}T^{1/2}\big)\sup_{t\in[0,T)}\big\|\mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)(t)\big\|_{H_x^{m-2}}.$$

By Sobolev's embedding, we have

$$\sup_{t\in[0,T)}\big\|\mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)(t)\big\|_{H_x^{m-2}} \leqslant C\Big(1 + \sup_{t\in[0,T)}\big\|\psi(t)\big\|_{H_x^m}^2\Big)\sup_{t\in[0,T)}\big\|\psi(t)\big\|_{H_x^m}^3.$$

Therefore,

$$\sup_{t\in[0,T)} \left\| \Phi(\psi_\epsilon)(t) \right\|_{H_x^m} \leqslant r + C\left(T + \epsilon^{-1/2}T^{1/2}\right)\left(1+r^2\right)r^3.$$

We can easily check that $\Phi(\psi_\epsilon) \in C([0,T); H^m(\mathbf{T}))$. Therefore, by choosing $T_\epsilon > 0$ sufficiently small so that $C(T_\epsilon + \epsilon^{-1/2}T_\epsilon^{1/2})(1+r^2)r^2 < 1$ we have $\psi_\epsilon \in X_T^r$. By a similar way, for $\psi_\epsilon^1, \psi_\epsilon^2 \in X_T^r$, we have

$$\begin{aligned}\sup_{t\in[0,T)} \left\| \Phi\left(\psi_\epsilon^1\right)(t) - \Phi\left(\psi_\epsilon^2\right)(t) \right\|_{H_x^m} &\leqslant C\left(T + \epsilon^{-1/2}T^{1/2}\right)\left(1+r^2\right)r^2 \sup_{t\in[0,T)} \left\| \psi_\epsilon^1(t) - \psi_\epsilon^2(t) \right\|_{H_x^m} \\ &< \sup_{t\in[0,T)} \left\| \psi_\epsilon^1(t) - \psi_\epsilon^2(t) \right\|_{H_x^m}.\end{aligned}$$

Consequently, we have that Φ is a contraction on X_T^r. The Banach fixed point theorem implies the unique existence of solution to (4) in X_T^r which completes the proof of Lemma 2.2. □

3. Modified energy

In this section, introducing the modified energy, we give a priori estimates for the solution to (4) obtained by Lemma 2.2.

Let $m \geqslant 1$ be an integer. We introduce the modified energy:

$$\begin{aligned}\left[E_m(\psi)\right](t) &= \left\| \partial_x^m \psi(t) \right\|_{L_x^2}^2 + \left\| \psi(t) \right\|_{L_x^2}^2 + C_m \left\| \psi(t) \right\|_{L_x^2}^{4m+2} \\ &\quad + \frac{\lambda_5}{\nu} \operatorname{Re} \int_{\mathbf{T}} \left(\partial_x^{m-1}\psi\right)^2 \overline{\psi}^2\,dx + \frac{2\lambda_3 + \lambda_4 + 2(m-1)\lambda_6}{4\nu} \int_{\mathbf{T}} \left|\partial_x^{m-1}\psi\right|^2 |\psi|^2\,dx,\end{aligned}$$

where C_m is a sufficiently large constant depending only on m so that $E_m(\psi)$ is positive. This is possible because of the following reason. The Gagliardo–Nirenberg inequality (Lemma 2.1) implies

$$\begin{aligned}&\frac{\lambda_5}{\nu} \operatorname{Re} \int_{\mathbf{T}} \left(\partial_x^{m-1}\psi\right)^2 \overline{\psi}^2\,dx + \frac{2\lambda_3 + \lambda_4 + 2(m-1)\lambda_6}{4\nu} \int_{\mathbf{T}} \left|\partial_x^{m-1}\psi\right|^2 |\psi|^2\,dx \\ &\quad \geqslant -\frac{1}{2}\left\| \partial_x^m \psi(t) \right\|_{L_x^2}^2 - \frac{1}{2}\left\| \psi(t) \right\|_{L_x^2}^2 - D_m \left\| \psi(t) \right\|_{L_x^2}^{4m+2}\end{aligned}$$

with some positive constant D_m depending only on ν, λ_3, λ_4, λ_5, λ_6 and m. Hence we obtain

$$\left[E_m(\psi)\right](t) \geqslant \frac{1}{2}\left\| \partial_x^m \psi(t) \right\|_{L_x^2}^2 + \frac{1}{2}\left\| \psi(t) \right\|_{L_x^2}^2 + (C_m - D_m)\left\| \psi(t) \right\|_{L_x^2}^{4m+2}.$$

Choosing C_m so large that $C_m > D_m$, we have $[E_m(\psi)](t) > 0$. We notice that

$$\frac{1}{2}\left\| \psi(t) \right\|_{H_x^m}^2 \leqslant \left[E_m(\psi)\right](t) \leqslant C\left(\left\| \psi(t) \right\|_{L_x^2}^{4m} + 1\right)\left\| \psi(t) \right\|_{H_x^m}^2. \tag{7}$$

Lemma 3.1. *Let $\psi_\epsilon \in C([0,T_\epsilon), H^m(\mathbf{T}))$ be a solution to* (4)*. Then, there exist positive constants C and $T = T(\|\phi\|_{H_x^m})$ which are independent of ϵ such that*

$$\left\| \psi_\epsilon(t) \right\|_{H_x^m} \leqslant C\left(T, \|\phi\|_{L_x^2}\right)\|\phi\|_{H_x^m},$$

for any $t \in [0,T)$.

Proof. We first evaluate $[E_m(\psi)](t)$. Applying the m-th derivative to the both sides of (4), taking the inner product of the resultant equation with $\partial_x^m \psi$, and adding the complex conjugation of the produce, we obtain

$$\frac{d}{dt}\big\|\partial_x^m \psi_\epsilon(t)\big\|_{L_x^2}^2 + 2\epsilon\big\|\partial_x^{m+2}\psi_\epsilon(t)\big\|_{L_x^2}^2 = 2\,\mathrm{Im}\int_{\mathbf{T}} \partial_x^m \mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)\partial_x^m\overline{\psi}_\epsilon\,dx. \tag{8}$$

Using the Leibniz rule, we obtain

$$\begin{aligned}
&\partial_x^m \mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)\\
&\quad = \big\{(2\lambda_3 + m\lambda_6)\overline{\psi}_\epsilon\partial_x\psi_\epsilon + (\lambda_4 + m\lambda_6)\psi_\epsilon\partial_x\overline{\psi}_\epsilon\big\}\partial_x^{m+1}\psi_\epsilon\\
&\qquad + (\lambda_4 + 2m\lambda_5)\psi_\epsilon\partial_x\psi_\epsilon\partial_x^{m+1}\overline{\psi}_\epsilon + \lambda_6|\psi_\epsilon|^2\partial_x^{m+2}\psi_\epsilon + \lambda_5\psi_\epsilon^2\partial_x^{m+2}\overline{\psi}_\epsilon\\
&\qquad + P_1\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^m\psi_\epsilon, \partial_x^m\overline{\psi}_\epsilon\big),
\end{aligned} \tag{9}$$

where P_1 is a linear combination of the cubic terms $\partial_x^{j_1}\psi_\epsilon\partial_x^{j_2}\overline{\psi}_\epsilon\partial_x^{j_3}\psi_\epsilon$ with $j_1 + j_2 + j_3 = m$, or $j_1 + j_2 + j_3 = m + 2$ and $j_1 + j_2 + j_3 \leqslant m$, and the quintic terms $\partial_x^{j_1}\psi_\epsilon\partial_x^{j_2}\overline{\psi}_\epsilon\partial_x^{j_3}\psi_\epsilon\partial_x^{j_4}\overline{\psi}_\epsilon\partial_x^{j_5}\psi_\epsilon$ with $j_1 + j_2 + j_3 + j_4 + j_5 = m$. Hence the Hölder and Gagliardo–Nirenberg (Lemma 2.1) inequalities imply

$$\begin{aligned}
\|P_1\|_{L_x^2} &\leqslant C\big(\|\psi_\epsilon\|_{L_x^2}^{(2m-1)/m}\|\psi_\epsilon\|_{H_x^m}^{(m+1)/m} + \|\psi_\epsilon\|_{L_x^2}^{(4m-2)/m}\|\psi_\epsilon\|_{H_x^m}^{(m+2)/m} + \|\psi_\epsilon\|_{L_x^2}^{(2m-3)/m}\|\psi_\epsilon\|_{H_x^m}^{(m+3)/m}\big)\\
&\leqslant C\big[E_m(\psi_\epsilon)\big](t)^{3/2}.
\end{aligned} \tag{10}$$

In the last inequality we used the inequalities

$$\|\psi\|_{L_x^2} \leqslant \big[E_m(\psi)\big](t)^\alpha \quad \text{for any } \frac{1}{4m+2} \leqslant \alpha \leqslant \frac{1}{2},$$
$$\|\psi\|_{H_x^m} \leqslant \big[E_m(\psi)\big](t)^{1/2}.$$

Substituting (9) and (10) into (8), we have

$$\begin{aligned}
&\frac{d}{dt}\big\|\partial_x^m\psi_\epsilon(t)\big\|_{L_x^2}^2 + 2\epsilon\big\|\partial_x^{m+2}\psi_\epsilon(t)\big\|_{L_x^2}^2\\
&\quad = 2(2\lambda_3 + m\lambda_6)\,\mathrm{Im}\int_{\mathbf{T}} \overline{\psi}_\epsilon\partial_x\psi_\epsilon\partial_x^m\overline{\psi}_\epsilon\partial_x^{m+1}\psi_\epsilon\,dx\\
&\qquad + 2(\lambda_4 + m\lambda_6)\,\mathrm{Im}\int_{\mathbf{T}} \psi_\epsilon\partial_x\overline{\psi}_\epsilon\partial_x^m\overline{\psi}_\epsilon\partial_x^{m+1}\psi_\epsilon\,dx\\
&\qquad + 2(\lambda_4 + 2m\lambda_5)\,\mathrm{Im}\int_{\mathbf{T}} \psi_\epsilon\partial_x\psi_\epsilon\partial_x^m\overline{\psi}_\epsilon\partial_x^{m+1}\overline{\psi}_\epsilon\,dx\\
&\qquad + 2\lambda_6\,\mathrm{Im}\int_{\mathbf{T}} |\psi_\epsilon|^2\partial_x^m\overline{\psi}_\epsilon\partial_x^{m+2}\psi_\epsilon\,dx + 2\lambda_5\,\mathrm{Im}\int_{\mathbf{T}} \psi_\epsilon^2\partial_x^m\overline{\psi}_\epsilon\partial_x^{m+2}\overline{\psi}_\epsilon\,dx
\end{aligned}$$

$$+ 2\,\mathrm{Im}\int_{\mathbf{T}} P_1\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^m\psi_\epsilon, \partial_x^m\overline{\psi}_\epsilon\big)\partial_x^m\overline{\psi}_\epsilon\,dx$$

$$\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \tag{11}$$

The inequality (10) and the Schwarz inequality imply

$$|I_6| \leqslant \|P_1\|_{L_x^2}\big\|\partial_x^m\psi\big\|_{L_x^2} \leqslant C\big[E_m(\psi_\epsilon)\big](t)^2.$$

An integration by parts yields

$$|I_3| \leqslant C\big[E_m(\psi_\epsilon)\big](t)^2.$$

We can express I_2, I_4 and I_5 in terms of I_1 by using an integration by parts:

$$I_2 = 2(\lambda_4 + m\lambda_6)\,\mathrm{Im}\int_{\mathbf{T}} \overline{\psi}_\epsilon\partial_x\psi_\epsilon\partial_x^m\overline{\psi}_\epsilon\partial_x^{m+1}\psi_\epsilon\,dx + R_1\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^m\psi_\epsilon, \partial_x^m\overline{\psi}_\epsilon\big),$$

$$I_4 = -4\lambda_6\,\mathrm{Im}\int_{\mathbf{T}} \overline{\psi}_\epsilon\partial_x\psi_\epsilon\partial_x^m\overline{\psi}_\epsilon\partial_x^{m+1}\psi_\epsilon\,dx + R_2\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^m\psi_\epsilon, \partial_x^m\overline{\psi}_\epsilon\big),$$

$$I_5 = -2\lambda_5\,\mathrm{Im}\int_{\mathbf{T}} \psi_\epsilon^2\big(\partial_x^{m+1}\overline{\psi}_\epsilon\big)^2\,dx + R_3\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^m\psi_\epsilon, \partial_x^m\overline{\psi}_\epsilon\big),$$

where R_1, R_2 and R_3 satisfy

$$|R_1| + |R_2| + |R_3| \leqslant C\big[E_m(\psi_\epsilon)\big](t)^2.$$

Substituting the above equations into (11), we have

$$\begin{aligned}
&\frac{d}{dt}\big\|\partial_x^m\psi_\epsilon(t)\big\|_{L_x^2}^2 + 2\epsilon\big\|\partial_x^{m+2}\psi(t)\big\|_{L_x^2}^2 \\
&\quad = 2\big\{2\lambda_3 + \lambda_4 + 2(m-1)\lambda_6\big\}\,\mathrm{Im}\int_{\mathbf{T}} \overline{\psi}_\epsilon\partial_x\psi_\epsilon\cdot\partial_x^m\overline{\psi}_\epsilon\partial_x^{m+1}\psi_\epsilon\,dx \\
&\qquad - 2\lambda_5\,\mathrm{Im}\int_{\mathbf{T}} \psi_\epsilon^2\big(\partial_x^{m+1}\overline{\psi}_\epsilon\big)^2\,dx + R_4\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^m\psi_\epsilon, \partial_x^m\overline{\psi}_\epsilon\big),
\end{aligned} \tag{12}$$

where R_4 satisfies

$$|R_4| \leqslant C\big[E_m(\psi_\epsilon)\big](t)^2.$$

On the other hand, from Eq. (1), we have

$$
\begin{aligned}
&\frac{d}{dt}\operatorname{Re}\int_{\mathbf{T}}\left(\partial_x^{m-1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon^2\,dx \\
&= 2\operatorname{Re}\int_{\mathbf{T}}\partial_x^{m-1}\psi_\epsilon\,\partial_t\partial_x^{m-1}\psi_\epsilon\cdot\overline{\psi}_\epsilon^2\,dx + 2\operatorname{Re}\int_{\mathbf{T}}\left(\partial_x^{m-1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon\,\partial_t\overline{\psi}_\epsilon\,dx \\
&= -2\epsilon\operatorname{Re}\int_{\mathbf{T}}\partial_x^{m-1}\psi_\epsilon\,\partial_x^{m+3}\psi_\epsilon\cdot\overline{\psi}_\epsilon^2\,dx - 2\operatorname{Im}\int_{\mathbf{T}}\partial_x^{m-1}\psi_\epsilon\,\partial_x^{m+1}\psi_\epsilon\cdot\overline{\psi}_\epsilon^2\,dx \\
&\quad - 2\nu\operatorname{Im}\int_{\mathbf{T}}\partial_x^{m-1}\psi_\epsilon\,\partial_x^{m+3}\psi_\epsilon\cdot\overline{\psi}_\epsilon^2\,dx \\
&\quad + 2\operatorname{Im}\int_{\mathbf{T}}\partial_x^{m-1}\psi_\epsilon\,\partial_x^{m-1}\mathcal{N}\left(\psi_\epsilon,\overline{\psi}_\epsilon,\dots,\partial_x^2\psi_\epsilon,\partial_x^2\overline{\psi}_\epsilon\right)\cdot\overline{\psi}_\epsilon^2\,dx \\
&\quad - 2\epsilon\operatorname{Re}\int_{\mathbf{T}}\left(\partial_x^{m-1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon\,\partial_x^4\overline{\psi}_\epsilon\,dx + 2\operatorname{Im}\int_{\mathbf{T}}\left(\partial_x^{m-1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon\cdot\partial_x^2\overline{\psi}_\epsilon\,dx \\
&\quad + 2\nu\operatorname{Im}\int_{\mathbf{T}}\left(\partial_x^{m-1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon\cdot\partial_x^4\overline{\psi}_\epsilon\,dx \\
&\quad - 2\operatorname{Im}\int_{\mathbf{T}}\left(\partial_x^{m-1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon\cdot\overline{\mathcal{N}}\left(\psi_\epsilon,\overline{\psi}_\epsilon,\dots,\partial_x^2\psi_\epsilon,\partial_x^2\overline{\psi}_\epsilon\right)dx \\
&\equiv I_7+I_8+I_9+I_{10}+I_{11}+I_{12}+I_{13}+I_{14}. \qquad (13)
\end{aligned}
$$

An integration by parts yields

$$
\begin{aligned}
I_7 &= -2\epsilon\operatorname{Re}\int_{\mathbf{T}}\left(\partial_x^{m+1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon^2\,dx + R_5\left(\psi_\epsilon,\overline{\psi}_\epsilon,\dots,\partial_x^m\psi_\epsilon,\partial_x^m\overline{\psi}_\epsilon\right), \\
I_9 &= -2\nu\operatorname{Im}\int_{\mathbf{T}}\left(\partial_x^{m+1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon^2\,dx + R_6\left(\psi_\epsilon,\overline{\psi}_\epsilon,\dots,\partial_x^m\psi_\epsilon,\partial_x^m\overline{\psi}_\epsilon\right),
\end{aligned}
$$

where R_5 and R_6 satisfy

$$
|R_5|+|R_6| \leqslant C\|\psi_\epsilon\|_{L_x^2}^{(2m-3)/m}\|\psi_\epsilon\|_{H_x^m}^{(2m+3)/m} \leqslant C\left[E_m(\psi_\epsilon)\right](t)^2.
$$

Integrating by parts, we also obtain

$$
\begin{aligned}
|I_8|+|I_{12}| &\leqslant C\|\psi_\epsilon\|_{L_x^2}^{(2m-1)/m}\|\psi_\epsilon\|_{H_x^m}^{(2m+1)/m} \leqslant C\left[E_m(\psi_\epsilon)\right](t)^2, \\
|I_{11}|+|I_{13}| &\leqslant C\|\psi_\epsilon\|_{L_x^2}^{(2m-3)/m}\|\psi_\epsilon\|_{H_x^m}^{(2m+3)/m} \leqslant C\left[E_m(\psi_\epsilon)\right](t)^2,
\end{aligned}
$$

$$
\begin{aligned}
|I_{10}|+|I_{14}| &\leqslant C\left(\|\psi_\epsilon\|_{L_x^2}^4\|\psi_\epsilon\|_{H_x^m}^2 + \|\psi_\epsilon\|_{L_x^2}^{(6m-1)/m}\|\psi_\epsilon\|_{H_x^m}^{(2m+1)/m} + \|\psi_\epsilon\|_{L_x^2}^{(4m-2)/m}\|\psi_\epsilon\|_{H_x^m}^{(2m+2)/m}\right) \\
&\leqslant C\left[E_m(\psi_\epsilon)\right](t)^2.
\end{aligned}
$$

Substituting the above equations into (13), we have

$$\frac{d}{dt}\operatorname{Re}\int_{\mathbf{T}}\left(\partial_x^{m-1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon^2\,dx = -2\nu\operatorname{Im}\int_{\mathbf{T}}\left(\partial_x^{m+1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon^2\,dx + 2\epsilon\operatorname{Re}\int_{\mathbf{T}}\left(\partial_x^{m+1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon^2\,dx + R_7\left(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^m\psi_\epsilon, \partial_x^m\overline{\psi}_\epsilon\right), \tag{14}$$

where R_7 satisfies

$$|R_7| \leqslant C\left[E_m(\psi_\epsilon)\right](t)^2.$$

By an argument similar to (14), we obtain

$$\frac{d}{dt}\int_{\mathbf{T}}\left|\partial_x^{m-1}\psi_\epsilon\right|^2|\psi_\epsilon|^2\,dx = -8\nu\operatorname{Im}\int_{\mathbf{T}}\overline{\psi}_\epsilon\partial_x\psi_\epsilon\cdot\partial_x^m\overline{\psi}_\epsilon\partial_x^{m+1}\psi_\epsilon\,dx - 2\epsilon\int_{\mathbf{T}}\left|\partial_x^{m+1}\psi_\epsilon\right|^2|\psi_\epsilon|^2\,dx + R_8\left(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^m\psi_\epsilon, \partial_x^m\overline{\psi}_\epsilon\right), \tag{15}$$

where R_8 satisfies

$$|R_8| \leqslant C\left[E_m(\psi_\epsilon)\right](t)^2.$$

Finally, we obtain

$$\frac{d}{dt}\left\|\psi_\epsilon(t)\right\|_{L_x^2}^{4m+2} + 2(2m+1)\epsilon\left\|\psi_\epsilon(t)\right\|_{L_x^2}^{4m}\left\|\partial_x^2\psi_\epsilon(t)\right\|_{L_x^2}^2 \leqslant C\left[E_m(\psi_\epsilon)\right](t)^2. \tag{16}$$

Collecting (13)–(15), and (16), we have

$$\begin{aligned}
&\frac{d}{dt}E_m(t) + 2\epsilon\left\|\partial_x^{m+2}\psi_\epsilon(t)\right\|_{L_x^2}^2 + 2(2m+1)\epsilon\left\|\psi_\epsilon(t)\right\|_{L_x^2}^{4m}\left\|\partial_x^2\psi_\epsilon(t)\right\|_{L_x^2}^2 \\
&\quad= -\frac{2\lambda_5}{\nu}\epsilon\operatorname{Re}\int_{\mathbf{T}}\left(\partial_x^{m+1}\psi_\epsilon\right)^2\overline{\psi}_\epsilon^2\,dx - \frac{2\lambda_3+\lambda_4+2(m-1)\lambda_6}{2\nu}\epsilon\int_{\mathbf{T}}\left|\partial_x^{m+1}\psi_\epsilon\right|^2|\psi_\epsilon|^2\,dx \\
&\qquad+ R_9\left(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^m\psi_\epsilon, \partial_x^m\overline{\psi}_\epsilon\right),
\end{aligned} \tag{17}$$

where R_9 satisfies

$$|R_9| \leqslant C\left[E_m(\psi_\epsilon)\right](t)^2.$$

Since the sum of the first and second terms in the right hand side of (17) is bounded by $\epsilon\|\partial_x^{m+2}\psi_\epsilon\|_{L_x^2}^2 + C[E_m(\psi_\epsilon)](t)^2$, we obtain

$$\frac{d}{dt}\left[E_m(\psi_\epsilon)\right](t) + \epsilon\left\|\partial_x^{m+2}\psi_\epsilon(t)\right\|_{L_x^2}^2 + 2(2m+1)\epsilon\left\|\psi_\epsilon(t)\right\|_{L_x^2}^{4m}\left\|\partial_x^2\psi_\epsilon(t)\right\|_{L_x^2}^2 \leqslant C\left[E_m(\psi_\epsilon)\right](t)^2.$$

Therefore

$$\frac{d}{dt}\left[E_m(\psi_\epsilon)\right](t) \leqslant C\left[E_m(\psi_\epsilon)\right](t)^2.$$

We note that the constant C is independent of $\epsilon \in (0, 1]$. From the above inequality we have

$$\big[E_m(\psi_\epsilon)\big](t) \leqslant \frac{[E_m(\psi_\epsilon)](0)}{1 - Ct[E_m(\psi_\epsilon)](0)},$$

for $0 \leqslant t < \min\{T_\epsilon, C^{-1}[E_m(\psi_\epsilon)](0)^{-1}\}$. Combining this inequality, $\|\phi_\epsilon\|_{H^m} \leqslant \|\phi\|_{H^m}$ for any $\epsilon \in (0,1]$ and (7), we see

$$\|\psi_\epsilon\|_{H_x^m}^2 \leqslant \frac{C(\|\phi\|_{L_x^2}^{4m} + 1)\|\phi\|_{H_x^m}^2}{1 - Ct(\|\phi\|_{L_x^2}^{4m} + 1)\|\phi\|_{H_x^m}^2},$$

for $0 \leqslant t < \min\{T_\epsilon, C^{-1}(\|\phi\|_{L_x^2}^{4m} + 1)^{-1}\|\phi\|_{H_x^m}^{-2}\}$. Let $T \equiv (2C)^{-1}(\|\phi\|_{L_x^2}^{4m} + 1)^{-1}\|\phi\|_{H_x^m}^{-2}$. Then for any $0 < t < \min\{T_\epsilon, T\}$, we have

$$\|\psi_\epsilon\|_{H_x^m}^2 \leqslant 2C\big(\|\phi\|_{L_x^2}^{4m} + 1\big)\|\phi\|_{H_x^m}^2.$$

If $T_\epsilon < T$, we can apply Lemma 2.2 to extend the solution in the same class to the interval $[0, T)$. Therefore we obtain the desired result. □

Using Lemma 3.1 we obtain the existence of the solution to (1):

Lemma 3.2. *Let $m \geqslant 4$ be an integer. For any $\phi \in H^m(\mathbf{T})$, there exist a time $T = T(\|\phi\|_{H^m}) > 0$ and a solution ψ of* (1) *satisfying*

$$\psi \in L^\infty\big([0, T); H^m(\mathbf{T})\big).$$

Proof. Let $\phi \in H^m(\mathbf{T})$ and let $\{\phi_\epsilon\}_\epsilon \subset H^\infty(\mathbf{T})$ be a Bona–Smith approximation of ϕ. Then by Lemma 2.2 there exists a unique solution $\psi_\epsilon \in C([0, T_\epsilon); H^m(\mathbf{T}))$ to (4). Lemma 3.1 yields that there exists $T = T(\|\phi\|_{H_x^m}) > 0$ which is independent of ϵ such that $\{\psi_\epsilon\}_\epsilon$ is uniformly bounded in $L^\infty(0, T; H^m(\mathbf{T}))$ with respect to $\epsilon \in (0, 1]$. By a standard limiting argument, it is inferred that a subsequence of ψ^ϵ converges in $L^\infty(0, T; H^m(\mathbf{T}))$ weak* to a solution ψ of (1) such that $\psi^\epsilon \in L^\infty(0, T; H^m(\mathbf{T}))$. We omit the details. □

4. Proof of Theorem 1.1

In the preceding sections, we proved the existence of the solution to (1). In this section, we complete the proof of Theorem 1.1 by showing the following three assertions:

(i) uniqueness of the solution,
(ii) persistent properties of the solution,
(iii) continuous dependence of the solution upon initial data.

4.1. Uniqueness

Let ψ_1 and ψ_2 be two solutions to (1) with the same initial data satisfying $\sup_{t\in[0,T)} \|\psi_j(t)\|_{H_x^m} < \infty$, $j = 1, 2$. We shall show that $\psi_1 \equiv \psi_2$ for $t \in [0, T)$. To prove this, it suffices to show that $\psi = \psi_2 - \psi_1$ satisfies $\|\psi(t)\|_{H_x^1} \equiv 0$ because this identity and $\psi(0) \equiv 0$ imply $\psi \equiv 0$. The reason we prove $\|\psi(t)\|_{H_x^1} \equiv 0$ instead of driving $\|\psi(t)\|_{L_x^2} \equiv 0$ is that the corresponding modified energy for L^2 involves the anti-derivatives of ψ.

The standard energy estimate yields

$$\begin{aligned}\frac{d}{dt}\|\psi(t)\|_{L_x^2}^2 &= 2\operatorname{Im}\int_{\mathbf{T}}\big\{\mathcal{N}\big(\psi+\psi_1,\overline{\psi}+\overline{\psi}_1,\ldots,\partial_x^2\psi+\partial_x^2\psi_1,\partial_x^2\overline{\psi}+\partial_x^2\overline{\psi}_1\big)\\&\quad-\mathcal{N}\big(\psi_1,\overline{\psi}_1,\ldots,\partial_x^2\psi_1,\partial_x^2\overline{\psi}_1\big)\big\}\overline{\psi}\,dx\\&\leqslant C\big(\|\psi_1\|_{H_x^2}^2+\|\psi_1\|_{H_x^2}^4+\|\psi_2\|_{H_x^2}^2+\|\psi_2\|_{H_x^2}^4\big)\|\psi\|_{H_x^1}^2,\end{aligned}\tag{18}$$

$$\begin{aligned}\frac{d}{dt}\|\partial_x\psi(t)\|_{L_x^2} &= 2\operatorname{Im}\int_{\mathbf{T}}\partial_x\big\{\mathcal{N}\big(\psi+\psi_1,\overline{\psi}+\overline{\psi}_1,\ldots,\partial_x^2\psi+\partial_x^2\psi_1,\partial_x^2\overline{\psi}+\partial_x^2\overline{\psi}_1\big)\\&\quad-\mathcal{N}\big(\psi_1,\overline{\psi}_1,\ldots,\partial_x^2\psi_1,\partial_x^2\overline{\psi}_1\big)\big\}\partial_x\overline{\psi}\,dx\\&=2(2\lambda_3+\lambda_4)\operatorname{Im}\int_{\mathbf{T}}\partial_x\psi_1\overline{\psi}_1\cdot\partial_x^2\psi\,\partial_x\overline{\psi}\,dx-2\lambda_5\operatorname{Im}\int_{\mathbf{T}}\psi_1^2\big(\partial_x^2\overline{\psi}\big)^2\,dx\\&\quad+R_{10}\big(\psi_1,\overline{\psi}_1,\ldots,\partial_x^3\psi_1,\partial_x^3\overline{\psi}_1,\psi_2,\overline{\psi}_2,\ldots,\partial_x^3\psi_2,\partial_x^3\overline{\psi}_2\big),\end{aligned}\tag{19}$$

where R_{10} satisfies

$$|R_{10}|\leqslant C\big(\|\psi_1\|_{H_x^3}^2+\|\psi_1\|_{H_x^3}^4+\|\psi_2\|_{H_x^3}^2+\|\psi_2\|_{H_x^3}^4\big)\|\psi\|_{H_x^1}^2.$$

On the other hand, a direct calculation yields

$$\begin{aligned}\frac{2\lambda_3+\lambda_4}{4\nu}\frac{d}{dt}\int_{\mathbf{T}}|\psi_1|^2|\psi|^2\,dx &= -2(2\lambda_3+\lambda_4)\operatorname{Im}\int_{\mathbf{T}}\partial_x\psi_1\overline{\psi}_1\cdot\partial_x^2\psi\,\partial_x\overline{\psi}\,dx\\&\quad+R_{11}\big(\psi_1,\overline{\psi}_1,\ldots,\partial_x^3\psi_1,\partial_x^3\overline{\psi}_1,\psi_2,\overline{\psi}_2,\ldots,\partial_x^3\psi_2,\partial_x^3\overline{\psi}_2\big),\end{aligned}\tag{20}$$

$$\begin{aligned}\frac{\lambda_5}{\nu}\frac{d}{dt}\operatorname{Re}\int_{\mathbf{T}}\psi_1^2(\overline{\psi})^2\,dx &= 2\lambda_5\operatorname{Im}\int_{\mathbf{T}}\psi_1^2\big(\partial_x^2\overline{\psi}\big)^2\,dx\\&\quad+R_{12}\big(\psi_1,\overline{\psi}_1,\ldots,\partial_x^3\psi_1,\partial_x^3\overline{\psi}_1,\psi_2,\overline{\psi}_2,\ldots,\partial_x^3\psi_2,\partial_x^3\overline{\psi}_2\big),\end{aligned}\tag{21}$$

where R_{11} and R_{12} satisfy

$$|R_{11}|+|R_{12}|\leqslant C\big(\|\psi_1\|_{H_x^3}^2+\|\psi_1\|_{H_x^3}^6+\|\psi_2\|_{H_x^3}^2+\|\psi_2\|_{H_x^3}^6\big)\|\psi\|_{H_x^1}^2.$$

Here we set

$$\big[\tilde{E}_1(\psi)\big](t)=\|\partial_x\psi(t)\|_{L_x^2}^2+\tilde{C}_1\|\psi(t)\|_{L_x^2}^2+\frac{2\lambda_3+\lambda_4}{4\nu}\int_{\mathbf{T}}|\psi_1|^2|\psi|^2\,dx+\frac{\lambda_5}{\nu}\operatorname{Re}\int_{\mathbf{T}}\psi_1^2(\overline{\psi})^2\,dx,$$

where $\tilde{C}_1$ is a sufficiently large constant depending only on m, $\sup_{t\in[0,T)}\|\psi_1(t)\|_{H_x^1}$ and $\sup_{t\in[0,T)}\|\psi_2(t)\|_{H_x^1}$ so that $\tilde{E}_1(\psi)$ is positive. Then, from (18)–(20) and (21), we obtain

$$\frac{d}{dt}\big[\tilde{E}_1(\psi)\big](t)\leqslant C\big(\|\psi_1\|_{H_x^3}^2+\|\psi_1\|_{H_x^3}^6+\|\psi_2\|_{H_x^3}^2+\|\psi_2\|_{H_x^3}^6\big)\|\psi\|_{L_x^2}^2\leqslant C\big[\tilde{E}_1(\psi)\big](t).$$

Hence Gronwall's lemma yields

$$\big[\tilde{E}_1(\psi)\big](t) \leqslant \big[\tilde{E}_1(\psi)\big](0)e^{ct}. \tag{22}$$

Since $[\tilde{E}_m(\psi)](0) = 0$, Gronwall's lemma yields $[\tilde{E}_1(\psi)](t) \equiv 0$. Combination of this identity and the equality $0 \leqslant \|\psi(t)\|_{H_x^1} \leqslant [\tilde{E}_1(\psi)](t)$ implies $\psi \equiv 0$, which completes the proof of the uniqueness.

4.2. *Persistence of solution*

To prove the persistent property of the solution to (1) which is obtained by Lemma 3.2, we employ the Bona–Smith approximation. We denote ϕ_ϵ the Bona–Smith approximation of ϕ.

Lemma 4.1. *Let $\phi \in H^m(\mathbf{T})$ with $m \geqslant 3$ and let ψ_α, ψ_ϵ denote the solution to* (1) *corresponding to the initial data ϕ_α and ϕ_ϵ, respectively. Then there exists $C = C(T, \|\phi\|_{H_x^m}) > 0$ such that for $0 \leqslant \alpha < \epsilon \leqslant 1$,*

$$\sup_{t\in[0,T)} \big\|\psi_\alpha(t) - \psi_\epsilon(t)\big\|_{H_x^m} \leqslant C\big(\epsilon^{m-3} + \|\phi - \phi_\alpha\|_{H_x^m} + \|\phi - \phi_\epsilon\|_{H_x^m}\big). \tag{23}$$

Proof. We put $\psi = \psi_\alpha - \psi_\epsilon$. We first evaluate $\|\psi(t)\|_{H_x^1}$. Replacing $\psi = \psi_1 - \psi_2$ by $\psi = \psi_\alpha - \psi_\epsilon$ in (22), we have

$$\big[\tilde{E}_1(\psi)\big](t) \leqslant \big[\tilde{E}_1(\psi)\big](0)e^{Ct}, \tag{24}$$

for $t \in [0, T)$, where

$$\big[\tilde{E}_1(\psi)\big](t) = \big\|\partial_x\psi(t)\big\|_{L_x^2}^2 + \tilde{C}_1\big\|\psi(t)\big\|_{L_x^2}^2 + \frac{2\lambda_3 + \lambda_4}{4\nu}\int_{\mathbf{T}} |\psi_\alpha|^2|\psi|^2\,dx + \frac{\lambda_5}{\nu}\operatorname{Re}\int_{\mathbf{T}} \psi_\alpha^2(\overline{\psi})^2\,dx,$$

$\tilde{C}_1$ is a sufficiently large constant depending only on m, $\sup_{t\in[0,T)}\|\psi(t)\|_{H_x^m}$[1] so that $\tilde{E}_1(\psi)$ is positive and C in (24) depends only on $\sup_{t\in[0,T)}\|\psi(t)\|_{H_x^3}$. Since

$$\big\|\psi(t)\big\|_{H_x^1}^2 \leqslant \big[\tilde{E}_1(\psi)\big](t), \tag{25}$$

$$\begin{aligned}\big[\tilde{E}_1(\psi)\big](0) &\leqslant C\|\phi_\alpha - \phi_\epsilon\|_{H_x^1}^2 \leqslant C\big(\|\phi - \phi_\alpha\|_{H_x^1}^2 + \|\phi - \phi_\epsilon\|_{H_x^1}^2\big)\\ &\leqslant C\big(\alpha^{2(m-1)} + \epsilon^{2(m-1)}\big) \leqslant C\epsilon^{2(m-1)},\end{aligned} \tag{26}$$

the inequalities (24), (25) and (26) lead to the inequality

$$\big\|\psi(t)\big\|_{H_x^1} \leqslant C\epsilon^{m-1}. \tag{27}$$

Next, we evaluate $\|\psi(t)\|_{H_x^m}$. By an argument similar to (8),

$$\begin{aligned}\frac{d}{dt}\big\|\partial_x^m\psi(t)\big\|_{L_x^2}^2 = 2\operatorname{Im}\int_{\mathbf{T}} \partial_x^m\big\{&\mathcal{N}\big(\psi + \psi_\epsilon, \overline{\psi} + \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi + \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi} + \partial_x^2\overline{\psi}_\epsilon\big)\\ &- \mathcal{N}\big(\psi_\epsilon, \overline{\psi}_\epsilon, \ldots, \partial_x^2\psi_\epsilon, \partial_x^2\overline{\psi}_\epsilon\big)\big\}\partial_x^m\overline{\psi}\,dx.\end{aligned} \tag{28}$$

[1] By the inequality $\|\phi_\epsilon\|_{H_x^m} \leqslant \|\phi\|_{H^m}$ and Lemma 3.2, we can choose $\tilde{C}_1$ independently of α.

Using the Leibniz rule, we have

$$\begin{aligned}
&\partial_x^m\big\{\mathcal{N}\big(\psi+\psi_\epsilon,\overline{\psi}+\overline{\psi}_\epsilon,\ldots,\partial_x^2\psi+\partial_x^2\psi_\epsilon,\partial_x^2\overline{\psi}+\partial_x^2\overline{\psi}_\epsilon\big)-\mathcal{N}\big(\psi_\epsilon,\overline{\psi}_\epsilon\ldots,\partial_x^2\psi_\epsilon,\partial_x^2\overline{\psi}_\epsilon\big)\big\}\\
&\quad=\big\{(2\lambda_3+m\lambda_6)\overline{\psi}_\alpha\partial_x\psi_\alpha+(\lambda_4+m\lambda_6)\psi_\alpha\partial_x\overline{\psi}_\alpha\big\}\partial_x^{m+1}\psi+(\lambda_4+2m\lambda_5)\psi_\alpha\partial_x\psi_\alpha\partial_x^{m+1}\overline{\psi}\\
&\qquad+\lambda_6|\psi_\alpha|^2\partial_x^{m+2}\psi+\lambda_5\psi_\alpha^2\partial_x^{m+2}\overline{\psi}\\
&\qquad+P_2\big(\psi_\alpha,\overline{\psi}_\alpha,\ldots,\partial_x^m\psi_\alpha,\partial_x^m\overline{\psi}_\alpha,\psi_\epsilon,\overline{\psi}_\epsilon,\ldots,\partial_x^{m+2}\psi_\epsilon,\partial_x^{m+2}\overline{\psi}_\epsilon\big),
\end{aligned}\tag{29}$$

and

$$\begin{aligned}
&P_2\big(\psi_\alpha,\overline{\psi}_\alpha,\ldots,\partial_x^m\psi_\alpha,\partial_x^m\overline{\psi}_\alpha,\psi_\epsilon,\overline{\psi}_\epsilon,\ldots,\partial_x^{m+2}\psi_\epsilon,\partial_x^{m+2}\overline{\psi}_\epsilon\big)\\
&\quad=\big\{(2\lambda_3+m\lambda_6)(\overline{\psi}_\alpha\partial_x\psi+\partial_x\psi_\epsilon\overline{\psi})+(\lambda_4+m\lambda_6)\psi_\alpha\partial_x\overline{\psi}+\partial_x\overline{\psi}_\epsilon\psi\big\}\partial_x^{m+1}\psi_\epsilon\\
&\qquad+(\lambda_4+2m\lambda_5)(\psi_\alpha\partial_x\psi+\partial_x\psi_\epsilon\psi)\partial_x^{m+1}\overline{\psi}_\epsilon+\lambda_6(\overline{\psi}_\alpha\psi+\psi_\epsilon\overline{\psi})\partial_x^{m+2}\psi_\epsilon\\
&\qquad+\lambda_5(\psi_\alpha\psi+\psi_\epsilon\psi)\partial_x^{m+2}\overline{\psi}_\epsilon+P_3\big(\psi_\alpha,\ldots,\partial_x^m\overline{\psi}_\alpha,\psi_\epsilon,\ldots,\partial_x^m\overline{\psi}_\epsilon\big),
\end{aligned}$$

where P_3 satisfies

$$\|P_3\|_{L_x^2}\leqslant C\big(\|\psi_\alpha\|_{H_x^m}^2+\|\psi_\alpha\|_{H_x^m}^4+\|\psi_\epsilon\|_{H_x^m}^2+\|\psi_\epsilon\|_{H_x^m}^4\big)\|\psi\|_{H_x^m}\leqslant C\|\psi\|_{H_x^m}.\tag{30}$$

Combining the inequalities (27) and (30) and $\|\psi_\epsilon\|_{H_x^{m+2}}\leqslant C\epsilon^{-2}\|\psi\|_{H_x^m}$, we have

$$\|P_2\|_{L_x^2}\leqslant C\big(\|\psi_\alpha\|_{H_x^1}+\|\psi_\epsilon\|_{H_x^1}\big)\|\psi_\epsilon\|_{H_x^{m+2}}\|\psi\|_{H_x^1}+C\|\psi\|_{H_x^m}\leqslant C\epsilon^{m-3}+C\|\psi\|_{H_x^m}.$$

The identity (29) and a standard energy estimate yield

$$\begin{aligned}
&\frac{d}{dt}\big\|\partial_x^m\psi(t)\big\|_{L_x^2}^2\\
&\quad=2\big\{2\lambda_3+\lambda_4+2(m-1)\lambda_6\big\}\operatorname{Im}\int_{\mathbf{T}}\partial_x\psi_\alpha\overline{\psi}_\alpha\partial_x^{m+1}\psi\,\partial_x^m\overline{\psi}\,dx-2\lambda_5\operatorname{Im}\int_{\mathbf{T}}\psi_\alpha^2\big(\partial_x^{m+1}\overline{\psi}\big)^2\,dx\\
&\qquad+R_{13}\big(\psi_\alpha,\overline{\psi}_\alpha,\ldots,\partial_x^m\psi_\alpha,\partial_x^m\overline{\psi}_\alpha,\psi_\epsilon,\overline{\psi}_\epsilon,\ldots,\partial_x^{m+2}\psi_\epsilon,\partial_x^{m+2}\overline{\psi}_\epsilon\big),
\end{aligned}\tag{31}$$

where R_{13} satisfies

$$|R_{13}|\leqslant C\epsilon^{m-3}+C\|\psi\|_{H_x^m}^2.$$

On the other hand, a direct calculation yields

$$\begin{aligned}
&\frac{2\lambda_3+\lambda_4+2(m-1)\lambda_6}{4\nu}\frac{d}{dt}\int_{\mathbf{T}}|\psi_\alpha|^2\big|\partial_x^{m-1}\psi\big|^2\,dx\\
&\quad=-2\big\{2\lambda_3+\lambda_4+2(m-1)\lambda_6\big\}\operatorname{Im}\int_{\mathbf{T}}\partial_x\psi_\alpha\overline{\psi}_\alpha\partial_x^{m+1}\psi\,\partial_x^m\overline{\psi}\,dx\\
&\qquad+R_{14}\big(\psi_\alpha,\overline{\psi}_\alpha,\ldots,\partial_x^m\psi_\alpha,\partial_x^m\overline{\psi}_\alpha,\psi_\epsilon,\overline{\psi}_\epsilon,\ldots,\partial_x^m\psi_\epsilon,\partial_x^m\overline{\psi}_\epsilon\big)
\end{aligned}\tag{32}$$

and

$$\frac{\lambda_5}{\nu}\frac{d}{dt}\operatorname{Re}\int_{\mathbf{T}}\psi_\alpha^2\big(\partial_x^{m-1}\overline{\psi}\big)^2\,dx$$
$$=2\lambda_5\operatorname{Im}\int_{\mathbf{T}}\psi_\alpha^2\big(\partial_x^{m+1}\overline{\psi}\big)^2\,dx+R_{15}\big(\psi_\alpha,\overline{\psi}_\alpha,\dots,\partial_x^m\psi_\alpha,\partial_x^m\overline{\psi}_\alpha,\psi_\epsilon,\overline{\psi}_\epsilon,\dots,\partial_x^m\psi_\epsilon,\partial_x^m\overline{\psi}_\epsilon\big),\tag{33}$$

where R_{14} and R_{15} satisfy

$$|R_{14}|+|R_{15}|\leqslant C\|\psi\|_{H_x^m}^2.$$

From (31), (32) and (33), we obtain

$$\frac{d}{dt}\left\{\big\|\partial_x^m\psi(t)\big\|_{L_x^2}^2+\frac{2\lambda_3+\lambda_4+2(m-1)\lambda_6}{4\nu}\int_{\mathbf{T}}|\psi_\alpha|^2\big|\partial_x^{m-1}\psi\big|^2\,dx+\frac{\lambda_5}{\nu}\operatorname{Re}\int_{\mathbf{T}}\psi_\alpha^2\big(\partial_x^{m-1}\overline{\psi}\big)^2\,dx\right\}$$
$$\leqslant C\|\psi\|_{H_x^m}^2+C\epsilon^{m-3}.\tag{34}$$

Here we set

$$\big[\tilde{E}_m(\psi)\big](t)=\int_{\mathbf{T}}\big|\partial_x^m\psi\big|^2\,dx+\tilde{C}_m\int_{\mathbf{T}}|\psi|^2\,dx+\int_{\mathbf{T}}|\psi_\alpha|^2\big|\partial_x^{m-1}\psi\big|^2\,dx+\operatorname{Re}\int_{\mathbf{T}}\psi_\alpha^2\big(\partial_x^{m-1}\overline{\psi}\big)^2\,dx,$$

where $\tilde{C}_m$ is a sufficiently large constant depending only on m, $\sup_{t\in[0,T)}\|\psi(t)\|_{H_x^m}$ so that $\tilde{E}_m(\psi)$ is positive. Then the inequality (34) is expressed in terms of $\tilde{E}_m$:

$$\frac{d}{dt}\big[\tilde{E}_m(\psi)\big](t)\leqslant C\big[\tilde{E}_m(\psi)\big](t)+C\epsilon^{m-3}.$$

Therefore Gronwall's lemma leads to the inequality

$$\big[\tilde{E}_m(\psi)\big](t)\leqslant C\big(\big[\tilde{E}_m(\psi)\big](0)+\epsilon^{m-3}\big)e^{CT}.$$

Therefore we have (23) which completes Lemma 4.1. □

Let us prove the persistent property of the solution to (1). Let $\phi\in H^m(\mathbf{T})$ and $\{\phi_\epsilon\}_{\epsilon>0}\subset H^\infty(\mathbf{T})$ be a Bona–Smith approximation of ϕ. Lemma 3.2 yields there exist $T=T(\|\phi_\epsilon\|_{H_x^m})$ and a unique solution $\psi_\epsilon(t)\in L^\infty(0,T;H^\infty(\mathbf{T}))$ to (1). Since $\|\phi_\epsilon\|_{H^m}\leqslant\|\phi\|_{H^m}$, we can choose T independently of ϵ. By Lemma 4.1, $\{\psi_\epsilon(t)\}_\epsilon$ is Cauchy sequence in $C(0,T;H^m(\mathbf{T}))$. Consequently we see that $\phi\in C(0,T;H^m(\mathbf{T}))$. This guarantees the persistent property of the solution in Theorem 1.1.

4.3. Continuity of data-to-solution map

As the final step of the proof of Theorem 1.1, we prove that the data-to-solution map $S_t : H^m(\mathbf{T}) \to C([0,T); H^m(\mathbf{T}))$ $(\phi \mapsto \psi(t))$ associated to (1) is continuous. To this end, we shall prove the following: Let $\phi \in H^m(\mathbf{T})$. For any $\eta > 0$ there exists $\delta > 0$ such that if $\tilde{\phi} \in H^m(\mathbf{T})$ satisfies

$$\|\phi - \tilde{\phi}\|_{H_x^m} < \delta,$$

then

$$\sup_{t\in[0,T)} \big\| S_t(\phi) - S_t(\tilde{\phi})\big\|_{H_x^m} < \eta.$$

Let $\{\phi_\epsilon\}_{\epsilon>0}$ and $\{\tilde{\phi}_\epsilon\}_{\epsilon>0}$ be the Bona–Smith approximations of ϕ and $\tilde{\phi}$, respectively. By the triangle inequality, we have

$$\begin{aligned}\sup_{t\in[0,T)} \big\| S_t(\phi) - S_t(\tilde{\phi})\big\|_{H_x^m} \leqslant{}& \sup_{t\in[0,T)} \big\| S_t(\phi) - S_t(\phi_\epsilon)\big\|_{H_x^m} + \sup_{t\in[0,T)} \big\| S_t(\phi_\epsilon) - S_t(\tilde{\phi}_\epsilon)\big\|_{H_x^m} \\ &+ \sup_{t\in[0,T)} \big\| S_t(\tilde{\phi}_\epsilon) - S_t(\tilde{\phi})\big\|_{H_x^m}. \end{aligned} \tag{35}$$

Letting α tend to 0 in (23), we have

$$\sup_{t\in[0,T)} \big\| S_t(\phi) - S_t(\phi_\epsilon)\big\|_{H_x^m} \leqslant C\big(\epsilon^{m-3} + \|\phi - \phi_\epsilon\|_{H_x^m}\big), \tag{36}$$

$$\sup_{t\in[0,T)} \big\| S_t(\tilde{\phi}_\epsilon) - S_t(\tilde{\phi})\big\|_{H_x^m} \leqslant C\big(\epsilon^{m-3} + \|\tilde{\phi} - \tilde{\phi}_\epsilon\|_{H_x^m}\big). \tag{37}$$

By a similar argument as the derivation of (23), we obtain

$$\sup_{t\in[0,T)} \big\| S_t(\phi_\epsilon) - S_t(\tilde{\phi}_\epsilon)\big\|_{H_x^m} \leqslant C\big(\epsilon^{m-3} + \|\phi_\epsilon - \tilde{\phi}_\epsilon\|_{H_x^m}\big).$$

Combining the above inequality with the triangle inequality

$$\|\phi_\epsilon - \tilde{\phi}_\epsilon\|_{H_x^m} \leqslant \|\phi_\epsilon - \phi\|_{H_x^m} + \|\phi - \tilde{\phi}\|_{H_x^m} + \|\tilde{\phi} - \tilde{\phi}_\epsilon\|_{H_x^m},$$

we have

$$\sup_{t\in[0,T)} \big\| S_t(\phi_\epsilon) - S_t(\tilde{\phi}_\epsilon)\big\|_{H_x^m} \leqslant C\big(\epsilon^{m-3} + \|\phi_\epsilon - \phi\|_{H_x^m} + \|\phi - \tilde{\phi}\|_{H_x^m} + \|\tilde{\phi} - \tilde{\phi}_\epsilon\|_{H_x^m}\big). \tag{38}$$

Substituting (36), (37) and (38) into (35), we obtain

$$\sup_{t\in[0,T)} \big\| S_t(\phi) - S_t(\tilde{\phi})\big\|_{H_x^m} \leqslant C\big(\epsilon^{m-3} + \|\phi_\epsilon - \phi\|_{H_x^m} + \|\phi - \tilde{\phi}\|_{H_x^m} + \|\tilde{\phi} - \tilde{\phi}_\epsilon\|_{H_x^m}\big). \tag{39}$$

We first choose $\delta > 0$ so that $C\delta < \eta/4$. Since $\phi_\epsilon \to \phi$ and $\tilde{\phi}_\epsilon \to \tilde{\phi}$ in H^m as $\epsilon \to 0$, there exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leqslant \epsilon_0$,

$$\|\phi - \phi_\epsilon\|_{H_x^m} < \frac{\eta}{4}, \qquad \|\tilde{\phi} - \tilde{\phi}_\epsilon\|_{H_x^m} < \frac{\eta}{4}.$$

Further choosing ϵ_0 sufficiently small so that $C\epsilon_0^{m-3} < \eta/4$, we have that if $\tilde{\phi} \in H^m(\mathbf{T})$ satisfies $\|\phi - \tilde{\phi}\|_{H_x^m} < \delta$, then taking $0 < \epsilon \leqslant \epsilon_0$ in (39), we have

$$\sup_{t\in[0,T)} \left\| S_t(\phi) - S_t(\tilde{\phi}) \right\|_{H_x^m} < \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{4} = \eta.$$

The proof of Theorem 1.1 is now complete.

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