Transition from real to virtual polarized photon structures

Takahiro Ueda a, Tsuneo Uematsu b, Ken Sasaki a,*

a Department of Physics, Faculty of Engineering, Yokohama National University, Yokohama 240-8501, Japan
b Department of Physics, Graduate School of Science, Kyoto University, Yoshida, Kyoto 606-8501, Japan

Received 27 June 2006; accepted 15 July 2006
Available online 10 August 2006
Editor: T. Yanagida

Abstract

We investigate the transition of the polarized photon structure function \( g_{1}^{\gamma}(x, Q^2, P^2) \) when the target photon shifts from on-shell \((P^2 = 0)\) to far off-shell \((P^2 \gg \Lambda^2)\) region. The analysis is performed to the next-to-leading order in QCD. The first moment of \( g_{1}^{\gamma} \) which vanishes for the real photon, turns to be a negative value when target photon becomes off-shell. The explicit \( P^2 \)-dependence of the first moment sum rule as well as of the structure function \( g_{1}^{\gamma}(x, Q^2, P^2) \) as a function of \( x \) is studied in the framework of the vector meson dominance model.

© 2006 Elsevier B.V. Open access under CC BY license.

The photon structure has been investigated both theoretically and experimentally for decades [1–4]. And in recent years there has been growing interest in the spin structure of the photon. In particular, the first moment of the polarized photon structure function \( g_{1}^{\gamma}(x, Q^2, P^2) \), where \( q^2 = -Q^2 \) \((p^2 = -P^2)\) is the mass squared of the probe (target) photon, can be measured in the experiments either of the polarized ep collision [10,11] or of the polarized e+e− collision in the future linear collider ILC [12,13].

A unique and interesting feature of the photon structure functions is that, in contrast with the nucleon case, the target mass squared \(-P^2\) is not fixed but can take various values and that the structure functions show different behaviors depending on the values of \( P^2 \). The QCD analysis of \( g_{1}^{\gamma} \) for a real photon \((P^2 = 0)\) target was performed in the leading order (LO) [15], and in the next-to-leading order (NLO) [11,18]. In the case of a virtual photon target \((P^2 \neq 0)\), \( g_{1}^{\gamma}(x, Q^2, P^2) \) was investigated up to the NLO in QCD by the present authors in [14], and in the second paper of [18]. Moreover, the polarized parton distributions inside the virtual photon were analyzed in various factorization schemes [16], and the target mass effects of \( g_{1}^{\gamma}(x, Q^2, P^2) \) was studied in [17]. In Refs. [14,16] the structure function \( g_{1}^{\gamma}(x, Q^2, P^2) \) was analyzed in the kinematic region

\[ A^2 \ll P^2 \ll Q^2, \tag{1} \]

where \( \Lambda \) is the QCD scale parameter. The advantage in studying the virtual photon target in the kinematic region (1) is that we can calculate structure functions by the perturbative method without any experimental data input [22], which is contrasted with the case of a real photon target where in the NLO there exist nonperturbative pieces [20–28].

In the present Letter we analyze the transition of the polarized photon structure function \( g_{1}^{\gamma} \) when the target photon shifts from on-shell \((P^2 = 0)\) to far off-shell in the region (1). In fact, \( g_{1}^{\gamma} \) exhibits an interesting \( P^2 \)-dependence. At \( P^2 = 0 \), the structure function
function $g_1^\gamma$ satisfies a remarkable sum rule [5–9]:

$$\int_0^1 dx g_1^\gamma(x, Q^2) dx = 0. \tag{2}$$

But when the target photon becomes off-shell, $P^2 \neq 0$, the first moment of the corresponding photon structure function $g_1^\gamma(x, Q^2, P^2)$ does not vanish any more. For the case $A^2 \ll P^2 \ll Q^2$, the first moment has been calculated up to the NLO [7,14], and quite recently up to the next-to-next-to-leading order (NNLO) in QCD [19]. The NLO result is

$$\frac{1}{\pi} \alpha s(P^2) \sum_{n=1}^{N_f} g_{\gamma n}(\bar{Q}_0^2, \mu^2) = \frac{3\alpha}{\pi} \left( \frac{\alpha s(Q^2)}{\pi} \right) - \frac{\alpha s(Q^2)}{\pi} \left( \frac{\alpha s(Q^2)}{2\beta_0} - \frac{\alpha s(Q^2)}{\pi} \right), \tag{3}$$

with $\beta_0 = 11 - 2N_f/3$ being the one-loop QCD $\beta$ function. Here $\alpha (\alpha s(Q^2))$ is the QED (QCD running) coupling constant, $n_f (\bar{Q}_0^2) = \sum_{i=1}^{N_f} e_i^2$ and $n_f (\bar{Q}_0^2) = \sum_{i=1}^{N_f} e_i^2$ with $e_i$ being the electric charge of the active quark (i.e., the massless quark) with flavor $i$ in the unit of proton charge and $n_f$ the number of active quarks. In order to investigate the transition of $g_1^\gamma$ from on-shell to far-off-shell region, we derive a formula which accommodates a unified description of both regions. And we explain the transition from the vanishing first moment sum rule (2) at $P^2 = 0$ to the nonvanishing sum rule (3) for off-shell $P^2 \neq 0$. For the explicit $P^2$-dependence we resort to the vector-meson-dominance (VMD) model [31].

In the framework of the operator product expansion supplemented by the renormalization group method, the $n$th moment of $g_1^\gamma(x, Q^2, P^2)$ is given as follows [14]:

$$\int_0^1 dx x^{n-1} g_1^\gamma(x, Q^2, P^2) = \sum_{j=\psi, G, NS, \gamma} C_j^\psi(Q^2/\mu^2, \bar{g}(\mu^2), \alpha)|\gamma(p)|\bar{R}_n^\psi(\mu^2)|\gamma(p)|. \tag{4}$$

where $|\gamma(p)|$ is the “target” virtual photon state with momentum $p$, $\bar{R}_n^\psi$ and $C_j^\psi$ are the twist-2 spin-$n$ operators and their coefficient functions with $\mu$ being the renormalization point. The indices $\psi$, $G$, $NS$ and $\gamma$ stand for singlet quark, gluon, nonsinglet quark and photon, respectively. The photon structure functions are defined in the lowest order of the QED coupling constant $\alpha$, and in this order we have $\langle\gamma(p)|\bar{R}_n^\psi(\mu^2)|\gamma(p)\rangle = 1$ for the photon matrix element of the photon operator $\bar{R}_n^\psi$.

In the previous work [14,16] we took the renormalization point at $\mu^2 = P^2$, where $P^2$ is much larger than $A^2$, so that we could calculate perturbatively the photon matrix elements of the hadronic operators $\bar{R}_n = (\bar{R}_n^\psi, \bar{R}_n^G, \bar{R}_n^{NS})$ [22]. In the present case, $P^2$ varies from the deeply virtual region down to $P^2 = 0$. Therefore we take the renormalization point at $\mu^2 = \bar{Q}_0^2$, where $\bar{Q}_0^2$ satisfies the condition

$$A^2 \ll \bar{Q}_0^2 \ll Q^2. \tag{5}$$

Then the $n$th moment of $g_1^\gamma(x, Q^2, P^2)$ is expressed as (see Eq. (3.13) of Ref. [14]),

$$\int_0^1 dx x^{n-1} g_1^\gamma(x, Q^2, P^2) = \frac{\alpha}{4\pi} \tilde{A}_n(\bar{Q}_0^2, P^2) \cdot M_n(Q^2/\bar{Q}_0^2, \bar{g}(\bar{Q}_0^2)) \tilde{C}_n(1, \bar{g}(\bar{Q}_0^2))$$

$$+ \tilde{X}_n(Q^2/\bar{Q}_0^2, \bar{g}(\bar{Q}_0^2), \alpha) \cdot \tilde{C}_n(1, \bar{g}(\bar{Q}_0^2)) + C_n^\gamma, \tag{6}$$

where $\tilde{A}_n = (A_n^\psi, A_n^G, A_n^{NS})$ denote the photon matrix elements of the hadronic operators $\bar{R}_n$ renormalized at $\mu^2 = \bar{Q}_0^2$ and are defined as

$$\langle\gamma(p)|\bar{R}_n(\mu)|\gamma(p)\rangle|_{\mu^2 = \bar{Q}_0^2} = \frac{\alpha}{4\pi} \tilde{A}_n(\bar{Q}_0^2, P^2). \tag{7}$$

The evolution factors $M_n$ and $\tilde{X}_n$ are given in terms of $T$-ordered exponential as [20]

$$M_n(Q^2/\bar{Q}_0^2, \bar{g}(\bar{Q}_0^2)) = T \exp \left[ \int \frac{dg}{\bar{g}(Q^2)} \frac{\hat{y}_n(g)}{\beta(g)} \right],$$

$$\tilde{X}_n(Q^2/\bar{Q}_0^2, \bar{g}(\bar{Q}_0^2), \alpha) = \int \frac{dg}{\bar{g}(Q^2)} \frac{\hat{K}_n(g, \alpha)}{\beta(g)} \times T \exp \left[ \int \frac{dg}{\bar{g}(Q^2)} \frac{\hat{y}_n(g')}{\beta(g')} \right]. \tag{8}$$
with \( \tilde{y}_n \) and \( \tilde{K}_n \) being the hadronic anomalous dimension matrix and the off-diagonal element representing the mixing between the photon and hadronic operators, respectively. The coefficient functions are expanded up to the one loop level as (see Eq. (3.15) of Ref. [14])

\[
\tilde{C}_n(1, \tilde{g}(Q^2)) = \left[ C_n^\beta(1, \tilde{g}(Q^2)), C_n^\gamma(1, \tilde{g}(Q^2)), C_n^{\bar{n}S}(1, \tilde{g}(Q^2)) \right] \equiv \left< (e^2)(1 + \frac{2 \alpha_s(Q^2)}{4 \pi} B_n^\beta) \right> \\
\left< (e^2) \frac{\alpha_s(Q^2)}{4 \pi} B_n^\gamma \right> \\
\left< 1 + \frac{\alpha_s(Q^2)}{4 \pi} B_n^{\bar{n}S} \right>.
\tag{9}
\]

It is noted that, in Eq. (6), the \( P^2 \)-dependence only resides in \( \tilde{A}_n(Q_0^2; P^2) \), the photon matrix elements of the hadronic operators. When the photon state becomes far off-shell and \( P^2 \) approaches \( Q_0^2 \), \( \tilde{A}_n(Q_0^2; P^2) \) are considered to be point-like and can be evaluated perturbatively. Indeed, the point-like component has been calculated in the \( \bar{\text{MS}} \) scheme [29], and we get in the leading order,

\[
\tilde{A}_n(Q_0^2; P^2 = Q_0^2) = \tilde{A}_n^{(0)}(Q_0^2) = \langle (e^2), 0, (e^4) - (e^2)^2 \rangle 12n_f \left[ \frac{n-1}{n(n+1)} \sum_{k=1}^{n} \frac{1}{k} + \frac{4}{(n+1)^2} - \frac{1}{n^2} - \frac{1}{n} \right].
\tag{10}
\]

For an arbitrary \( P^2 \) in the range \( 0 \leq P^2 \leq Q_0^2 \), we divide \( \tilde{A}_n(Q_0^2; P^2) \) into two pieces such that \( \tilde{A}_n(Q_0^2; P^2) = \tilde{A}_n(Q_0^2; P^2) + \tilde{A}_n^{(0)} \) with

\[
\tilde{A}_n(Q_0^2; P^2) = (\tilde{A}_n(Q_0^2; P^2) - \tilde{A}_n^{(0)}) = (\tilde{A}_n^\beta(Q_0^2; P^2), \tilde{A}_n^\gamma(Q_0^2; P^2), \tilde{A}_n^{\bar{n}S}(Q_0^2; P^2)).
\tag{11}
\]

Note that \( \tilde{A}_n(Q_0^2; P^2) \) contain nonperturbative contributions (i.e., hadronic components) when \( P^2 \) is in the range \( 0 \leq P^2 \leq Q_0^2 \), and satisfy the following boundary condition by definition:

\[
\tilde{A}_n(Q_0^2; P^2 = Q_0^2) = 0.
\tag{12}
\]

Then following the same procedures as we did in Ref. [14], we obtain the formula for the \( n \)th moment of \( g_1^n \) up to the NLO in QCD,

\[
\int_0^1 dx x^{n-1} g_1^n(x, Q^2, P^2) = \alpha \left[ \frac{1}{2} \frac{4 \pi}{2 \beta_0} \sum_{i=+,-,NS} L_i^n \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} \left[ 1 - \left( \frac{\alpha_s(Q_0^2)}{\alpha_s(Q_0^2)} \right) d_{i}^{n+1} \right] + \sum_{i=+,-,NS} A_i^n \left[ 1 - \left( \frac{\alpha_s(Q_0^2)}{\alpha_s(Q_0^2)} \right) \right] \right] + \sum_{i=+,-,NS} B_{i}^n \left[ 1 - \left( \frac{\alpha_s(Q_0^2)}{\alpha_s(Q_0^2)} \right) \right] + C^n + 2 \beta_0 \tilde{A}_n(Q_0^2; P^2) \sum_{i=+,-,NS} P_i^n \tilde{C}_n(1, 0) \left( \frac{\alpha_s(Q_0^2)}{\alpha_s(Q_0^2)} \right) \right],
\tag{13}
\]

which is applicable for any values of the target mass \( P^2 \) in the range \( 0 \leq P^2 \leq Q_0^2 \). Here the coefficients \( L_i^n, A_i^n \) and \( B_{i}^n \) are computed from the one- and two-loop anomalous dimensions together with one-loop coefficient functions and their explicit expressions are given in Ref. [14]. The coefficient \( C^n \) is expressed as

\[
C^n = 2 \beta_0 (3n_f (e^4) B_{i}^n + \tilde{A}_n^{(0)}. \tilde{C}_n(1, 0)).
\tag{14}
\]

The exponents \( d_{i}^{n} \) are given by \( d_{i}^{n} = \lambda_{i}^n \) \( / \) \( 2 \beta_0 \) \( (i = +, -, NS) \) where \( \lambda_{i}^n \) are the eigenvalues of the one-loop anomalous dimension matrix \( \tilde{y}_n^{(0)} \), which is expanded as \( \tilde{y}_n^{(0)} = \sum_{i} \lambda_{i}^n P_{i}^{n} \) with \( P_{i}^{n} \) being the projection operators (see Appendix A of Ref. [14] for more detail). It is noted that all the terms in the square brackets except the last one with \( \tilde{A}_n(Q_0^2; P^2) \) are calculable by the perturbative QCD. Eq. (13) is the master formula which can be applied to the kinematical region where \( Q_0^2 \) satisfies the condition (5) and \( P^2 \) takes any values between 0 and \( Q_0^2 \). When \( P^2 \) approaches \( Q_0^2 \), the last term with \( \tilde{A}_n(Q_0^2; P^2) \) vanishes, and we recover our previous result (Eq. (3.16) of Ref. [14]) for the \( n \)th moment of \( g_1^n (x, Q^2, P^2) \) which is applicable to the case \( A^2 \ll P^2 \ll Q^2 \).

Now let us study the first moment in the leading order. To this end we take the \( n \rightarrow 1 \) limit of the formula (13). As shown in Ref. [14], the terms proportional to \( L_i^n, A_i^n \) and \( B_{i}^n \) all vanish in the \( n = 1 \) limit. But the last two terms in (13) contribute to the first moment. Since \( B_{i}^{n=1} = 0 \) and \( \tilde{C}_n(1, 0) = (\langle e^2 \rangle, 0, 1)^T \) in Eq. (9), we find from Eqs. (10) and (14)

\[
C^{n=1} = -24 \beta_0 n_f (e^4).
\tag{15}
\]
we obtain the following relations: the eigenvalues $\lambda_n$ where $\alpha$ Thus we obtain for the first moment of $\hat{g}^\gamma_1(x, Q^2, P^2)$ up to the order of $\alpha$

\[
\int_0^1 dx \ g_1^\gamma(x, Q^2, P^2) = -\frac{3\alpha}{\pi} \langle \sigma \rangle + \frac{\alpha}{4\pi} \langle (\sigma^2) \rangle \hat{A}^\gamma_1(\mathbf{Q}_0^2, P^2) + \hat{A}^\gamma_{NS}(\mathbf{Q}_0^2, P^2).
\]

which holds for the range of the target photon mass squared: $0 \leq P^2 \leq Q_0^2$. It is emphasized that apart from the QED coupling constant $\alpha$, the leading order of the first moment is $O(1)$ not of order $1/\alpha_s(Q^2)$, which is the case for the general moments of $g_1^\gamma$ with $n > 1$.

In Eq. (17), the last two terms will vanish as we go to higher $P^2 \gg \Lambda^2$, while at $P^2 = 0$ they cancel out the first term which arises from the pure QED point-like interaction, as we see from Eq. (2). Now we investigate the $P^2$-dependence of these two terms. An argument which leads to Eq. (2) goes as follows [6]. Only the quark operators contribute to the first moment of $g_1^\gamma$, since gauge-invariant twist-two gluon and photon operators with spin one are absent. This remark is supported also by the expression of Eq. (17). The relevant $n = 1$ quark operators $R^\gamma_{n=1,\sigma}$ and $R^\gamma_{n=1,NS}$ (we added the Lorentz index $\sigma$) are, in fact, the flavor singlet $J^\psi_\sigma = \bar{\psi}\gamma_\sigma \gamma_5 \psi$ and nonsinglet $J^{NS}_\sigma = \bar{\psi}\gamma_\sigma (Q^2_{ch} - \langle \sigma^2 \rangle) \psi$ axial currents, respectively, where $Q_{ch}(1)$ is the $n_f \times n_f$ quarkcharge (unit) matrix. Consider the photon matrix element of these axial vector currents, which are expressed as

\[
\langle \gamma(l) | J_{\sigma_\alpha}^\psi(p) \rangle = R^\gamma_{\sigma_\alpha}(p,l) e^{\psi \beta}(l) e^\alpha(p), \quad k = \psi, \text{NS},
\]

where $p$ and $l$ are external photon momenta, and $e^\alpha(p)$ and $e^{\psi \beta}(l)$ are photon polarization vectors as shown in Fig. 1. Then the covariance, parity and crossing symmetry lead to the following tensor decomposition for the vertex function [6,30]:

\[
R^\gamma_{\sigma_\alpha}(p,l) = \epsilon_{\sigma_\alpha \beta}(p + l)^\gamma A^\beta(s^2, p^2, l^2) + \epsilon_{\sigma_\alpha \psi \beta} p^\psi l^\beta \{ B^1_\alpha(s^2, p^2, l^2) p_\beta + B^2_\alpha(s^2, p^2, l^2) l_\beta \} + \epsilon_{\sigma_\psi \beta \psi} p^\psi l^\beta \{ B^1_\psi(s^2, p^2, l^2) l_\alpha + B^2_\psi(s^2, p^2, l^2) p_\alpha \},
\]

where $s^2 = (p - l)^2$. Imposing the current conservation,

\[
p^\alpha R^\gamma_{\sigma_\alpha} = l^\beta R^\gamma_{\sigma_\alpha} = 0,
\]

we obtain the following relations:

\[
A^\beta(s^2, p^2, l^2) = p \cdot B^1_\beta(s^2, p^2, l^2) + p^2 B^1_\psi(s^2, p^2, l^2) = p \cdot l B^2_\beta(s^2, p^2, l^2) + l^2 B^2_\psi(s^2, p^2, l^2).
\]

Now restricting ourselves to the case of the forward scattering where we put $p = l$ and $s^2 = (p - l)^2 = 0$, we get

\[
R^\gamma_{\sigma_\alpha}(p,p) = 2\epsilon_{\sigma_\alpha \beta} p^\nu A^\beta(0, p^2, p^2) = \{ 2\epsilon_{\sigma_\alpha \beta} p^\nu \} p^2 (B^1_\beta(0, p^2, p^2) + B^2_\beta(0, p^2, p^2)).
\]

Since there is a relation $R^\gamma_{\sigma_\alpha}(p,p) = [2\epsilon_{\sigma_\alpha \beta} p^\nu \langle \gamma(p) | R^\gamma_{\sigma_\alpha}(0, \mu^2) \langle \gamma(p) \rangle \rangle$ the matrix element $\langle \gamma(p) | R^\gamma_{\sigma_\alpha}(0, \mu^2) \langle \gamma(p) \rangle$ is proportional to $p^2 (= -P^2)$. Hence we conclude from Eq. (4) that the first moment of $g_1^\gamma$ is expressed in the form as

\[
\int_0^1 dx \ g_1^\gamma(x, Q^2, P^2) = P^2 \tilde{B}(P^2) \equiv B(P^2).
\]

At $P^2 = 0$ we have the vanishing sum rule (2), unless there appears a massless pole like a Nambu–Goldstone boson in $\tilde{B}(P^2)$.
In order to examine the explicit $P^2$-dependence of the hadronic terms $\tilde{A}^V_{n=1}(Q^2_0; P^2)$ and $\tilde{A}^{NS}_{n=1}(Q^2_0; P^2)$ in Eq. (17), let us adopt the VMD model [31]. In this model the photon matrix element of the axial vector current $J_{S\sigma}$ is given by

$$\langle \gamma(p) | J_{S\sigma}(\mu) | \gamma(p) \rangle = 4\pi \alpha \int_0^\infty dm^2 \rho(m^2) \left( \frac{1}{m^2 + P^2} \right)^2 \langle V(m) | J_{S\sigma}(\mu) | V(m) \rangle,$$

where $|V(m)\rangle$ are the vector meson states with mass $m$, and $\rho(m^2)$ is the spectral function for the generalized vector dominance model [32], which can be written as

$$\rho(m^2) = \sum_{V=\rho,\omega,\phi} \left( \frac{m_V^2}{f_V^2} \right)^2 \delta(m^2 - m_V^2) + \rho_0(m^2).$$

The first term corresponds to the contribution from the low-lying vector mesons, $\rho$, $\omega$, $\phi$, and can be interpreted as the double-pole singularities $\delta'(m^2 - m_V^2)$ in the dispersion integral discussed in Refs. [33–35]. The second term $\rho_0$ refers to the continuous part of the spectral function. Here we assume the dominance of the low-lying vector mesons. Remembering that $C(\rho)$ to $C(\phi)$ are the vertex functions of the quark-loop diagrams so that the flavor nonsinglet part $\tilde{A}^{NS}_{n=1}(Q^2_0; P^2)$ and $\tilde{A}^{NS}_{n=1}(Q^2_0; P^2)$ may be expressed as

$$\tilde{A}^\phi_{n=1}(Q^2_0; P^2) = \sum_{V=\rho,\omega,\phi} \left( \frac{m_V^2}{f_V^2} \right)^2 \langle V | J_5(Q^2_0) | V \rangle,$$

$$\tilde{A}^{NS}_{n=1}(Q^2_0; P^2) = \sum_{V=\rho,\omega,\phi} \left( \frac{(e^4 - m_V^2)}{f_V^2} \right)^2 \langle V | J_5(Q^2_0) | V \rangle,$$

where $\langle V | J_5(Q^2_0) | V \rangle$ is a common factor the magnitude of which will be determined later. In Eq. (26), we have taken the viewpoint that the photon matrix elements of the axial vector current arise dominantly from the quark-loop diagrams so that the flavor nonsinglet part $\tilde{A}^{NS}_{n=1}$ is proportional to $\text{Tr}(Q^2_0 - (e^4) 1) = 3n_f(e^4 - (e^2)^2)$, while the singlet part $\tilde{A}^\phi_{n=1}$ is proportional to $\text{Tr}(Q^2_0 1) = 3n_f(e^2)$.

In order for $\tilde{A}^\phi_{n=1}(Q^2_0; P^2)$ and $\tilde{A}^{NS}_{n=1}(Q^2_0; P^2)$ to satisfy the boundary condition (12), we either subtract the values at $P^2 = Q^2_0$ or introduce a sharp cutoff at $P^2 = Q^2_0$. Here we are considering the case where $Q^2_0$ is sufficiently large ($Q^2_0 \gg \Lambda^2$) and thus the subtraction terms are negligible. On the other hand, the vanishing of the first moment of $g^Y_1$ at $P^2 = 0$ gives a condition on $\langle V | J_5(Q^2_0) | V \rangle$. We find from Eqs. (17) and (26),

$$\sum_{V=\rho,\omega,\phi} C^{(V)}(Q^2_0) = 1,$$

with $C^{(V)}(Q^2_0) = 1/12n_f(\langle V | J_5(Q^2_0) | V \rangle)$. Since $Q^2_0$ is sufficiently large, we expect that the $Q^2_0$ dependence of $C^{(V)}(Q^2_0)$ is rather mild, and hence we neglect the $Q^2_0$ dependence of $C^{(V)}(Q^2_0)$ from now on and write simply as $C^{(V)}$. Then the sum rule is now written as

$$\int_0^1 dx g^Y_1(x, Q^2, P^2) = -\frac{3\alpha}{\pi} n_f(e^4) + \frac{3\alpha}{\pi} n_f(e^4) \sum_{V=\rho,\omega,\phi} C^{(V)} \left( \frac{m_V^2}{m_V^2 + P^2} \right)^2.$$
Next we pursue the QCD corrections of order \(\alpha_s\) to the first moment of \(g_1^\gamma(x, Q^2, P^2)\) for an arbitrary target mass \(P^2\) in the range \(0 \leq P^2 \leq Q_0^2\). We now take into account the \(\alpha_s\) corrections to the coefficient functions in Eq. (9) as well as the \(\mathcal{O}(\alpha_s)\) terms in the evolution factor \(M_n (Q^2 / Q_0^2, \bar{g} (Q^2))\) in Eq. (8) (see Ref. [14] for detail). We can use the same \(\tilde{A}^{\psi}_{n=1}(Q_0^2; P^2)\) and \(\tilde{A}^{NS}_{n=1}(Q_0^2; P^2)\) given in Eq. (26) and we get

\[
\int_0^1 dx g_1^\gamma(x, Q^2, P^2) = -\frac{3\alpha}{\pi} n_f \langle \epsilon^4 \rangle \left( 1 - \frac{\alpha_s(Q^2)}{\pi} \right) + \frac{3\alpha}{\pi} n_f \langle \epsilon^2 \rangle \frac{2n_f}{\beta_0} \left( \frac{\alpha_s(Q_0^2)}{\pi} - \frac{\alpha_s(Q^2)}{\pi} \right) \\
+ \frac{\alpha}{4\pi} \langle \epsilon^2 \rangle \tilde{A}^{\psi}_{n=1}(Q_0^2, P^2) \left( 1 - \frac{\alpha_s(Q^2)}{\pi} - \frac{2n_f}{\beta_0} \left( \frac{\alpha_s(Q_0^2)}{\pi} - \frac{\alpha_s(Q^2)}{\pi} \right) \right) \\
+ \frac{\alpha}{4\pi} \tilde{A}^{NS}_{n=1}(Q_0^2; P^2) \left( 1 - \frac{\alpha_s(Q^2)}{\pi} \right).
\]

At \(P^2 = Q_0^2\), we have \(\tilde{A}^{\psi}_{n=1}(Q_0^2 = P^2; P^2) = \tilde{A}^{NS}_{n=1}(Q_0^2 = P^2; P^2) = 0\), and we recover the previous result (3). For the real photon target \((P^2 = 0)\), we obtain from Eq. (26) and the condition (27),

\[
\tilde{A}^{\psi}_{n=1}(Q_0^2; P^2 = 0) = 12 n_f \langle \epsilon^2 \rangle, \quad \tilde{A}^{NS}_{n=1}(Q_0^2; P^2 = 0) = 12 n_f \langle \epsilon^4 \rangle - \langle \epsilon^2 \rangle^2,
\]

and thus we see that the first moment of \(g^\gamma_1\) to the order \(\alpha_s\) indeed vanishes for the real photon target. If we use the vertex function \(B(P^2)\) given in Eq. (29), the first moment (30) can be rewritten as

\[
\int_0^1 dx g_1^\gamma(x, Q^2, P^2) = B(P^2) \left[ \left( 1 - \frac{\alpha_s(Q^2)}{\pi} \right) - \frac{n_f \langle \epsilon^2 \rangle^2}{\langle \epsilon^4 \rangle} \frac{2\alpha_s(Q_0^2)}{\pi} - \frac{\alpha_s(Q^2)}{\pi} \right].
\]

Note that the order \(\alpha_s\) QCD effect is factorizable and amounts to reduce the leading order result by about 7% for \(Q_0^2 = 1.0\) GeV\(^2\) and \(Q^2 = 30\) GeV\(^2\). This analysis would be easily extended to the case for the order \(\alpha_s^2\) QCD effect on the first moment [19]. We also note here that there has been an analysis of the first moment sum rule of \(g_1^\gamma(x, Q^2, P^2)\) for the intermediate values of \(P^2\) based on the more general principle like chiral symmetry and making a connection with the off-shell radiative couplings of the pseudo-scalar mesons [13].

Finally we investigate the \(x\)-dependence of the structure function \(g_1^\gamma(x, Q^2, P^2)\) for an arbitrary value of \(P^2\) between 0 and \(Q_0^2\). Once we know all the moments, we can perform the inverse Mellin transform of the moments to get \(g_1^\gamma\) as a function of \(x\). The \(n\)th moment of \(g_1^\gamma(x, Q^2, P^2)\) up to the NLO in QCD is given by the formula (13). All the quantities in there are already known and are collected in Ref. [14], except for \(\tilde{A}^{\psi}_{n}(Q_0^2; P^2), \tilde{A}^{G}_{n}(Q_0^2; P^2)\) and \(\tilde{A}^{NS}_{n}(Q_0^2; P^2)\). Since photon does not couple to gluon field directly, the contribution of the term with \(\tilde{A}^{G}_{n}(Q_0^2; P^2)\) is expected to be much smaller than those with \(\tilde{A}^{\psi}_{n}(Q_0^2; P^2)\) and \(\tilde{A}^{NS}_{n}(Q_0^2; P^2)\) and thus we assume \(\tilde{A}^{G}_{n}(Q_0^2; P^2) = 0\). To estimate the \(P^2\)-dependence of \(\tilde{A}^{\psi}_{n}(Q_0^2; P^2)\) and \(\tilde{A}^{NS}_{n}(Q_0^2; P^2)\), we adopt the VMD model again. We have seen in Fig. 2 that as regards the \(P^2\)-dependence of the first moment of \(g_1^\gamma\), the \(\rho\)-dominance
where the authors presented the maximal and minimal values of $g_{1\gamma}$ assume to be a binomial function as follows:

$$P = \text{multiple factor of the renormalization scale.}$$

is a good approximation to the case where all the three low-lying vector mesons $\rho, \omega, \phi$ are included. Therefore, we assume the $\rho$-dominance and consider only the contribution of $\rho$ meson taking $m_\rho = m_\rho$. Now recalling that $\tilde{A}_{\gamma}^y(Q^2_0; P^2)$ and $\tilde{A}_{\gamma}^{NS}(Q^2_0; P^2)$ satisfy the boundary condition (12) at $P^2 = Q^2_0$, we take the following forms for $0 \leq P^2 \leq Q^2_0$:

$$\tilde{A}_{\gamma}^y(Q^2_0; P^2) = 12n_f|e|^2 \left[ \left( \frac{m_\rho^2}{m_\rho^2 + P^2} \right)^2 - \left( \frac{m_\rho^2}{m_\rho^2 + Q^2_0} \right)^2 \right] \left[ 1 - \left( \frac{m_\rho^2}{m_\rho^2 + Q^2_0} \right)^2 \right]^{-1} \times \tilde{f}(n),$$

$$\tilde{A}_{\gamma}^{NS}(Q^2_0; P^2) = 12n_f \langle |e^4| - \langle e^2 \rangle^2 \rangle \left[ \left( \frac{m_\rho^2}{m_\rho^2 + P^2} \right)^2 - \left( \frac{m_\rho^2}{m_\rho^2 + Q^2_0} \right)^2 \right] \left[ 1 - \left( \frac{m_\rho^2}{m_\rho^2 + Q^2_0} \right)^2 \right]^{-1} \times \tilde{f}(n),$$

where we have inserted a multiplication factor $(1 - (m_\rho^2/(m_\rho^2 + Q^2_0))^2)^{-1}$ so that $\tilde{A}_{\gamma}^y$ and $\tilde{A}_{\gamma}^{NS}$ with $n = 1$ fulfill the conditions (31) at $P^2 = 0$. Here $\tilde{f}(n)$ is the Mellin transform of the quark parton distribution function $f(x)$ inside the vector meson, which we assume to be a binomial function as follows:

$$f(x) = B(p, q)^{-1}x^{p-1}(1-x)^{q-1}, \quad \tilde{f}(n) = \frac{1}{x} \int_0^1 x^{n-1} f(x) \, dx, \quad \tilde{f}(n = 1) = 1.$$

Although there might be more sophisticated VMD inputs [35,38], we consider the simplest case with $p = q = 2$.

In Fig. 3, we have plotted $g_1^\gamma(x, Q^2, P^2)$ in units of $3n_f(\alpha/\pi)(e^4)$ for various values of $P^2$. We took $P^2 = 0, 0.1, 0.3, 0.6, 1.0$ GeV$^2$ with $Q^2 = 30$ GeV$^2$, $Q^2_0 = 1.0$ GeV$^2$, $n_f = 3$ and $\Lambda = 0.2$ GeV. Note that we can read off the tendency where the first moment sum rule vanishes for the real photon ($P^2 = 0$), and then turns to be more negative as the mass squared of the virtual photon $P^2$ increases.

Here we also note that the $g_1^\gamma$ of real and virtual photons have been studied in NLO QCD using the positivity constraints in [18], where the authors presented the maximal and minimal values of $g_1^\gamma$ for real ($P^2 = 0$) and virtual ($P^2 = 1$ GeV$^2$) photons, which appear to be consistent with our present analysis.

In order to make sure that our analysis is stable under the change of the renormalization scale $Q^2_0$, we show in Fig. 4 the $Q^2_0$-dependence of $g_1^\gamma(x, Q^2, P^2)$ in units of $3n_f(\alpha/\pi)(e^4)$ for the case of a real photon ($P^2 = 0$), with $Q^2 = 30$ GeV$^2$, $n_f = 3$ and $\Lambda = 0.2$ GeV. Three curves with $Q^2_0 = 0.75, 1.00, 1.25$ GeV$^2$ almost overlap in the whole $x$ region and we see that there appears no sizable dependence on $Q^2_0$.

In summary, we have investigated in QCD the transition of the polarized photon structure function $g_1^\gamma$ when the target photon shifts from on-shell to far off-shell region up to the NLO. The first moment of $g_1^\gamma$ vanishes for the real photon, turns to a negative value when the target photon becomes off-shell. Although our estimate of the non-perturbative effects of the photon matrix elements relies on the VMD model, we have studied the explicit $P^2$-dependence not only for the first moment sum rule but also for the structure function $g_1^\gamma(x, Q^2, P^2)$, in particular, as a function of $x$. It turns out that the results are not so sensitive to the choice of the renormalization scale.
Acknowledgements

This work is supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, Japan No. 18540267.

References