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Exceptional Properties of Second and Third Order Ordinary Differential Equations of Maximal Symmetry

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The Riccati transformation is used in the reduction of order of second and third order ordinary differential equations of maximal symmetry. The $sl(2, R)$ subalgebra is preserved under this transformation. The Riccati transformation is itself associated with the symmetry that is annihilated in the reduction of order. The solution symmetries and the intrinsically contact symmetries become nonlocal symmetries under the Riccati transformation. We investigate the fate and origins of the contact symmetries arising from the Riccati transformation. The exceptional properties of the second and third order equations of maximal symmetry are indicated. In the context of generalised symmetries we express the solution symmetries, contact symmetries, and the $sl(2, R)$ elements in terms of a Jacobian. We show that a basis for the solution set of equations of maximal symmetry is given in terms of the solution set of a second order ordinary differential equation. © 2000 Academic Press

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1. INTRODUCTION

Second order ordinary differential equations differ from third order ordinary differential equations in their symmetry properties. Lie [9] showed that the maximum number of point symmetries for second order ordinary differential equations is $2 + 6$ and for higher order equations ($n \geq 3$) $n + 4$ [10]. Lie [9] further showed that an nth order equation which possesses $n + 4$ point symmetries is equivalent to

$$y^{(n)} = 0, \tag{1.1}$$

where $^{(n)}$ denotes $\mathrm{d}^n/\mathrm{d}x^n$, under a point transformation

$$X = F(x, y), \qquad Y = G(x, y). \tag{1.2}$$

(For $n \geq 3$, $X = F(x)$ in (1.2) which is a restricted type of point transformation called a fibre-preserving transformation [4].) He also proved that all linear second order ordinary differential equations are equivalent to $y'' = 0$. A linear higher order equation can have $n + 1$, $n + 2$ or, the maximum number, $n + 4$ point symmetries (see Mahomed and Leach [12]). Krause and Michel [6] proved that the maximum number of point symmetries for nth order equations is $n + 4$ iff the equation is iterative; i.e., it can be written in the form

$$L[y] \equiv r(x)y' + q(x)y = 0, \qquad L^n[y] \equiv L^{n-1}[L[y]]. \tag{1.3}$$

They further showed that equations possessing $n + 4$ point symmetries were equivalent to (1.1) under a suitable transformation of the form (1.2). In this paper we confine ourselves to equations of maximal symmetry written in an equivalent but more general form, i.e., the self adjoint form. We do not transform them to their equivalent form (1.1) so that we maintain the structure of the equations. This enables us to extract more information and make new inferences than when the equations are in the simpler form (1.1). The exceptional properties of second and third order ordinary differential equations of maximal symmetry are studied and the generalised symmetries of both the second and third order equations are used to show that under a Riccati transformation the $sl(2, R)$ subalgebra is preserved. The structure of the symmetries in both cases is of particular interest. We rewrite the symmetries in terms of a Jacobian and show that this Jacobian is also a solution of the original equation. The first integrals associated with the $sl(2, R)$ subalgebra are computed and the general expression for them in terms of the Jacobian is calculated. Some interesting aspects arising from the analysis are discussed.

2. GENERALISED SYMMETRIES

An operator

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}, \tag{2.1}$$

where ξ and η are analytic functions of x, y and derivatives of y, is a generalized symmetry of the differential equation written in normal form

$$y^{(n)} = E(x, y, \dots, y^{n-1}) \tag{2.2}$$

if the symmetry condition

$$G^{[n]}\left[y^{(n)} - E(x, y, \dots, y^{n-1}) \right] \Big|_{y^{(n)}=E} = 0 \tag{2.3}$$

is satisfied. It is well known that second order equations have an infinite number of generalized symmetries [13]. There is no totally inclusive approach to determine generalized symmetries. One must put a restriction on the y' dependence that is systematic in form to make a conclusive calculation possible.

A one parameter Lie group of contact transformations with infinitesimal generator

$$G = \xi(x, y, y') \frac{\partial}{\partial x} + \eta(x, y, y') \frac{\partial}{\partial y} + \zeta(x, y, y') \frac{\partial}{\partial y'} \tag{2.4}$$

is a contact transformation provided the contact condition,

$$\zeta = \eta' - y'\xi', \tag{2.5}$$

is preserved. The absence of the y'' term in ζ implies that

$$\frac{\partial \eta}{\partial y'} = y' \frac{\partial \xi}{\partial y'}. \tag{2.6}$$

This puts a constraint on the y' dependence in η and ξ. We can describe the symmetry in terms of a generating function, $W(x, y, y')$ [1]. We have that

$$W = \xi y' - \eta \tag{2.7}$$

$$\xi = \partial W / \partial y' \tag{2.8}$$

$$\eta = y' \partial W / \partial y' - W \tag{2.9}$$

$$\zeta = -\partial W / \partial x - y' \partial W / \partial y. \tag{2.10}$$

It is evident that a contact symmetry is a restricted type of generalised symmetry.

3. REDUCTION USING THE RICCATI TRANSFORMATION

The well known Riccati transformation

$$u = \frac{y'(x)}{y(x)} \tag{3.1}$$

is associated with the homogeneity symmetry

$$G = y\frac{\partial}{\partial y} \tag{3.2}$$

since the zeroth and first order differential invariants of (3.2) are $x = x$ and $u = y'/y$, respectively. The homogeneity symmetry is the only symmetry of a general homogeneous linear equation which can be obtained without prior knowledge of a solution to the equation or its adjoint. The symmetry (2.1) under the Riccati transformation (3.1) is transformed to

$$\overline{G} = \xi\frac{\partial}{\partial x} + \left[\left(\frac{\eta}{y}\right)' - u\xi'\right]\frac{\partial}{\partial u}. \tag{3.3}$$

4. GENERAL PROPERTIES

The general form of equations with maximal symmetry [12] is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\prod_{i=1}^{(n-1)/2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(2i)^2}{\binom{n+1}{3}}\nu_{n-2}\right)\right]y = 0 \tag{4.1}$$

for n odd, and

$$\left[\prod_{i=1}^{n/2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(2i-1)^2}{\binom{n+1}{3}}\nu_{n-2}\right)\right]y = 0 \tag{4.2}$$

for n even. We present the point symmetry structure for the second and third order equations with maximal symmetry, i.e., with $n = 2$ and $n = 3$ in (4.2) and (4.1), viz.

$$y'' + \nu(x)y = 0 \tag{4.3}$$

and

$$y''' + \nu(x)y' + \tfrac{1}{2}\nu'(x)y = 0, \tag{4.4}$$

respectively (The subscript in ν has been ignored for simplicity.) The symmetry structure is

$$G_1 = \omega_1 \frac{\partial}{\partial y} \tag{4.5}$$

$$\vdots$$

$$G_n = \omega_n \frac{\partial}{\partial y} \tag{4.6}$$

$$G_{n+1} = y\frac{\partial}{\partial y} \tag{4.7}$$

$$G_{n+2} = \beta_1 \frac{\partial}{\partial x} + \frac{n-1}{2}\beta_1' y \frac{\partial}{\partial y} \tag{4.8}$$

$$G_{n+3} = \beta_2 \frac{\partial}{\partial x} + \frac{n-1}{2}\beta_2' y \frac{\partial}{\partial y} \tag{4.9}$$

$$G_{n+4} = \beta_3 \frac{\partial}{\partial x} + \frac{n-1}{2}\beta_3' y \frac{\partial}{\partial y}, \tag{4.10}$$

where $n = 2, 3$, $\omega_1, \ldots, \omega_n$ are solutions of the differential equation and $\beta_1, \beta_2, \beta_3$ are solutions of a third order equation. In addition to these symmetries it is a simple calculation to show that the second order ordinary differential equation has the two non-Cartan symmetries

$$G_7 = \omega_1 y \frac{\partial}{\partial x} + \omega_1' y^2 \frac{\partial}{\partial y} \tag{4.11}$$

$$G_8 = \omega_2 y \frac{\partial}{\partial x} + \omega_2' y^2 \frac{\partial}{\partial y} \tag{4.12}$$

while the third order ordinary differential equation has three intrinsically contact symmetries (ICS) since their ξ's and η's contain y' when the other seven symmetries are point symmetries of the structure in (4.5) and (4.10). In the symmetry structure, $G_1 - G_n$ represent the solution symmetries since their coefficients are solutions of the original differential equation. The presence of the homogeneity symmetry G_{n+1} comes from the fact that the differential equation is homogeneous in y while $G_{n+2} - G_{n+4}$ represent the $sl(2, R)$ subalgebra. The β_i's, $i = 1, 2, 3$, are solutions of the third order differential equation

$$\frac{(n+1)!}{(n-2)!4!}\beta_i''' + \nu\beta_i' + \tfrac{1}{2}\nu'\beta_i = 0. \tag{4.13}$$

Under the transformation $\beta_i = \rho_i^2$ and integration *wrt* x (4.13) becomes

$$\frac{4(n+1)!}{(n-2)!4!}\rho_i'' + \nu\rho_i = \frac{k}{\rho_i^3}, \tag{4.14}$$

where k is a constant. Therefore the $sl(2, R)$ coefficients are related to the solution of (4.14) which is the Ermakov–Pinney equation [2, 15] the general solution of which was given by Pinney in terms of the solution set of

$$\frac{4(n+1)!}{(n-2)!4!}\rho_i'' + \nu\rho_i = 0. \tag{4.15}$$

Under the Riccati transformation (3.1) the $sl(2, R)$ symmetries take the form

$$\overline{G}_{sli} = \beta_i\frac{\partial}{\partial x} + \left[\left(\frac{n-1}{2}\right)\beta_i'' - \beta_i' u\right]\frac{\partial}{\partial u}. \tag{4.16}$$

The homogeneity symmetry G_{n+1} is lost while the solution symmetries become

$$\overline{G}_{si} = \exp\left[\int u\,\mathrm{d}x\right](\omega_i' - \omega_i u)\frac{\partial}{\partial u}. \tag{4.17}$$

(Note that in this notation the G_{si} represent solution symmetries and the G_{sli} the $sl(2, R)$ symmetries.) We observe that the solution symmetries become nonlocal symmetries, since ξ and η now contain integrals, with a positive exponent under the Riccati transformation. The $sl(2, R)$ symmetries (4.16) remain as point symmetries under the Riccati transformation (3.1).

The general nth order ordinary differential equation in the form of (4.1) and (4.2) can be transformed to

$$V^{(n)}(t) = 0 \tag{4.18}$$

via the transformation

$$y = \rho^{n-1}(x)V(t), \qquad t = \int \rho^{-2}(x)\,\mathrm{d}x \tag{4.19}$$

and ρ satisfies

$$\frac{4(n+1)!}{(n-2)!4!}\rho'' + \nu(x)\rho = 0. \tag{4.20}$$

The solution set for y in (4.19) then gives the solution set for (4.1) and (4.2), respectively. (The subscripts in ν and ρ have been ignored for simplicity.)

In the case of the third order ordinary differential equation (4.4) we have $n = 3$ and (4.20) becomes

$$4\rho'' + \nu(x)\rho = 0. \tag{4.21}$$

The solution set for ρ from (4.21) with $\nu = 4$ is $\{\cos x, \sin x\}$. The transformation of (4.4) to (4.18) is given by (4.19) with $n = 3$, i.e.,

$$t = \tan x, \qquad V = \{1, \tan x, \tan^2 x\}. \tag{4.22}$$

Therefore the third order equation (4.4) has the fundamental solution set

$$\{y\} = \{\cos^2 x, \cos x \sin x, \sin^2 x\}. \tag{4.23}$$

If $\vartheta = \cos x$ and $\upsilon = \sin x$ then (4.23) becomes $\{\vartheta^2, \vartheta\upsilon, \upsilon^2\}$, where ϑ and υ are any two independent solutions of the second order ordinary differential equation (4.21). The solution set for the third order ordinary differential equation (4.4),

$$y''' + 4y' = 0, \tag{4.24}$$

with $\nu = 4$ is

$$\{y\} = \{1, \sin 2x, \cos 2x\} \tag{4.25}$$

which is a linear combination of the fundamental set of solutions. As another example consider the second order ordinary differential equation (4.3) which is

$$y'' + \nu(x)y = 0. \tag{4.26}$$

Equation (4.20) with $\nu = 1$ becomes

$$\rho'' + \rho = 0. \tag{4.27}$$

The solution set for (4.27) is $\{\rho\} = \{\cos x, \sin x\}$. From (4.19) the transformation to $V^{(2)} = 0$ gives

$$t = \tan x, \qquad \{V\} = \{1, \tan x\}. \tag{4.28}$$

This means that

$$\{y\} = \{\cos x, \sin x\}. \tag{4.29}$$

If we put $\vartheta = \cos x$, $\upsilon = \sin x$ then the fundamental solution set for (4.26) is $\{\vartheta, \upsilon\}$. This illustrates the connection of the solutions of the equation to those of the second order equation.

We therefore have the following proposition:

PROPOSITION 1. *A basis for the solution set of an nth order ordinary differential equation in the form of* (4.1) *or* (4.2) *can be given as a basis for the solution of the second order ordinary differential equation* (4.20). *If the solution set for* (4.20) *is say* $\{\rho\} = \{\vartheta, \upsilon\}$, *where* ϑ *and* υ *are linearly independent solutions of* (4.20), *then*

$$\{\omega_i\} = \{\vartheta^{n-i}\upsilon^{i-1}\}, \qquad i = 1, \dots, n \tag{4.30}$$

is the basis of the solution set for the nth order ordinary differential equation (4.1) *and* (4.2), *respectively.*

Remark. It is easy to see with this representation why we get the Lie bracket relations for the $sl(2, R)$ subalgebra. Consider

$$G_1 = \vartheta^2 \frac{\partial}{\partial x} + (n-1)\vartheta\vartheta' y \frac{\partial}{\partial y} \tag{4.31}$$

$$G_2 = \vartheta\upsilon \frac{\partial}{\partial x} + \frac{n-1}{2}(\vartheta'\upsilon + \upsilon'\vartheta) y \frac{\partial}{\partial y} \tag{4.32}$$

$$G_3 = \upsilon^2 \frac{\partial}{\partial x} + (n-1)\upsilon\upsilon' y \frac{\partial}{\partial y}. \tag{4.33}$$

The Lie bracket relations are

$$[G_1, G_2] = -WG_1, \tag{4.34}$$

$$[G_1, G_3] = -2WG_2 \tag{4.35}$$

$$[G_2, G_3] = -WG_1, \tag{4.36}$$

where $W = \vartheta' \upsilon - \vartheta \upsilon'$ is the Wronskian of ϑ and υ. For instance in the case of the third order ordinary differential equation the coefficients of the $\partial/\partial x$ term in the $sl(2, R)$ subalgebra are precisely the solution set (4.23) with $\vartheta = \cos x$, $\upsilon = \sin x$, and $sl(2, R)$ representation (4.31)–(4.33). Recall that if $(n-1)$ solutions of the linear differential equation are known, Abel's formula can be used to give the nth solution [5]. In our case we use the basis for the solution set of the second order ordinary differential equation to find the solution set of the higher order equations with maximal symmetry.

5. SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

We refer to (4.3) which is given by

$$y'' + \nu(x)y = 0. \tag{5.1}$$

If we seek the point symmetries of (5.1), i.e., ξ and η in (2.1) are functions of x and y only, the operation of the second extension of (2.1) of (5.1) leads to

$$\xi\nu' y + \eta\nu + (\eta'' - 2y''\xi' - y'\xi'') = 0. \tag{5.2}$$

Expanding (5.2) and separating by powers of y' we obtain to the following system of partial differential equations,

$$y^3: \quad \xi = a(x)y + b(x) \tag{5.3}$$

$$y'^2: \quad \eta = a'(x)y^2 + c(x)y + d(x) \tag{5.4}$$

$$y'^1: \quad 0 = 3a''(x)y + 3y\nu(x)a(x) - b'' + 2c' \tag{5.5}$$

$$\begin{aligned} y'^0: \quad 0 = {} & (y^2 a + by)\nu' + (a'y^2 + cy + 2yb' + d)\nu \\ & + (a'''y^2 + c''y + d''). \end{aligned} \tag{5.6}$$

Separation of (5.5) by powers of y gives

$$y^1: \quad 0 = a'' + \nu a \tag{5.7}$$

$$y^0: \quad 0 = b'' - 2c' \tag{5.8}$$

and (5.6) leads to

$$y^2: \quad 0 = a\nu' + a'\nu + a''' \tag{5.9}$$

$$y^1: \quad 0 = b\nu' + 2\nu b' + c'' \tag{5.10}$$

$$y^0: \quad 0 = d\nu + d''. \tag{5.11}$$

The use of (5.8) in (5.10) results in

$$b''' + 4b'\nu + 2b\nu' = 0. \tag{5.12}$$

The substitution of $b = \rho^2$ after multiplying throughout by b in (5.12) leads to

$$\rho'' + \nu\rho = \frac{k}{\rho^3}, \tag{5.13}$$

which is just (4.20) with $n = 2$. The $sl(2, R)$ symmetries come from the solutions of (5.12). If ϑ and υ are solutions of (5.1), we can write the symmetries of (5.1) as

$$G_1 = \vartheta\frac{\partial}{\partial y} \tag{5.14}$$

$$G_2 = \upsilon\frac{\partial}{\partial y} \tag{5.15}$$

$$G_3 = y\frac{\partial}{\partial y} \tag{5.16}$$

$$G_4 = \vartheta^2\frac{\partial}{\partial x} + \vartheta\vartheta' y\frac{\partial}{\partial y} \tag{5.17}$$

$$G_5 = \vartheta\upsilon\frac{\partial}{\partial x} + \tfrac{1}{2}(\vartheta'\upsilon + \upsilon'\vartheta)y\frac{\partial}{\partial y} \tag{5.18}$$

$$G_6 = \upsilon^2\frac{\partial}{\partial x} + \upsilon\upsilon' y\frac{\partial}{\partial y} \tag{5.19}$$

$$G_7 = \vartheta y\frac{\partial}{\partial x} + \vartheta' y^2\frac{\partial}{\partial y} \tag{5.20}$$

$$G_8 = \upsilon y\frac{\partial}{\partial x} + \upsilon' y^2\frac{\partial}{\partial y} \tag{5.21}$$

of which the first six are in Cartan form and the last two are non-Cartan. A symmetry is non-Cartan if ξ is not a function of x only. Under the Riccati transformation (3.1) the second order ordinary differential equation (5.1) becomes

$$u' + u^2 + \nu = 0. \tag{5.22}$$

The symmetries G_1 through to G_8 are transformed as

$$\overline{G}_1 = \exp\left[-\int u\,\mathrm{d}x\right](\vartheta' - \vartheta u)\frac{\partial}{\partial u} \tag{5.23}$$

$$\overline{G}_2 = \exp\left[-\int u\,\mathrm{d}x\right](v' - vu)\frac{\partial}{\partial u} \tag{5.24}$$

$\overline{G}_3$ vanishes

$$\overline{G}_4 = \vartheta^2\frac{\partial}{\partial x} + (\vartheta'^2 + \vartheta\vartheta'' - 2\vartheta\vartheta' u)\frac{\partial}{\partial u} \tag{5.25}$$

$$\overline{G}_5 = \vartheta v\frac{\partial}{\partial x} + \left(\tfrac{1}{2}[\vartheta'' v + v''\vartheta + 2\vartheta' v'] - (\vartheta' v + v'\vartheta)u\right)\frac{\partial}{\partial u} \tag{5.26}$$

$$\overline{G}_6 = v^2\frac{\partial}{\partial x} + (v'^2 + vv'' - 2vv' u)\frac{\partial}{\partial u} \tag{5.27}$$

$$\overline{G}_7 = \exp\left[\int u\,\mathrm{d}x\right]\left(\vartheta\frac{\partial}{\partial x} + (\vartheta'' - \vartheta u^2)\frac{\partial}{\partial u}\right) \tag{5.28}$$

$$\overline{G}_8 = \exp\left[\int u\,\mathrm{d}x\right]\left(v\frac{\partial}{\partial x} + (v'' - vu^2)\frac{\partial}{\partial u}\right). \tag{5.29}$$

We have already noted in the general properties of the symmetry structure that the solution symmetries G_1 and G_2 become nonlocal symmetries and G_3 vanishes. The $sl(2, R)$ subalgebra $G_4 - G_6$ remains as point symmetries while G_7 and G_8 become nonlocal symmetries. If ϑ is a solution of (5.1), a first integral can be obtained by multiplying (5.1) throughout by ϑ and integrating by parts. A first integral is

$$I_1 = \vartheta y' - \vartheta' y. \tag{5.30}$$

Similarly, if v is another solution, then

$$I_2 = vy' - v'y \tag{5.31}$$

is also a first integral.

Remarks.

- We observe that the elements of the $sl(2, R)$ subalgebra, namely G_4, G_5, and G_6, remain as point symmetries of the reduced equation.
- The symmetry used in the reduction $G_3 = y\partial/\partial y$ vanishes.
- The non-Cartan symmetries, $\overline{G}_7$ and $\overline{G}_8$, have a positive exponent while $\overline{G}_1$ and $\overline{G}_2$ have a negative exponent.

We note that the standard representation of the $sl(2, R)$ subalgebra is

$$G_4 = \beta_1 \frac{\partial}{\partial x} + \frac{n-1}{2} \beta_1' y \frac{\partial}{\partial y} \tag{5.32}$$

$$G_5 = \beta_2 \frac{\partial}{\partial x} + \frac{n-1}{2} \beta_2' y \frac{\partial}{\partial y} \tag{5.33}$$

$$G_6 = \beta_3 \frac{\partial}{\partial x} + \frac{n-1}{2} \beta_3' y \frac{\partial}{\partial y}, \tag{5.34}$$

where β_1, β_2, and β_3 are solutions of the third order ordinary differential equation (4.13) with the appropriate value of n. Therefore, in transforming from the second order equation to the first order equation, the $sl(2, R)$ subalgebra is preserved. The corresponding first integrals associated with each $sl(2, R)$ subalgebra can be computed easily. Consider the first extension of the element of the $sl(2, R)$ subalgebra for the second order ordinary differential equation (5.1) given by

$$G^{[1]} = \beta \frac{\partial}{\partial x} + \tfrac{1}{2} \beta' y \frac{\partial}{\partial y} + \tfrac{1}{2} (\beta'' y - \beta' y') \frac{\partial}{\partial y'}. \tag{5.35}$$

The associated Lagrange system is

$$\frac{\mathrm{d}x}{\beta} = \frac{\mathrm{d}y}{(1/2)\beta' y} = \frac{\mathrm{d}y'}{(1/2)(\beta'' y - \beta' y')}. \tag{5.36}$$

The characteristics are

$$p = \frac{y}{\beta^{\frac{1}{2}}}, \qquad q = y' \beta^{\frac{1}{2}} - \tfrac{1}{2} \frac{\beta'}{\beta^{\frac{1}{2}}} y. \tag{5.37}$$

Now

$$\frac{\beta\, \mathrm{d}p}{q} = \frac{\mathrm{d}q}{0}. \tag{5.38}$$

Therefore the first integral for the second order equation is

$$I = y' \beta^{\frac{1}{2}} - \tfrac{1}{2} \beta' \beta^{-\frac{1}{2}} y. \tag{5.39}$$

Just as a little note we observe that β is $\vartheta^2, \vartheta v, v^2$ and so we have

$$I_1 = y' \vartheta - y \vartheta' \tag{5.40}$$

$$I_3 = y' v - y v' \tag{5.41}$$

and then

$$I_2 = y'(\vartheta v)^{\frac{1}{2}} - y(\vartheta v)'^{\frac{1}{2}}. \tag{5.42}$$

The I_2 is curious. Both I_1 and I_3 are well known from quadratic time-dependent Hamiltonian systems [7].

6. THIRD ORDER EQUATION

The general third order equation of maximal symmetry (4.4) is

$$y''' + \nu(x)y' + \tfrac{1}{2}\nu'(x)y = 0. \tag{6.1}$$

The Riccati transformation (3.1), $y = \exp(\int u\,\mathrm{d}x)$, transforms (6.1) to the second order differential equation

$$u'' + 3u'u + u^3 + \nu u + \tfrac{1}{2}\nu' = 0. \tag{6.2}$$

Equation (6.1) has ten contact symmetries. If ϑ^2, ϑv, and v^2 are solutions of (6.1), we can list the ten contact symmetries as

$$G_1 = \vartheta^2 \frac{\partial}{\partial y} \tag{6.3}$$

$$G_2 = \vartheta v \frac{\partial}{\partial y} \tag{6.4}$$

$$G_3 = v^2 \frac{\partial}{\partial y} \tag{6.5}$$

$$G_4 = \vartheta^2 \frac{\partial}{\partial x} + 2\vartheta\vartheta' y \frac{\partial}{\partial y} \tag{6.6}$$

$$G_5 = \vartheta v \frac{\partial}{\partial x} + (\vartheta' v + v'\vartheta) y \frac{\partial}{\partial y} \tag{6.7}$$

$$G_6 = v^2 \frac{\partial}{\partial x} + 2vv' y \frac{\partial}{\partial y} \tag{6.8}$$

$$G_7 = y \frac{\partial}{\partial y} \tag{6.9}$$

$$G_8 = 2\left(y'\vartheta^2 - 2y\vartheta\vartheta'\right)\frac{\partial}{\partial x} + \left(y'^2\vartheta^2 - 2y^2(\nu\vartheta^2 + \vartheta''\vartheta + \vartheta'^2)\right)\frac{\partial}{\partial y} \tag{6.10}$$

$$G_9 = 2(y'\vartheta v - y(\vartheta' v + \vartheta v'))\frac{\partial}{\partial x} + \left(y'^2\vartheta v - 2y^2(\nu\vartheta v + \vartheta'' v + 2\vartheta' v' + \vartheta v'')\right)\frac{\partial}{\partial y} \quad (6.11)$$

$$G_{10} = 2\left(y'v^2 - 2yvv'\right)\frac{\partial}{\partial x} + \left(y'^2 v^2 - 2y^2(\nu v^2 + v'' v + v'^2)\right)\frac{\partial}{\partial y}. \quad (6.12)$$

Under the Riccati transformation (6.3)–(6.12) become

$$\overline{G}_1 = \exp\left[-\int u\,\mathrm{d}x\right](2\vartheta\vartheta' - \vartheta^2 u)\frac{\partial}{\partial u} \quad (6.13)$$

$$\overline{G}_2 = \exp\left[-\int u\,\mathrm{d}x\right](\vartheta' v + v'\vartheta - \vartheta v u)\frac{\partial}{\partial u} \quad (6.14)$$

$$\overline{G}_3 = \exp\left[-\int u\,\mathrm{d}x\right](2vv' - v^2 u)\frac{\partial}{\partial u} \quad (6.15)$$

$$\overline{G}_4 = \vartheta^2\frac{\partial}{\partial x} + (2\vartheta'^2 + 2\vartheta\vartheta'' - 2\vartheta\vartheta' u)\frac{\partial}{\partial u} \quad (6.16)$$

$$\overline{G}_5 = \vartheta v\frac{\partial}{\partial x} + (\vartheta'' v + 2\vartheta' v' + \vartheta v'' - u(\vartheta' v + v'\vartheta))\frac{\partial}{\partial u} \quad (6.17)$$

$$\overline{G}_6 = v^2\frac{\partial}{\partial x} + (2v'^2 + 2vv'' - 2vv' u)\frac{\partial}{\partial u} \quad (6.18)$$

$\overline{G}_7$ vanishes

$$\overline{G}_8 = 2\exp\left[\int u\,\mathrm{d}x\right]\left[(u\vartheta^2 - 2\vartheta\vartheta')\frac{\partial}{\partial x} - (\nu'\vartheta^2 + 2\vartheta\vartheta'\nu + \vartheta'''\vartheta + 3\vartheta'\vartheta'')\frac{\partial}{\partial u}\right] - \exp\left[\int u\,\mathrm{d}x\right](2u\nu\vartheta^2 - 2u^2\vartheta\vartheta' - 2u\vartheta'^2 + \vartheta^2 u^3 - 2u\vartheta\vartheta'')\frac{\partial}{\partial u} \quad (6.19)$$

$$\overline{G}_9 = 2\exp\left[\int u\,\mathrm{d}x\right]\left[(u\vartheta v - \vartheta' v - \vartheta v')\frac{\partial}{\partial x}\right.$$
$$\left. -(\nu\vartheta' v + \nu\vartheta v' + \nu'\vartheta v + \vartheta''' v)\frac{\partial}{\partial u}\right]$$
$$+ \exp\left[\int u\,\mathrm{d}x\right]\left[u^2(\vartheta' v + \vartheta v') - u^3\vartheta v - 2u\nu\vartheta v\right.$$
$$\left. -2\vartheta v''' - 6\vartheta' v'' - 6\vartheta'' v'\right]\frac{\partial}{\partial u} \tag{6.20}$$

$$\overline{G}_{10} = 2\exp\left[\int u\,\mathrm{d}x\right]\left[(uv^2 - 2vv')\frac{\partial}{\partial x}\right.$$
$$\left. -(\nu' v^2 + 2vv'\nu + v''' v + 3v' v'')\frac{\partial}{\partial u}\right]$$
$$- \exp\left[\int u\,\mathrm{d}x\right](2u\nu v^2 - 2u^2 vv' - 2uv'^2 + v^2u^3 - 2uvv'')\frac{\partial}{\partial u}. \tag{6.21}$$

We observe that the $sl(2, R)$ subalgebra remains invariant under the Riccati transformation. In the case of the third order ordinary differential equation the solution symmetries, the $sl(2, R)$ symmetries and the contact symmetries originate from the same equation which is just the original equation (6.1).

7. FATE OF THE CONTACT SYMMETRIES

Consider the first extension of the intrinsically contact symmetry, G_{ics}, of (4.4) given by

$$G_{ics}^{[1]} = (\beta' y - \beta y')\frac{\partial}{\partial x} + \left(\beta'' y^2 + \tfrac{1}{2}\beta y^2\nu - \tfrac{1}{2}\beta y'^2\right)\frac{\partial}{\partial y}$$
$$+\left(\beta''' y^2 + \beta'' yy' - \tfrac{1}{2}\beta' y'^2 + \left(\beta yy' + \tfrac{1}{2}\beta' y^2\right)\nu + \tfrac{1}{2}\beta y^2\nu'\right)\frac{\partial}{\partial y'}, \tag{7.1}$$

where β is a solution of the third order equation

$$\beta''' + \beta\nu' + \tfrac{1}{2}\beta'\nu = 0. \tag{7.2}$$

Multiplication of (4.4) by y and one integration lead to

$$yy'' - \tfrac{1}{2}y'^2 + \tfrac{1}{2}y^2\nu = k, \tag{7.3}$$

where k is a constant. The coefficient of $\partial/\partial y$ in (7.1) is

$$\eta = \beta'' y^2 + \tfrac{1}{2}\beta y^2\nu - \tfrac{1}{2}\beta y'^2. \tag{7.4}$$

Making use of (7.3) in (7.4) we have that

$$\eta = y(\beta'' y - \beta y'') + k\beta. \tag{7.5}$$

The expression of η in (7.5) enables us to express (7.1) without the first extension as

$$G_{ics} = (\beta' y - \beta y')\frac{\partial}{\partial x} + (\beta' y - \beta y')' y\frac{\partial}{\partial y} + k\beta\frac{\partial}{\partial y}. \tag{7.6}$$

Recall that β is a solution of (7.2) and so $\beta\partial/\partial y$ is a solution symmetry of (6.1). We can therefore remove the term $k\beta\partial/\partial y$ in (7.6). This means that we can write the intrinsically contact symmetry as

$$G_{ics} = (\beta' y - \beta y')\frac{\partial}{\partial x} + (\beta' y - \beta y')' y\frac{\partial}{\partial y}, \tag{7.7}$$

where β is a solution of (7.2). This is possible because we have taken into account the integral consequences of the original differential equation [14]. In the case of the third order equation (4.4), the solution symmetries, the $sl(2, R)$ and the ICS originate from the same differential equation which is a third order equation. From (7.7) we note that the coefficient of $\partial/\partial x$ is the Jacobian of β and y. Under the Riccati transformation $u = y'/y$ the ICS (7.7) becomes

$$\overline{G}_{ics} = \exp\left(\int u\,\mathrm{d}x\right)(\beta' - \beta u)\left[\frac{\partial}{\partial y} - \tfrac{1}{2}(u^2 + \nu)\frac{\partial}{\partial u}\right] \tag{7.8}$$

when (7.2) and (7.3) are taken into account. The coefficient of $\partial/\partial u$ in (7.8) is in the form of the first order Riccati equation

$$u' + u^2 + \nu = 0 \tag{7.9}$$

obtained from the reduction of (4.3) with $u = y'/y$. That of $\partial/\partial x$ is

$$\exp\left(\int u\,\mathrm{d}x\right)[\beta' - \beta u] = \beta' y - \beta y'. \tag{7.10}$$

Now $W = \beta' y - \beta y'$ is the Jacobian of β and y. This will enable us to express the symmetries in terms of the Jacobian. We denote the symmetries of the third order ordinary differential equation as

$$G_{si} = \omega_i \frac{\partial}{\partial y} \tag{7.11}$$

$$G_{sli} = \omega_i \frac{\partial}{\partial y} + \omega_i' y \frac{\partial}{\partial y} \tag{7.12}$$

$$G_{icsj} = \left(\omega_j' y - \omega_j y'\right) \frac{\partial}{\partial x} + \left(\omega_j'' y^2 + \tfrac{1}{2}\omega_j y^2 \nu - \tfrac{1}{2}\omega_j y'^2\right) \frac{\partial}{\partial y}, \tag{7.13}$$

where the G_{si} are the solution symmetries, the G_{sli} are the $sl(2, R)$ symmetries, and the G_{icsj} are the ICS in the usual representation. We now introduce the generalised symmetries. In the case of the second order ordinary differential equation we can write

$$G_{si} = \omega_i \frac{\partial}{\partial y} \tag{7.14}$$

$$G_{sli} = \left(\omega_i' y - \omega_i y'\right) \frac{\partial}{\partial y} \tag{7.15}$$

$$G_{icsj} = \left[\left(\omega_j' y - \omega_j y'\right)' y - \left(\omega_j' y - \omega_j y'\right) y'\right] \frac{\partial}{\partial y} \tag{7.16}$$

by using the fact that the generators (2.1) and

$$G = (\eta - y'\xi) \frac{\partial}{\partial y} \tag{7.17}$$

are equivalent [1] in the context of generalised symmetries. For the second order ordinary differential equation the $sl(2, R)$ subalgebra in terms of the generalised symmetries now becomes

$$\begin{aligned} G_{sl1} &= (\vartheta' y - \vartheta y')\vartheta \frac{\partial}{\partial y} \\ G_{sl2} &= \tfrac{1}{2}\left[(\vartheta' y - \vartheta y')\upsilon + \vartheta(\upsilon' y - \upsilon y')\right]\vartheta \frac{\partial}{\partial y} \\ G_{sl3} &= (\upsilon' y - \upsilon y')\upsilon \frac{\partial}{\partial y}. \end{aligned} \tag{7.18}$$

On the other hand in the case of the third order ordinary differential equation associated with each $sl(2, R)$ subalgebra, G_{sl1}, G_{sl2}, and G_{sl3}, are the integrals I_1, I_2, and I_3, respectively where

$$I_1 = \vartheta^2 y'' - (\vartheta^2)' y' + (\vartheta^2)'' y + \vartheta^2 \nu y \tag{7.19}$$

$$I_2 = \vartheta v y'' - (\vartheta v)' y' + (\vartheta v)'' y + \vartheta v \nu y \tag{7.20}$$

$$I_3 = v^2 y'' - (v^2)' y' + (v^2)'' y + v^2 \nu y. \tag{7.21}$$

The integral

$$J = I_1 I_3 - I_2^2 \tag{7.22}$$

which can be expressed as some explicit function of $x, y, y', \ldots$ has $sl(2, R)$ symmetry, viz.,

$$G_{sl1}^{[2]}\left(I_1 I_3 - I_2^2\right) = 0 \tag{7.23}$$

$$G_{sl2}^{[2]}\left(I_1 I_3 - I_2^2\right) = 0 \tag{7.24}$$

$$G_{sl3}^{[2]}\left(I_1 I_3 - I_2^2\right) = 0. \tag{7.25}$$

For the $sl(2, R)$ subalgebra the integral J is the common integral. The representation of J differs from that given by Leach *et al.* [8] by a factor of a half. The half is due to the use of a different representation of $sl(2, R)$. For $sl(2, R)$ symmetries $G_{sli} I_i = 0$. The actions of the symmetries on the integrals are

$$\begin{array}{lll} G_{sl1}^{[2]} I_1 = 0, & G_{sl1}^{[2]} I_2 = -W_2 I_1, & G_{sl1}^{[2]} I_3 = -2W_2 I_2 \\ G_{sl2}^{[2]} I_1 = W_2 I_1, & G_{sl2}^{[2]} I_2 = 0, & G_{sl2}^{[2]} I_3 = -W_2 I_3 \\ G_{sl3}^{[2]} I_1 = 2W_2 I_2, & G_{sl3}^{[2]} I_2 = W_2 I_3, & G_{sl3}^{[2]} I_3 = 0, \end{array} \tag{7.26}$$

where $W_2 = \vartheta' v - \vartheta v'$ and the form of G_{slj}, $j = 1, 2, 3$, used is that given by (7.12).

PROPOSITION 2. *The intrinsically generalised symmetry with "super" Jacobian as coefficient of the $\partial/\partial y$ term given by*

$$G_{assj} = \Big\{\big[(\omega_j' y - \omega_j y')' y - (\omega_j' y - \omega_j y') y'\big]' y - \big[(\omega_j' y - \omega_j y')' y - (\omega_j' y - \omega_j y') y'\big] y'\Big\} \frac{\partial}{\partial y}$$

is a symmetry of the differential equation (6.1).

When the differential equation is taken into account the symmetry G_{assj} becomes

$$G_{assj} = -2kW_j \frac{\partial}{\partial y}, \tag{7.27}$$

where $W_j = \omega_j' y - \omega_j y'$ and $k = yy'' - \frac{1}{2}y'^2 + \frac{1}{2}\nu y^2$. It is easy to show that $G_{assj}^{[3]}$ leaves (6.1) invariant.

PROPOSITION 3. *The Lie bracket relations corresponding to* (7.11)–(7.13) *are*

$$\begin{aligned} \left[G_{si}, G_{slj}\right] &= -(\omega_i'\omega_j - \omega_i\omega_j')\frac{\partial}{\partial y} \\ &= G_{si} \qquad (7.28) \\ \left[G_{si}, G_{icsj}\right] &= -\left[(\omega_i'\omega_j - \omega_i\omega_j')\frac{\partial}{\partial x} + (\omega_i'\omega_j - \omega_i\omega_j')' y\frac{\partial}{\partial y}\right] \\ &\quad + (\omega_i''\omega_j - \omega_i'\omega_j' + \omega_i\omega_j'' + \omega_i\omega_j\nu) y\frac{\partial}{\partial y} \\ &= G_{sli} \qquad (7.29) \\ \left[G_{sli}, G_{icsj}\right] &= G_{icsj}. \qquad (7.30) \end{aligned}$$

The proof of the relation (7.28) and (7.30) is trivial so we concentrate on (7.29). Let

$$C_{ij} = \omega_i''\omega_j - \omega_i'\omega_j' + \omega_i\omega_j'' + \omega_i\omega_j\nu. \tag{7.31}$$

It is easy to see that C_{ij} is symmetric. The ordinary differential equations for ω_i and ω_j are

$$\omega_i''' + \nu\omega_i' + \tfrac{1}{2}\nu'\omega_i = 0 \tag{7.32}$$

$$\omega_j''' + \nu\omega_j' + \tfrac{1}{2}\nu'\omega_j = 0. \tag{7.33}$$

Now

$$(7.32)\,\omega_j + \omega_i(7.33) = \omega_i'''\omega_j + \omega_j'''\omega_i + \nu(\omega_i'\omega_j + \omega_j'\omega_i) + \nu'\omega_i\omega_j. \tag{7.34}$$

Upon integrating both sides of (7.34) we observe that

$$\begin{aligned} C_{ij} &= \omega_i''\omega_j - \omega_i'\omega_j' + \omega_i\omega_j'' + \omega_i\omega_j\nu \\ &= \text{a constant}. \qquad (7.35) \end{aligned}$$

This means that we can rewrite

$$[G_{si}, G_{icsj}] = -\left[(\omega_i'\omega_j - \omega_i\omega_j')\frac{\partial}{\partial x} + (\omega_i'\omega_j - \omega_i\omega_j')' y\frac{\partial}{\partial y}\right] + C_{ij} y\frac{\partial}{\partial y}. \tag{7.36}$$

The C_{ij}'s vary depending upon the solution set used. To illustrate this we consider the third order ordinary differential equation (6.1) for $\nu = 1$ and $\nu = 0$, respectively. We have for the first case

$$y''' + y' = 0, \qquad \{y\} = \{1, \sin x, \cos x\}, \tag{7.37}$$

and

$$\begin{array}{lll} C_{11} = 1, & C_{21} = 0, & C_{31} = 0 \\ C_{12} = 0, & C_{22} = -1, & C_{32} = 0 \\ C_{13} = 0, & C_{23} = 0, & C_{33} = -1. \end{array} \tag{7.38}$$

The second case gives

$$y''' = 0, \qquad \{y\} = \left\{1, x, \tfrac{1}{2}x^2\right\}, \tag{7.39}$$

and

$$\begin{array}{lll} C_{11} = 0, & C_{21} = 0, & C_{31} = 1 \\ C_{12} = 0, & C_{22} = -1, & C_{32} = 0 \\ C_{13} = 1, & C_{23} = 0, & C_{33} = 0. \end{array} \tag{7.40}$$

These are in accordance with the table in [8]. We denote E_{ij} by

$$E_{ij} = (\omega_i'\omega_j - \omega_i\omega_j')\frac{\partial}{\partial x} + (\omega_i'\omega_j - \omega_i\omega_j')' y\frac{\partial}{\partial y}. \tag{7.41}$$

For $\nu = 0$ and $\{\omega_i\} = \{1, x, 1/2x^2\}$ we have

$$\begin{array}{lll} E_{11} = 0, & E_{21} = G_{sl1}, & E_{31} = G_{sl2} \\ E_{12} = -G_{sl1}, & E_{22} = 0, & E_{32} = G_{sl3} \\ E_{13} = -G_{sl2}, & E_{23} = -G_{sl3}, & E_{33} = 0. \end{array} \tag{7.42}$$

We also have that from the solution set $\{\omega_i\} = \{1, x, 1/2x^2\}$ the Jacobian gives

$$\omega_1'\omega_2 - \omega_1\omega_2' = -\omega_1 \tag{7.43}$$

$$\omega_1'\omega_3 - \omega_1\omega_3' = -\omega_2 \tag{7.44}$$

$$\omega_2'\omega_3 - \omega_2\omega_3' = -\omega_3. \tag{7.45}$$

It is clear to see that in this case the Jacobian is also a solution of the differential equation and that the bracket relation (7.36) is

$$\left[G_{si}, G_{icsj}\right] = G_{sli}. \tag{7.46}$$

PROPOSITION 4. *For the third order ordinary differential equation the Jacobian is a solution of the original differential equation and so the intrinsically contact symmetries* G_{icsj} *are of the same structure as the* $sl(2, R)$ *symmetries* G_{sli}.

Proof. The third order ordinary differential equation (6.1) has symmetries

$$G_h^{[1]} = y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} \tag{7.47}$$

$$G_{si}^{[1]} = \omega_i\frac{\partial}{\partial y} + \omega_i'\frac{\partial}{\partial y'} \tag{7.48}$$

$$G_{sli}^{[1]} = \omega_i\frac{\partial}{\partial x} + \omega_i' y\frac{\partial}{\partial y} + \omega_i'' y\frac{\partial}{\partial y'} \tag{7.49}$$

$$G_{icsj}^{[1]} = \left(\omega_j' y - \omega_j y'\right)\frac{\partial}{\partial x} + \left(\omega_j' y - \omega_j y'\right)' y\frac{\partial}{\partial y} + \left(\omega_j' y - \omega_j y'\right)'' y\frac{\partial}{\partial y'}, \tag{7.50}$$

which have been extended once and in which G_h is the homogeneity symmetry, G_{si} the solution symmetries, G_{sli} the $sl(2, R)$ symmetries, and G_{icsj} the ICS. Note that here we are using generalised symmetries and have taken a first integral into account [14]. We now have the following propositions.

PROPOSITION 5. *Let* $w_1, \ldots, w_{n-1}$ *be linearly independent solutions of the nth order ordinary differential equation in the form of* (4.1) *or* (4.2). *Then the Jacobian* $W(w_1, \ldots, w_{n-1})$ *is also a solution.*

PROPOSITION 6. *Let the n fundamental solutions of the equation in the form of* (4.1) *or* (4.2) *be arranged in the order of their correspondence to the fundamental set* $\{1, x, \frac{1}{2}x^2, \ldots, \frac{1}{n-1}x^{n-1}\}$ *of* $y^{(n)} = 0$. *Then*

$$W(w_1, \ldots, w_{n-1}) = cw_1 \tag{7.51}$$

$$W(w_2, \ldots, w_n) = cw_n, \tag{7.52}$$

where c is a constant.

PROPOSITION 7. *Let S_{w_i} be the Jacobian of the n first integrals of an equation in the form of* (4.1) *or* (4.2). *Then S_{w_i} is a solution of the original equation.*

$$S_{w_i} = c_i w_i, \qquad i \leq \left[\frac{n}{2}\right] \tag{7.53}$$

$$S_{w_i} = c_i w_{i+1}, \qquad i > \left[\frac{n}{2}\right] \tag{7.54}$$

provided that the wording can be changed to render it correct.

PROPOSITION 8. *None of the above propositions apply if the equation is not in the form of* (4.1) *or* (4.2). *The proofs are left as a nontrivial exercise.*

The super Jacobian arises from taking the integral consequences into account. This enables us to move from a contact to a generalised symmetry. This comes from the fact that in the context of generalised symmetries [1] the operator (2.1) is equivalent to taking

$$G = \zeta \frac{\partial}{\partial y}, \tag{7.55}$$

where $\zeta = \eta - y'\xi$.

8. A REMARK ON FIRST INTEGRALS OF MAXIMAL SYMMETRY

It is interesting to note that

$$y^{(iv)} = 0 \tag{8.1}$$

has the solution symmetries

$$G_1 = \frac{\partial}{\partial y} \tag{8.2}$$

$$G_2 = x\frac{\partial}{\partial y} \tag{8.3}$$

$$G_3 = \frac{1}{2}x^2\frac{\partial}{\partial y} \tag{8.4}$$

$$G_4 = \frac{1}{6}x^3\frac{\partial}{\partial y}. \tag{8.5}$$

(8.6)

Associated with each solution symmetry is a linear integral [3]. The four functionally independent integrals are

$$I_1 = \tfrac{1}{6}x^3y''' - \tfrac{1}{2}x^2y'' + xy' - y \tag{8.7}$$

$$I_2 = \tfrac{1}{2}x^2y''' - xy'' + y' \tag{8.8}$$

$$I_3 = xy''' - y'' \tag{8.9}$$

$$I_4 = y'''. \tag{8.10}$$

The two integrals I_1 and I_4 each have five point symmetries while the middle integrals I_2 and I_3 each have four point symmetries. When G_1 and G_4 are used in the reduction of (8.1) they each lead to a scalar third order ordinary differential equation $Y''' = 0$ which is also an equation of maximal symmetry. This gives the relationship between the first integrals of maximal symmetry associated with each symmetry G_1 and G_4 and the reduced equation $Y''' = 0$ associated with each symmetry, respectively. In the case of a second order ordinary differential equation $y'' = 0$ the first integrals are $I_1 = y'$ and $I_2 = xy' - y$. The ratio of the two first integrals $I_2/I_1 = I_3$ also has the same number of point symmetries as I_1 and I_2.

9. CONCLUSION

We have used the Riccati transformation in the reduction of order of the second and third order equations of maximal symmetry (4.3) and (4.4), respectively. In the general structure of the symmetries we find that the $sl(2, R)$ subalgebra is preserved under the Riccati transformation. On the other hand the solution symmetries and the ICS become nonlocal symmetries. The homogeneity symmetry vanishes under the Riccati transformation. In the case of the third order equation (4.4) it is observed that the solution symmetries, the $sl(2, R)$, and the ICS all originate from the same third order equation (4.4). This is curious since equations in normal form do not have this property. We also point out that a basis for the solution set of equations of maximal symmetry (4.1) or (4.2) can be given in terms of the solution set of a second order ordinary differential equation. In the context of generalised symmetries we are able to express $sl(2, R)$ symmetries and the ICS in terms of a Jacobian. We further showed that the "super" Jacobian G_{assj} is also a symmetry of the third order differential equation. The first integrals associated with each element of the $sl(2, R)$ subalgebra were calculated and it was shown that the combination $I_1I_3 - I_2^2$ has $sl(2, R)$ symmetry. This is in agreement with the results given by Leach *et al*. [8]. It is also interesting to note that in terms of the scalar equation $y^{iv} = 0$ the four solution symmetries G_1, G_2, G_3, and G_4 have the property that if G_1 and G_4 are used in the reduction, the resulting equation $Y''' = 0$

is also an equation of maximal symmetry (see Section 1). Both G_1 and G_4 correspond to the integrals with five point symmetries I_1 and I_4, respectively.

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