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# On the Extension of 2-Polynomials

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Let $X$ be a normed linear space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $P: X \to \mathbb{K}$ is said to be a *2-polynomial* if there is a bilinear functional $\Pi: X \times X \to \mathbb{K}$ such that $P(x) = \Pi(x, x)$ for every $x \in X$. The norm of $P$ is defined by

$$\|P\| = \sup\{|P(x)| : \|x\| = 1\}.$$

It is known that if $X$ is an inner product space, then every 2-polynomial defined in a linear subspace of $X$ can be extended to $X$ preserving the norm. On the other hand, there is a 2-polynomial $P$ defined in a two-dimensional subspace of $l_\infty^3$ such that every extension of $P$ to $l_\infty^3$ has norm greater than $\|P\|$ (see [1, 2]). Recently, Benítez and Otero [2] showed that if $X$ is a three-dimensional real Banach space $X$ such that the unit ball of $X$ is an intersection of two ellipsoids, then every 2-polynomial defined in a linear subspace of $X$ can be extended to $X$ preserving the norm. It is natural to ask [2] the following.

*Question* 1. Suppose $X$ is a normed space such that the unit ball of $X$ is an intersection of two ellipsoids. Can every 2-polynomial defined in a linear subspace of $X$ be extended to $X$ preserving the norm?

In this article, we show the answer is affirmative when $X$ is a finite dimensional space. First, we recall the following result in [2].

LEMMA 1. *If $P$ and $Q$ are 2-polynomials in $X = \mathbb{R}^2$ such that*

$$0 \le \{P(x), Q(x)\} \qquad (x \in X)$$



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*then there exists* $0 \le \alpha \le 1$ *such that*

$$0 \le \alpha P(x) + (1-\alpha)Q(x) \qquad (x \in X).$$

LEMMA 2. *Suppose* $X$ *is a real normed space and suppose* $P_1, P_2$ *are two positive semidefinite 2-polynomials on* $X$. *If* $Q$ *is a 2-polynomial such that*

$$|Q(x)| \le \max\{P_1(x), P_2(x)\} \qquad (x \in X),$$

*then there are* $0 \le \alpha, \beta < 1$ *such that for every* $x \in X$,

$$-((\beta P_1(x) + (1-\beta)P_2(x)) \le Q(x) \le \alpha P_1(x) + (1-\alpha)P_2(x).$$

*Proof.* For every $x \in S$ with $Q(x) \ge 0$ ($Q(x) \le 0$), let

$$A_x =: \{\gamma \in [0,1] : \gamma P_1(x) + (1-\gamma)P_2(x) \ge Q(x)\}$$
$$(B_x =: \{\gamma \in [0,1] : -(\gamma P_1(x) + (1-\gamma)P_2(x)) \le Q(x)\}.)$$

To prove this lemma, it is enough to show that

$$\bigcap_{\{x \in S : Q(x) \ge 0\}} A_x \ne \varnothing, \qquad \bigcap_{\{x \in S : Q(x) \le 0\}} B_x \ne \varnothing.$$

For any $x \in S$ with $Q(x) \ge 0$ (respectively, $Q(x) \le 0$), $A_x$ (respectively, $B_x$) is a nonempty closed subinterval of $[0, 1]$. So we only need to show that for any $x, y \in S$, if $\min\{Q(x), Q(y)\} \ge 0$ (respectively, $\max\{Q(x), Q(y)\} \le 0$), then $A_x \cap A_y \ne \varnothing$ (respectively, $B_x \cap B_y \ne \varnothing$).

Suppose that $Q(x) \ge 0$ and $Q(y) \ge 0$. For any $z \in \operatorname{span}\{x, y\}$, $R_1$ and $R_2$ are defined by

$$R_1(z) = P_1(z) - Q(z)$$
$$R_2(z) = P_2(z) - Q(z).$$

By Lemma 1, there is a $\gamma$, $0 \le \gamma \le 1$, such that

$$0 \le \gamma R_1(z) + (1-\gamma)R_2.$$

This implies $\gamma \in A_x \cap A_y$. Similarly, if $Q(x) \le 0$ and $Q(y) \le 0$, then there exists $0 \le \gamma \le 1$ such that $\gamma \in B_x \cap B_y$. We proved our lemma. ∎

THEOREM 3. *Let* $\Pi_1$ *and* $\Pi_2$ *be two inner products on* $\mathbb{R}^n$ *and let* $X = \mathbb{R}^n$ *be the space with the norm*

$$\|x\| = \sqrt{\max\{\Pi_1(x, x), \Pi_2(x, x)\}}.$$

*Then every* 2*-polynomial defined in a subspace of* $X$ *can be extended to* $X$ *preserving the norm.*

*Proof.* Let $P$ be any 2-polynomial on a subspace $Y$ of $X$ and let

$$\|x\|_1 = \sqrt{\Pi_1(x, x)}$$
$$\|x\|_2 = \sqrt{\Pi_2(x, x)}.$$

Without loss of generality, we may assume $\|P\| = 1$ and $Y$ is a hyperplane of $X$. So for any $x \in Y$,

$$|P(x)| \leq \|x\|^2 \leq \max\{\|x\|_1^2, \|x\|_2^2\}.$$

By Lemma 2, there exist $0 \leq \alpha, \beta \leq 1$ such that for every $x \in Y$,

$$-(\beta\|x\|_1^2 + (1 - \beta)\|x\|_2^2) \leq P(x) \leq \alpha\|x\|_1^2 + (1 - \alpha)\|x\|_2^2.$$

Replacing $\|x\|_1$ (respectively, $\|x\|_2$) by $(\alpha\Pi_1(x, x) + (1 - \alpha)\Pi_2(x, x))^{1/2}$ (respectively, $(\beta\Pi_1(x, x) + (1 - \beta)\Pi_2(x, x))^{1/2}$), we may assume that if $x \in Y$, then

$$-\|x\|_2^2 \leq P(x) \leq \|x\|_1^2.$$

Let $\Pi$ be the symmetric bilinear functional associated with $P$. Then there is a (bounded) symmetric operator $T_1$ (respectively, $T_2$) on $(Y, \|\cdot\|_1)$ (respectively, $(Y, \|\cdot\|_2)$) such that for any $y_1, y_2 \in Y$

$$\Pi(y_1, y_2) = \Pi_1(y_1, T_1(y_2)) = \Pi_2(y_1, T_2(y_2)).$$

Since $T_1$ and $T_2$ are symmetric, they are diagonalizable. It is known that

1. every eigenvalue of $T_1$ (respectively, $T_2$) is real;
2. if $x_1$ and $x_2$ are two eigenvectors of $T_1$ (respectively, $T_2$) associated with two distinct eigenvalues, then

$$\Pi(x_1, x_2) = 0.$$

Let $Y_1$ (respectively, $Y_3$) be the subspace spanned by all eigenvectors of $T_1$ (respectively, $T_2$) associated with non-negative eigenvalues, and $Y_2$ (respectively, $Y_4$) be the subspace spanned by all eigenvectors of $T_1$ (respectively, $T_2$) associated with negative eigenvalues. Then $Y_1$, $Y_2$, $Y_3$, $Y_4$ satisfy the following conditions:

3. $Y = Y_1 \oplus Y_2 = Y_3 \oplus Y_4$;
4. $T_1(Y_1) \subseteq Y_1$, $T_1(Y_2) \subseteq Y_2$, $T_2(Y_3) \subseteq Y_3$, and $T_2(Y_4) \subseteq Y_4$;
5. for any $y_i \in Y_i \setminus \{0\}$,

$$\Pi_1(y_1, T_1(y_1)) \geq 0 > \Pi_1(y_2, T_1(y_1)),$$
$$\Pi_2(y_3, T_1(y_3)) \geq 0 > \Pi_2(y_4, T_2(y_4)),$$
$$\Pi_1(y_1, y_2) = 0 = \Pi_2(y_3, y_4).$$

We claim that $Y_1 \cap Y_4 = \{0\}$. Suppose it is not true. Let $y \in (Y_1 \cap Y_4)\setminus\{0\}$. Then

$$0 > \Pi_2(y, T_2(y)) = \Pi(y, y) = \Pi_1(y, T_1(y)) \geq 0.$$

We get a contradiction. Similarly, $Y_2 \cap Y_3 = \{0\}$. Hence, we have

6. $\dim(Y_1) = \dim(Y_3)$, $\dim(Y_2) = \dim(Y_4)$, and

$$Y_1 \oplus Y_4 = Y = Y_2 \oplus Y_3.$$

Let

$$M_1 = \{z \in X : \Pi_1(z, x) = 0 \text{ for all } x \in Y_1\},$$
$$M_2 = \{z \in X : \Pi_2(z, x) = 0 \text{ for all } x \in Y_4\},$$

By (5) and (6), $\dim(M_1) = \dim(Y_2) + 1$ and $\dim(M_2) = \dim(Y_3) + 1$. This implies there is a non-zero vector $z \in M_1 \cap M_2$. Let $\phi$ be any non-zero linear functional on $X$ such that $\ker \phi = Y$. For any $x \in X$, define

$$\tilde{P}(x) = P\left(x - \frac{\phi(x)}{\phi(z)} z\right).$$

We claim that if $0 < \tilde{P}(x)$, then $\tilde{P}(x) \leq \|x\|^2$.

*Case* 1. $x - (\phi(x)/\phi(z))\, z \in Y_1$. Since $z \in M_1$, we have $\Pi_1(x - (\phi(x)/\phi(z))\, z, z) = 0$. So

$$\tilde{P}(x) = P\left(x - \frac{\phi(x)}{\phi(z)} z\right) \leq \left\| x - \frac{\phi(x)}{\phi(z)} z \right\|_1^2 \leq \|x\|_1^2.$$

*Case* 2. $x - (\phi(x)/\phi(z))\, z \notin Y_1$. Then there exist $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $x - (\phi(x)/\phi(z))\, z = y_1 + y_2$. (Note. $\Pi_1(y_1, y_2) = 0 = \Pi_1(y_1, z)$.) So

$$\begin{aligned}\tilde{P}(x) &= P\left(x - \frac{\phi(x)}{\phi(z)} z\right) = \Pi(y_1 + y_2, y_1 + y_2)\\ &= P(y_1) + P(y_2) \quad \leq P(y_1)\\ &\leq \|y_1\|_1^2 \qquad\qquad \leq \|x\|_1^2\\ &\leq \|x\|^2.\end{aligned}$$

We proved our claim. Similarly, if $\tilde{P}(x) \leq 0$, then $|\tilde{P}(x)| \leq \|x\|^2$. Hence, $\tilde{P}$ is an extension of $P$ preserving the norm. ∎

In [2, Lemma 2 and Proposition 2], Benítez and Otero proved that the problem of extension preserving the norm can be reduced to the real case. Hence, we have the following theorem.

THEOREM 4. *Let* $\Pi_1$ *and* $\Pi_2$ *be two inner products on* $\mathbb{C}^n$ *and let* $X = \mathbb{C}^n$ *be the space with the norm*

$$\|x\| = \sqrt{\max\{\Pi_1(x, x), \Pi_2(x, x)\}}.$$

*Then every 2-polynomial defined in a subspace of X can be extended to X preservng the norm.*

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