Monomial Patterns in the Sequence $A^k b$

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ABSTRACT

We consider the pattern of zero and nonzero elements in the sequence $A^k b$, where A is an $n \times n$ nonnegative matrix and b is an $n \times 1$ nonnegative column vector. We establish a tight bound of $k < n$ for the first occurrence of a given monomial pattern, and we give a graph theoretic characterization of triples (A, b, i) such that there exists a k, $k \geqslant n$, for which $A^k b$ is an i-monomial. The appearance of monomial patterns with a single nonzero entry is linked to controllability of discrete n-dimensional linear dynamic systems with positivity constraints on the state and control.

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1. INTRODUCTION

In the course of investigating a control theoretic question, Coxson and Shapiro [1] showed that, for a nonnegative matrix A with a positive diagonal and for a nonnegative vector b, an i-monomial pattern consisting of a single nonzero entry in the ith position will appear in $A^k b$ for some $k < n$, or it will not appear at all. They conjectured that the result holds even without restriction on the diagonals of A.

It is the main purpose of this paper to prove the conjecture of Coxson and Shapiro: see Theorem 1 in Section 2. We call a triple (A, b, i) for which there exists a k, $k \geqslant n$, such that $A^k b$ is i-monomial a *monophil* triple. In Section 3 we describe some graph theoretic properties of monophil triples, and we determine, in Lemma 5, the set of k such that $A^k b$ is i-monomial. From these properties, we derive a graph theoretic characterization of monophil triples in Theorem 2, Section 4, and we pursue some of its consequences. In Section 5 we show that for monophil triples the bound $k < n$ for the least k such that $A^k b$ is i-monomial is tight, and we display matrices and vectors for which the first i-monomial power is $n-1$. A final Section 6 explains the control theoretic background.

Our proofs proceed by means of a translation of combinatorial matrix properties into graph theoretic terms. We have structured our paper so as to derive Theorem 1 as early as possible. An alternative approach would be first to prove the characterization of monophil triples contained in Theorem 2 and then to derive Theorem 1.

The investigation of the combinatorial properties of the powers of a nonnegative matrix A is classical and of importance in applications to areas such as the theory of Markov chains. It is known that there exist a positive c and a nonnegative K_0 such that A^{k+c} has the same pattern as A^k for all $k \geqslant K_0$. Many papers derive bounds for the least such c and the corresponding least K_0; see, for example, [2]–[10]. We do not use these results directly, though our paper is in the same spirit and contains theorems of a similar type.

2. THE MAIN RESULT

A matrix or vector M is nonnegative, denoted $M \geqslant 0$, if all of its entries are nonnegative real numbers. A nonnegative vector x will be called a *monomial column* if it has precisely one nonzero entry. If the nonzero entry of a monomial column is in the ith position, we will refer to it as an *i-monomial*. The following theorem is the focus of this paper.

THEOREM 1. *Let A be an $n \times n$ nonnegative matrix and b be an $n \times 1$ nonnegative column vector. If there is an i-monomial in the sequence $\{A^k b: k = n, n+1, n+2, \dots\}$, then there is an i-monomial in the sequence $\{A^k b: k = 0, 1, 2, \dots, n-1\}$.*

Of course, Theorem 1 implies the conjecture of Coxson and Shapiro [1] which is stated in the first paragraph of the introduction.

Our proof of this result is based on graph theoretic considerations. The (directed) graph of an $n \times n$ nonnegative matrix A, denoted $G(A)$, is the graph with n nodes having a directed arc from node i to node j if and only if the (i, j) entry of A is positive. For our purpose, it will be much more convenient to think in terms of the graph $G^-(A) := G(A^T)$, in which the direction of each arc is reversed.

A (directed) path of $G^-(A)$ is *simple* if it has no repeated nodes. Let S be a subset of $\{1, \dots, n\}$. A path of $G^-(A)$ that starts at a node of S is called an *S-path*. If i is a node of $G^-(A)$, then an $\{i\}$-path is called an *i-path*. An S-path that ends at a node i is called an *(S, i)-path*. By *cycle* we shall always mean a cycle without repeated nodes, except for the first and last. Two cycles with the same arc set are identified. An i-path that is a cycle is called an *i-cycle*. By the previous remark, an i-cycle is also a j-cycle for every node j on the cycle.

For an $n \times n$ nonnegative matrix A and an $n \times 1$ nonnegative column vector b, the product $A^k b$ has the following interpretation in terms of $G^-(A)$. Let $S = \{s_1, s_2, \dots, s_V\}$ denote the positions of all the nonzero entries of b. We call S the support of b, and we write $S = \text{supp}(b)$. We identify the set S with the corresponding subset of the nodes of $G^-(A)$. Then the jth entry of $A^k b$ is nonzero if and only if there is an (S, j)-path of length k in $G^-(A)$. Thus we obtain the following key proposition which allows us to restate Theorem 1 in graph theoretic terms. It will be used many times in our proofs, often without further reference.

PROPOSITION 1. *Let A be a nonnegative matrix, b a nonnegative vector, and i a node of $G^-(A)$. Let $S = \text{supp}(b)$. Then $A^k b$ is i-monomial if and only if there is at least one (S, i)-path of length k in $G^-(A)$ and every S-path of length k ends at i.*

We can now state Theorem 1 in the following equivalent form.

THEOREM 1G. *Let A be an $n \times n$ nonnegative matrix, and S a subset of nodes of $G^-(A)$. For some k, $k \geq n$, suppose there is an S-path of length k, and suppose that every S-path of length k terminates at the ith node of*

$G^{-}(A)$. Then there is a $j < n$ such that all S-paths of length j also terminate at node i.

Let A be an $n \times n$ nonnegative matrix, b a nonnegative vector, and i a positive integer, $1 \leqslant i \leqslant n$. We call the triple (A, b, i) *monophil* if there exists an integer k, $k \geqslant n$, such that $A^k b$ is i-monomial.

LEMMA 1. *Let the triple (A, b, i) be monophil, and let $S = \operatorname{supp}(b)$. Then no S-path can meet a cycle which is not an i-cycle.*

Proof. Suppose $A^k b$ is i-monomial, $k \geqslant n$, If the S-path P meets a cycle which is not an i-cycle, then there is a path of length k which ends on that cycle, contrary to Proposition 1. ■

A path *augmented* (*reduced*) by a cycle is a path with the same beginning and end but covering one more (less) cycle than the original path. A path of length 0 is identified with a node.

Proof of Theorem 1G. By assumption there is a k, $k \geqslant n$, such that all S-paths of length k end at node i, and such that there is at least one S-path P of length k which ends at i. Then, by Lemma 1, all cycles that can be reached from S must be i-cycles. Since the length of P is greater than $n-1$, the path P includes a cycle C. Let c denote the length of C. Then $c \leqslant n \leqslant k$. We claim that every S-path of length $k - c$ either ends at node i or is disjoint from C, for if it contains a node of C it can be augmented to a path of length k.

Now suppose that k, $k \geqslant n$, is the minimal integer such that all S-paths of that length end at i. Then there is an S-path R of length $k - c$ with endpoint other than i. By the above argument, the nodes of C and R are disjoint. In particular, i is not a node of R. Hence, by Lemma 1, R contains no cycle and therefore has $k - c + 1$ nodes. Since C has c nodes, it follows that $k + 1 = (k - c + 1) + c \leqslant n$, a contradiction. ■

The proof of Theorem 1G above does not explicitly contain the insights into the structure of S-paths in $G^{-}(A)$ which originally led us to a proof. Those insights are detailed next in our discussion of the graph theoretic properties of monophil triples. We found Lemma 3 and Figure 1 to be especially helpful in understanding the result.

3. PROPERTIES OF MONOPHIL TRIPLES (A, b, i)

Lemma 2 follows immediately from Proposition 1 and the definition of a monophil triple.

LEMMA 2. *Let the triple* (A, b, i) *be monophil, and let* $S = \text{supp}(b)$. *Then there is an S-path which contains a cycle.*

Let the triple (A, b, i) be monophil. By $k_0 = k_0(A, b, i)$ we shall always denote the *least* integer k for which $A^k b$ is i-monomial. If $S = \text{supp}(b)$, then $p_{\max}$ will denote the length of the longest S-path which does not meet a cycle. If there is no such path, we put $p_{\max} = -1$. The result of Theorem 1 may now be stated as $k_0 \leqslant n-1$. In fact, we have proved more, namely, $k_0 \leqslant p_{\max} + c$, where c is the length of a cycle contained in some S-path. In view of the next lemma, this result is of some interest.

LEMMA 3. *If* (A, b, i) *is monophil, then every simple cycle through the node* i *must have the same length* c.

Proof. Let $A^k b$ be i-monomial, $k \geqslant n$. Suppose there are two cycles containing node i of lengths c_1 and c_2, with $c_1 < c_2$. By Lemma 1, both cycles are i-cycles. All S-paths of length k terminate at node i. Select s in S such that there exists a path of length k originating at node s of $G^-(A)$. Consider the following two paths:

Path 1: Begin at node s in S. At the first occurrence of node i (say after p steps), follow the cycle of length c_1 until the first return to node i. Then follow along the longer cycle until a path of length k has been determined.

Path 2: Begin at node s, and take the same route as path 1 to reach node i for the first time. Now follow the cycle of length c_2 and continue to cycle until the path has length k.

At the $(p + c_1)$th step, path 1 reaches node i and path 2 arrives at a node which is a distance c_1 beyond node i on the cycle of length c_2. From this point on, paths 1 and 2 are always a distance c_1 apart on the cycle of length c_2. In particular, they cannot both terminate at node i as required. Thus, all cycles must have the same length, as claimed. ■

By c we shall denote the common length of all i-cycles if (A, b, i) is monophil. We now immediately have the following corollary to Theorem 1G.

COROLLARY 1. *Let (A, b, i) be monophil. Then*

$$k_0 \leqslant p_{\max} + c \leqslant n - 1. \tag{3.1}$$

We shall investigate the cases of equality in (3.1) in Section 5.

LEMMA 4. *Let (A, b, i) be monophil, and let P be a simple (i, j)-path of length d. Then $d < c$.*

Proof. If j lies on an i-cycle, then $d < c$, since the path is simple. Suppose j does not lie on an i-cycle, and suppose that $d \geqslant c$. Since the path is simple, the subpath P' of P of length c which also starts at i ends at a node j' with $j' \neq i$. Let Q be an (S, i)-path of length k, $k \geqslant n$. Reduce Q by an i-cycle, and then continue the reduced path by adjoining P'. We obtain an S-path of length k which does not end at i, contrary to assumption. ∎

LEMMA 5. *Let the triple (A, b, i) be monophil. Then the vector $A^k b$ is i-monomial if and only if $k = k_0 + mc$, where m is a nonnegative integer.*

Proof. Suppose $A^k b$ is i-monomial. Since by Lemmas 2 and 3 there exists an i-cycle of length c, and by Lemma 4 the only i-paths of length c are i-cycles, it follows immediately that $A^{k+c} b$ is i-monomial. Hence, by induction, $A^k b$ is i-monomial if $k = k_0 + mc$, where m is a nonnegative integer.

Conversely, suppose that $A^k b$ is i-monomial and that $0 < d < c$. By Lemmas 1 and 3 there exists an i-path of length d along an i-cycle which does not end at i. It follows that $A^{k+d} b$ is not i-monomial. The result follows. ∎

COROLLARY 2. *Let A be an $n \times n$ nonnegative matrix, b an $n \times 1$ nonnegative column vector, and i a node of $G^-(A)$. Then either all integers k such that $A^k b$ is i-monomial satisfy $k < n$, or else there are an infinity of integers k such that $A^k b$ is i-monomial.*

Proof. If there exists a $k \geqslant n$ such that $A^k b$ is i-monomial, then (A, b, i) is monophil, and the corollary follows from Lemma 5. ∎

In the next section, we use these results to specify necessary and sufficient conditions for a triple to be monophil.

4. GRAPH THEORETIC CHARACTERIZATION OF MONOPHIL TRIPLES

THEOREM 2. *Let A be an $n \times n$ nonnegative matrix, let b be a nonnegative vector, and let i be a node of $G^-(A)$. Then (A, b, i) is monophil if and only if the following conditions hold for $G^-(A)$:*

(a) *There exists a cycle which lies on an S-path, where $S = \mathrm{supp}(b)$.*
(b) *Every cycle that can be reached from S is an i-cycle.*
(c) *Every i-cycle has the same length c.*
(d) *Every simple i-path has length less than c.*
(e) *If P and Q are two simple (S, i)-paths of lengths p and q respectively, then $p \equiv q \pmod{c}$.*

Proof. Suppose $A^k b$ is i-monomial, where $k \geqslant n$. Conditions (a), (b), (c) and (d) are Lemmas 2, 1, 3, and 4, respectively. For (e), note that by (c) and Lemma 5 there must exist m and m' such that $p + mc = q + m'c = k$.

Conversely, suppose that (a)–(e) are satisfied. By (a) and (b) there exists a simple (S, i)-path P. Let p be its length, and let m be an integer such that $k = p + mc \geqslant n$. By (c), we can augment P by adjoining m i-cycles to obtain a path of length $p + mc = k$. Thus conditions (a), (b) and (c) imply that there is an (S, i) path of length k. Now suppose that R is an S-path of length k which is not an (S, i)-path. Since R contains a cycle, by (b) it contains an i-cycle and all of its cycles are i-cycles. Thus R is obtained from a simple (S, i)-path Q, repeatedly augmenting Q by an i-cycle, and then adjoining a simple i-path D not ending at i. Denote the lengths of Q and D by q and d, respectively. By (c) we have $p + mc = k = q + m'c + d$, for some positive integer m', so that $p \equiv q + d \pmod{c}$. Using the congruence in (e), we conclude $0 \equiv d \pmod{c}$. But, since D is nonempty, we have $d > 0$, and from (d) we know $d < c$, a contradiction. Hence the path R is an (S, i)-path, and, in view of Proposition 1, the theorem is proved. ∎

From Theorem 2, we see that an S-path Q for monophil (A, b, i) has very simple structure, as illustrated in Figure 1. It first traces a route of length p, which does not meet any cycle. (For consistency, we define p to be -1 if the initial node of Q is on a cycle.) Then it moves around the various cycles through i, all of length c. After the last occurrence of node i, it follows an i-path D of length $d < c$.

COROLLARY 3. *If (A, b, i) is monophil, then (A, e_i, i) is monophil, where e_i is the ith canonical unit vector.*

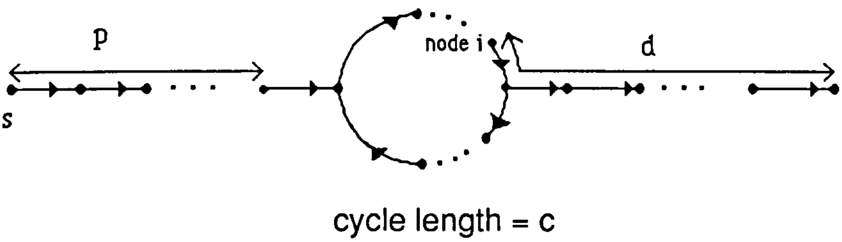


FIG. 1.

Proof. If (A, b, i) is monophil, then conditions (a)–(e) of Theorem 2 hold. Note that (c) and (d) are independent of the vector b. Conditions (a) and (b) for (A, e_i, i) follow from (a) and (b) for (A, b, i). Condition (e) also holds for $S = \{i\}$, since the only simple i-path which ends at i is empty and has length 0. Hence the result follows from the converse direction of Theorem 2. ■

In view of an application in Section 6, it is of interest to determine whether all i-monomials appear among the columns $\{A^k b_j : j = 1, \dots, m;\ k = 0, 1, \dots\}$. By Theorem 1, it is sufficient to consider $k \leqslant n-1$. If an i-monomial appears in $\{A^k b : k = 0, \dots, n-1\}$ we shall say that the triple (A, b, i) is *submonophil.* Every monophil triple is submonophil (Theorem 1), but the converse is false. Theorem 3 characterizes the graphs $G^-(A)$ and corresponding vectors b such that (A, b, i) is submonophil for all nodes i (here $m = 1$).

THEOREM 3. *Let A be an $n \times n$ nonnegative matrix, and let b be a nonnegative vector. Then the following are equivalent:*

(a) *(A, b, i) is submonophil for each i, $i = 1, \dots, n$.*

(b) *After a permutation of $(1, \dots, n)$, $\mathrm{supp}(b) = \{1\}$ and the set of arcs of $G^-(A)$ is the union of $\{(1,2),(2,3),\dots,(n-1,n)\}$ and an arbitrary subset of $\{(n,n),(n,n-1),\dots,(n,1)\}$ (Figure 2).*

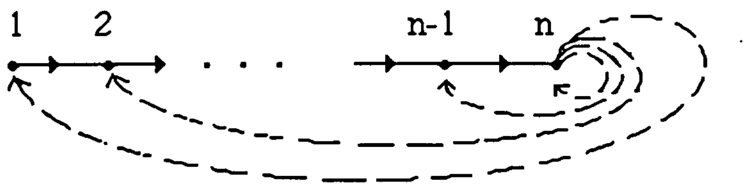


FIG. 2.

Proof. Suppose b and $G^{-}(A)$ satisfy condition (b). Then $A^{i-1}b$ is i-monomial, $i = 1,\dots,n$, so that condition (a) holds.

Conversely, suppose that A and b satisfy condition (a). Since $k_0(A, b, i) < n$ for each i, it follows that (after permutation of indices) $A^{i-1}b$ is i-monomial, $i = 1,\dots,n$. In particular, $\operatorname{supp}(b) = \{1\}$. Since Ab is 2-monomial, we see that $(1,2)$ is the only arc beginning at node 1. The result follows by a repetition of this argument. ∎

Corollary 4. *Let A be an $n \times n$ nonnegative matrix. Then the following conditions are equivalent:*

(a) *For every node i of $G^{-}(A)$ there is a nonnegative vector b_i such that (A, b_i, i) is monophil.*

(b) *$G^{-}(A)$ consists of a union of disjoint cycles which cover all nodes, and b_i is a monomial vector whose support is a node on the cycle containing node i.*

Proof. Suppose condition (b) holds. Let d_i denote the distance from $\operatorname{supp}(b_i)$ to node i, and let c_i denote the length of the cycle containing node i. Then $A^k b_i$ is i-monomial for all $k \equiv d_i \mod c_i$ and thus (A, b_i, i) is monophil.

For the converse, assume condition (a). From Theorem 2(a) and 2(b), it follows that every node of $G^{-}(A)$ is on a cycle. Suppose nodes i and j are on the same cycle. Then any i-cycle can be reached from any j-cycle, so Theorem 2(b) implies that the cycle through i (and j) is unique and the only simple (i, j)-path is the path along this cycle. Now suppose nodes i' and j' lie on different cycles. An arc (i', j') would join the i'-cycle to the j'-cycle, violating Theorem 2(b) for $(A, b_{i'}, i')$. The result follows. ∎

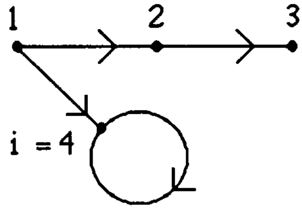
Evidently, if $G^{-}(A)$ is the union of m disjoint cycles, the minimum number of distinct b_i such that (A, b_i, i) is monophil for every node i of $G^{-}(A)$ is m. In particular, we have the following corollary.

Corollary 5. *Let A be a nonnegative matrix. Then there is a nonnegative vector b such that (A, b, i) is monophil for each i if and only if $G^{-}(A)$ consists of a single full cycle.*

5. THE BOUND IS SHARP

The bound $k_0 \leqslant p_{\max} + c \leqslant n - 1$ is best possible, as illustrated in Figure 3, where $k_0 = 3 = n - 1$.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$



$$\{A^k b\}_{k=0}^{3} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

FIG. 3.

The general structure of a monophil (A, b, i), $i = n$, having $k_0 = n - 1$ is illustrated in Figure 4 for the case $n = 10$, $p_{\max} = 6$, $c = 3$. The graph $G^-(A)$ has precisely one cycle of length c and precisely one cycle free S-path of length $p_{\max}$ which does not meet the cycle. The nodes are numbered consecutively, beginning with the initial node 1 of the cycle free path and ending with $i = n$ on the cycle. In the matrices A and b, 1's denote positions which *must* have nonzero character. The X character denotes positions which may be chosen to be zero or nonzero. The # character denotes positions which may be nonzero, but may be forced to be zero if certain X's

$$\overset{\longleftarrow\; p_{\max}+1 \;\longrightarrow}{}$$

$$A = \left(\begin{array}{ccccccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \# & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \# & \# & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdashline{8-10} x & \# & \# & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \# & x & \# & \# & 1 & 0 & 0 & 0 & 0 & \# \\ \# & \# & x & \# & \# & 1 & 0 & \# & 0 & \# \\ \hline x & 0 & 0 & x & 0 & 0 & x & 0 & 0 & 1 \\ 0 & x & 0 & 0 & x & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & x & 0 & 0 & 1 & 0 \end{array}\right) \begin{matrix} \\ \\ \\ \\ \updownarrow c \\ \\ \\ \updownarrow c \\ \\ \\ \end{matrix} \qquad b = \left(\begin{array}{c} 1 \\ \# \\ \# \\ x \\ \# \\ \# \\ x \\ \hline 0 \\ 0 \\ * \end{array}\right) \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \updownarrow c \\ \\ \\ \end{matrix}$$

FIG. 4.

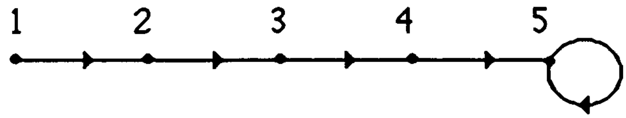


FIG. 5.

are nonzero. For example, if $a_{87} \neq 0$, then all positions marked # are zero. Note that the X's of A all lie on subdiagonals separated by the distance c. If all of the X's in the last three rows of A are zero—indicating that the cycle and the cycle free path are disjoint—then the single entry marked $*$ in b must be nonzero.

We note that it is easy to construct an example for which the first monomial column (of any kind) in the sequence $\{A^k b: k = 0,1,2,\dots\}$ occurs when $k = n-1$. For example, let $G^-(A)$ be given in Figure 5, and let $\mathrm{supp}(b) = \{1,2\}$.

6. MONOMIAL PATTERNS AND POSITIVE REACHABILITY OF LINEAR SYSTEMS

A discrete single input linear dynamic system is given by

$$x(k+1) = Ax(k) + Bu(k), \qquad x(0) = x_0, \tag{6.1}$$

$$x(k) \text{ in } \mathbf{R}^n \quad \text{and} \quad u(k) \text{ in } \mathbf{R}^m,$$

where A is an $n \times n$ real matrix and B is an $n \times m$ real matrix. $x(k)$ is a real $n \times 1$ column vector, called the *state* vector, for each $k = 0,1,2,\dots,$ and $u(k)$ is a real m-dimensional *input* to the system. For a given initial state $x(0) = x_0$, the solution of (6.1) is given by

$$x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^j B u(k-1-j).$$

We will be interested in the states x in $\mathbf{R}^n$ which can be reached from $x_0 = 0$. This set of reachable states is clearly the subspace of $\mathbf{R}^n$ spanned by the column vectors of $\{A^j B\}$. For each $j \geq n$, the Cayley-Hamilton theorem provides that $A^j B$ can be expressed as a linear combination of $\{A^i B: i = 0,1,\dots,n-1\}$. Thus, any state which can be reached in a finite number of steps can also be reached within n steps with an appropriate choice of inputs $u(i)$. The reachable set is all of $\mathbf{R}^n$ if and only if $\mathbf{R}^n$ is spanned by $\{A^jB: j = 0, 1, \dots, n-1\}$.

Next consider a positive discrete linear system,

$$x(k+1) = Ax(k) + Bu(k), \qquad x(0) = x_0,$$

$$x(k) \text{ in } \mathbf{R}^n_+ \quad \text{and} \quad u(k) \text{ in } \mathbf{R}^m_+,$$

with A, B, x_0, and $u(j)$ constrained to be nonnegative. Under these conditions, $x(k)$ will be nonnegative for all $k \geqslant 0$. Such nonnegativity constraints are an essential feature of a wide range of applications, including chemical reaction systems, where the underlying states are masses of chemicals which can never be negative.

Due to the nonnegativity of $\{A^kB\}$ and the input $u(k)$, the reachable set of the positive system is contained in the nonnegative orthant $\mathbf{R}^n_+$. This set is denoted by $\mathbf{R}_\infty$. The subset $\mathbf{R}_k$ of states which can be reached within k steps is the polyhedral cone generated by the nonnegative columns of $\{A^jB: j = 0, \dots, k-1\}$. The Cayley-Hamilton theorem again guarantees that A^kB can be expressed as a linear combination of $\{A^jB: j = 0, 1, \dots, n-1\}$. But the coefficients are not necessarily nonnegative, so we cannot conclude, as in the unconstrained case, that $\mathbf{R}_\infty = \mathbf{R}_n$.

$\mathbf{R}_k = \mathbf{R}^n_+$ if and only if each of the n independent i-monomials of $\mathbf{R}^n_+$ is an element of $\mathbf{R}_k$. We let e_i denote the i-monomial with a 1 in the ith position. Since e_i is extremal in $\mathbf{R}^n_+$, it is possible to obtain

$$e_i = \sum_{j=0}^{k-1} A^jBu(k-1-j)$$

with $u(j) \geqslant 0$, $j = 0, \dots, k-1$, if and only if A^jB has an i-monomial column for some nonnegative integer j. If A^jB has an i-monomial column, then e_i can be reached in $j+1$ steps.

Thus Theorem 1 applied to each column b of B implies that $\mathbf{R}_\infty = \mathbf{R}^n_+$ if and only if $\mathbf{R}_n = \mathbf{R}^n_+$; that is, if all nonnegative states are reachable, then they are reachable within n steps ($\mathbf{R}_\infty = \mathbf{R}_n$). Note, however, that if $\mathbf{R}_\infty \neq \mathbf{R}^n_+$, then it is possible that $\mathbf{R}_\infty \neq \mathbf{R}_n$. For example,

$$\text{if} \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{then} \quad A^kb = \begin{pmatrix} k \\ 1 \end{pmatrix},$$

and $\mathbf{R}_k$ is strictly contained in $\mathbf{R}_{k+1}$ for each k.

If $\mathbf{R}_\infty = \mathbf{R}^n_+$, then for each node i of $G^-(A)$ there is a column b of B such that (A, b, i) is submonophil. For the single input system ($B = b$, a column vector), a characterization of A and b such that $\mathbf{R}_\infty = \mathbf{R}^n_+$ is given in Theorem 3. If (A, b, i) is monophil, then i-monomials can be reached repeatedly over time. If (A, b, i) is submonophil but not monophil, then i-monomials can be reached only in the initial n steps following an input (see Corollary 2). A detailed discussion of the reachability problem and related control issues for positive systems can be found in Coxson and Shapiro [1].

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