Sign-Nonsingular Matrices and Even Cycles in Directed Graphs

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ABSTRACT

A *sign-nonsingular* matrix or *L-matrix* A is a real $m \times n$ matrix such that the columns of any real $m \times n$ matrix with the same sign pattern as A are linearly independent. The problem of recognizing square L-matrices is equivalent to that of finding an even cycle in a directed graph. In this paper we use graph theoretic methods to investigate L-matrices. In particular, we determine the maximum number of nonzero elements in square L-matrices, and we characterize completely the semicomplete L-matrices [i.e. the square L-matrices (a_{ij}) such that at least one of a_{ij} and a_{ji} is nonzero for any i, j] and those square L-matrices which are combinatorially symmetric, i.e., the main diagonal has only nonzero entries and $a_{ij} = 0$ iff $a_{ji} = 0$. We also show that for any $n \times n$ L-matrix there is an i such that the total number of nonzero entries in the ith row and ith column is less than n unless the matrix has a completely specified structure. Finally, we discuss the algorithmic aspects.

1. INTRODUCTION

Two $m \times n$ real matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to have *the same sign pattern* if $a_{ij} = 0$ if and only if $b_{ij} = 0$ and a_{ij} and b_{ij} have the same sign otherwise. When $b \in R^m$, the linear system $Ax = b$ is *sign-solvable* if it is solvable and, for each A' with the same sign pattern as A and each b' with the same sign pattern as b, the solutions of the two systems $Ax = b$, $A'x = b'$ have the same sign pattern. Sign-solvability problems occur in certain models in economics [11].

The difficulty in deciding whether or not the system $Ax = b$ is sign-solvable lies in recognizing an L-matrix associated with the system (see [6]). Klee, Ladner, and Manber [6] showed that it is NP-complete to decide whether or not an $m \times n$ matrix is an L-matrix if $m = n + \lfloor n^{1/k} \rfloor$, where k is any (fixed) natural number. However, they left the important special case $m = n$ open.

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As observed by Maybee [10], this problem can be formulated in terms of digraphs (directed graphs). Consider an $n \times n$ matrix $A = (a_{ij})$. A necessary condition for A to be an L-matrix is that its rows and columns can be permuted such that the entries in the main diagonal of the resulting matrix are all nonzero. These operations do not destroy the property of being an L-matrix, and the existence of such operations is precisely the problem of finding a perfect matching in a bipartite which can be solved by a fast (i.e. polynomially bounded) algorithm (see [6]). We shall therefore assume in what follows that $a_{ii} = 1$ for $1 \leqslant i \leqslant n$ (since multiplying a row of an L-matrix by a nonzero constant results in an L-matrix). Now $D(A)$ denotes the digraph whose vertices are $v_1, v_2, \dots, v_n$ and whose arcs are all $v_i v_j$ such that $a_{ij} \neq 0$, $i \neq j$. [In graph theoretic terms, $D(A)$ is the digraph whose adjacency matrix is $A - I$.] Moreover, we associate the weight 1 (respectively zero) to $v_i v_j$ if $a_{ij} > 0$ (respectively $a_{ij} < 0$). Such a digraph is called *weighted*. Clearly, A is an L-matrix iff the standard expression of $\det A$ has only positive terms, which is the case iff $D(A)$ has no cycle (i.e. directed cycle) of even weight. (The *weight* of a cycle is the sum of the arc weights. Our notation here is slightly different from that of [6, 8].) Clearly, $D(A)$ has a cycle of even weight iff there is a cycle of even length in the digraph obtained from $D(A)$ by subdividing all arcs of weight zero. So from an algorithmic point of view, the problem of recognizing a square L-matrix is equivalent with that of finding a cycle of even length in a digraph, which is equivalent with deciding whether or not a 0-1 square matrix is an L-matrix.

In this paper we use graph theoretic methods to investigate L-matrices. There are L-matrices with no zero elements, but we show than an $n \times n$ L-matrix has at least $\frac{1}{2}n(n-3)+1$ zero entries and we characterize the extremal matrices. We also characterize another class of dense $n \times n$ L-matrices, namely those L-matrices (a_{ij}) which are semicomplete, i.e., at least one of a_{ij} and a_{ji} is nonzero for all i, j. We describe completely the combinatorially symmetric $n \times n$ L-matrices. These are all sparse [they have at most $4(n-1)$ nonzero entries] and exist only if n is even. Finally, we observe that there are good algorithms for recognizing special classes of L-matrices.

Our terminology is the same as in [2]. In particular, a digraph is *semicomplete* if every two vertices are joined by an arc, and a digraph is *k-connected* if the removal of any set of fewer than k vertices leaves a *strong* (i.e. strongly connected) digraph. If S is a set of vertices of the digraph D, then $D(S)$ denotes the digraph induced by S. The *symmetric digraph* associated with the undirected graph G is denoted G^*. The *undirected cycle* of length n is denoted C_n. When we speak of a *cycle in a digraph* we always mean a *directed cycle*. A *double cycle* is a digraph of the form C_n^*.

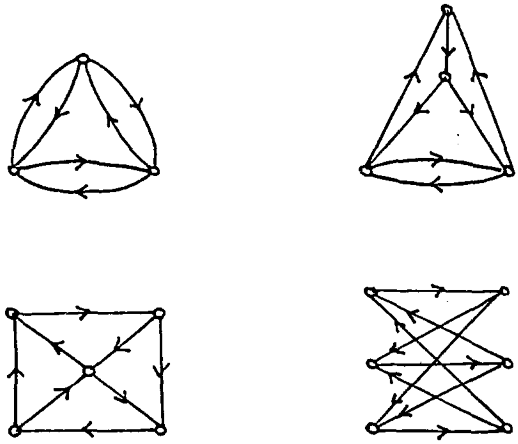

FIG. 1. Examples of even digraphs.

A *weighted digraph* is a digraph such that a weight (0 or 1) is associated to each arc. The weight of a cycle in a weighted digraph is the sum of its arc weights.

Finally, we say that a digraph D is *even* if D contains a cycle of even weight whenever weights are assigned to the arcs of D. Equivalently, D is even if and only if each subdivision of D contains a cycle of even length. A *subdivision* of a digraph D is a digraph obtained from D by replacing some (or all) arcs by directed paths. If x is a vertex in a digraph D, and we split x up into two vertices x_1 and x_2 such that x_1 (respectively x_2) is incident with all arcs directed towards x (respectively away from x), and we add the arc x_1x_2 and denote the resulting digraph by D', then it is easy to see that D is even if and only if D' is even. It is also easy to see that the double cycle C_n^* is even whenever n is odd, and so is any subdivision of every digraph obtained from C_n^* (n odd) by splitting vertices as indicated above. Figure 1 shows the even digraphs obtained from C_3^* in this way.

2. THE MAXIMUM NUMBER OF NONZERO ENTRIES IN L-MATRICES AND EVEN DIGRAPHS WITH LARGE DEGREES OR ARC DENSITY

Seymour [12] proved that a 3-color-critical hypergraph with n vertices must have at least n hyperedges. He showed that the problem of deciding whether or not a given hypergraph with n vertices and n hyperedges is 3-color-critical is equivalent with finding an even cycle in a certain digraph with n vertices. Using the aforementioned connection between L-matrices and digraphs without cycles of even weight, Seymour's result can be expressed as follows: Consider an $n \times n$ L-matrix A whose entries are all 0 or 1 (and whose main diagonal consists of ones). Suppose further that $D(A)$ is strong and contains no cycle of even weight. Then the rows of A are the incidence vectors of a 3-color-critical hypergraph. Moreover, every 3-color-critical hypergraph with n vertices and n hyperedges can be obtained in this way. Seymour [12] conjectured that any 3-color-critical hypergraph must have an edge with less than 10^{10} vertices (or, equivalently, any 0-1 square L-matrix has a row with at most 10^{10} ones). This conjecture, which was also mentioned by Lovász [7] (formulated as an even cycle problem), was disproved by the author [14, Theorem 3.1] as follows:

THEOREM 2.1. *For each natural number $n \geqslant 4$, there exists an $n \times n$ L-matrix A such that all entries are* 0 *or* 1 *and such that A has at least $\frac{1}{2}\log_2 n$ ones in each row.*

The bound $\frac{1}{2}\log_2 n$ is not sharp, but it has the right order of magnitude, as shown by the next result, which is sharp.

THEOREM 2.2 [14, Theorem 3.2]. *For each natural number $n \geqslant 2$ there exists an L-matrix with $\lfloor \log_2 n \rfloor + 1$ nonzero entries in each row. On the other hand, any $n \times n$ L-matrix has a row with at most $\lfloor \log_2 n \rfloor + 1$ nonzero entries.*

Klee et al. [6] showed that every special L-matrix (defined in [6]) has a row with at most three nonzero entries.

Theorem 2.2 asserts that any digraph with n vertices each of which has outdegree greater than $\log_2 n$ is even and that there are digraphs which are not even and in which all vertices have outdegree $\lfloor \log_2 n \rfloor$. For total degrees we have the following

PROPOSITION 2.3. *If D is a digraph with n vertices such that all vertices have degree at least $n+1$, then D is even.*

Proof. By a remark above it is sufficient to show that D contains a subdivision of C_3^*. We can assume that D is strong (otherwise we consider any strong component of D instead of D). Let D' be a strong subdigraph of D which has fewer vertices than D and which is maximal under this condition. It is easy to see that $D' = D - v$ for some vertex v in D. [In general, if D' is a strong subdigraph of a strong digraph D, and D' has fewer vertices than D and is maximal under these conditions, then $D \setminus V(D')$ is a path, and if it has length at least one, then all its vertices have indegree 1 and outdegree 1 in D.]

Now D' has $n-1$ vertices each of which has degree at least $n-1$ in D', and hence, by Ghouila–Houri's theorem (see [2, Theorem 1.1.2]), D' has a Hamiltonian cycle C. Also, v is incident with at least two 2-cycles which together with C form a subdivision of C_3^*. ■

We shall later extend Proposition 2.3, but first we apply it to give a short proof of the following:

PROPOSITION 2.4. *An $n \times n$ L-matrix has at least $\binom{n-1}{2}$ zero entries.*

Proof. Proposition 2.4 asserts that a digraph D with n vertices and more than $n(n-1)-\binom{n-1}{2}$ arcs is even. We prove this by induction on n, by showing that D contains a subdivision of C_3^*. For $n \leqslant 3$ there is nothing to prove, so we proceed to the induction step assuming $n \geqslant 4$. If all vertices of D have degree at least $n+1$, we apply (the proof of) Proposition 2.3. So assume that x is a vertex of degree at most n. Then $D - x$ has n vertices and more than $\binom{n-1}{2} + n - 2$ arcs, and hence $D - x$ contains a subdivision of C_3^*. ■

Proposition 2.4 cannot be extended to $m \times n$ L-matrices. Indeed, the matrix whose rows are the 2^n ± 1-vectors with n entries is an L-matrix with no zero elements. It also shows that an $m \times n$ L-matrix need not contain an $n \times n$ L-matrix when $n \geqslant 3$.

The bounds in Propositions 2.3 and 2.4 are sharp. To see this we consider a caterpillar T, i.e., a tree which consists of a paths $x_1 x_2 \cdots x_k$, $k \geqslant 3$, together with a (possibly empty) set of vertices each of which is joined to precisely one of $x_2, x_3, \ldots, x_{k-1}$. Now we consider T^*, and we add all arcs from x_i (and all end vertices joined to x_i in T) to x_j (and all end vertices joined to x_j) whenever $i > j$. Also we add arcs such that the end vertices joined to x_i induce a transitive tournament for each $i = 2, 3, \ldots, k-1$. The resulting semicomplete digraph is called an *extended caterpillar*, and $x_1 x_2 \cdots x_k$ its *basic path*. Clearly, any extended caterpillar or n vertices has $\binom{n}{2} + (n-1)$ arcs, and all vertices have degree n or more. Moreover, if we assign weights to the arcs in T^* such that all 2-cycles of T^* have odd weight, then each arc outside T^* can be assigned a weight (in only one way) such that the extended caterpillar has no cycle of even weight. We shall prove that the extended caterpillars are the extremal digraphs in Proposition 2.4, and that they contain all extremal digraphs in Proposition 2.3. For this we need the following result, which is of independent interest.

THEOREM 2.5. *Let D be a strong semicomplete digraph. Then D is even if and only if it is not a subdigraph of an extended caterpillar.*

Proof. Suppose D is not even. We prove, by induction on n, the number of vertices of D, that D is a subdigraph of an extended caterpillar. For $n = 3$ this is obvious, so assume $n \geqslant 4$. We consider first the case where D has a vertex x such that $D - x$ is not strong, i.e., the vertex set of $D - x$ can be partitioned into sets A, B such that no vertex in A dominates any vertex in B.

Since D is strong, x dominates some vertex b in B. If y is a vertex in A such that x does not dominate y, then we add the arc xy. The resulting digraph D' is not even, because we can assign the weight of the path xby (taken modulo 2) to xy. This will not create any cycle of even weight. Now the subdigraph of D' induced by $A \cup \{x\}$ is a subdigraph of an extended caterpillar with basic path $x_1x_2 \cdots x_k$ say. Since x dominates all vertices of A, we must have $x = x_k$ or $x = x_{k-1}$. Similarly, the subdigraph of D induced by $B \cup \{x\}$ is a subdigraph of an extended caterpillar with basic path $y_1y_2 \cdots y_m$ where $x = y_1$ or $x = y_2$. Hence D is a subdigraph of an extended caterpillar with basic path $x_1x_2 \cdots x_{k-1} \cdots y_m$, and the proof is complete.

We can assume that D is 2-connected. In particular, each vertex has indegree and outdegree at least 2. Let x be any vertex of D. By the induction hypothesis, $D - x$ is a subdigraph of an extended caterpillar with basic path $x_1x_2 \cdots x_k$, $k \geqslant 3$. Since all vertices in D have indegree and outdegree at least 2, x dominates x_k and all other vertices of indegree 1 in $D - x$, and is dominated by x_1 and all other vertices of outdegree 1 in $D - x$. Let D_1 be obtained from $D - x$ by adding an arc from x_1 to each vertex of indegree 1 in $D - x$. Then D_1 is not even, because any new arc x_1y can be assigned the same weight as in the path x_1xy. So D_1 is a subdigraph of an extended caterpillar. In particular, D_1 has two distinct vertices z_1 and z_2 of outdegree and indegree (respectively) 1 in D_1. Since each vertex in $D - x$ has indegree at least 1, and no vertex other than x_2 is dominated by x_1 in $D - x$, we must have $k = 3$ and $z_2 = x_2$, and in $D - x$, x_2 is dominated by x_1 only. We then form the digraph D_2 by adding to $D - x$ all arcs yx_3 where y has outdegree 1 in $D - x$, and, as above, we conclude that in $D - x$, x_2 dominates x_3 only. So $D - x$ is the 3-cycle $x_1x_2x_3x_1$. We have shown that the deletion of any vertex from D results in a 3-cycle. But this is impossible and the proof is complete. ■

We can now characterize the extremal L-matrices in Proposition 2.4 in terms of their associated digraphs.

THEOREM 2.6. *A digraph D with n vertices and $n(n-1)-\binom{n-1}{2}$ or more arcs is even if and only if D is not an extended caterpillar.*

Proof (by induction on n). For $n \leqslant 3$ the statement is easily verified. So assume that D has at least $n(n-1)-\binom{n-1}{2}$ arcs and is not even. As in the proof of Proposition 2.4, we consider a vertex of degree at most n. By the induction hypothesis, $D-x$ is an extended caterpillar and hence x has degree n. Since $D-x$ has a Hamiltonian cycle, x is not incident with two cycles of length 2. (Otherwise D contains a subdivision of C_3^*.) So x is adjacent to all vertices of $D-x$, and hence D is semicomplete. Now the result follows from Theorem 2.5. ■

We conclude this section by characterizing the extremal digraphs in Proposition 2.3.

THEOREM 2.7. *If D is a strong digraph with n vertices and of minimum degree at least n, and D is not even, then D is a subdigraph of an extended caterpillar or else D is isomorphic to C_2^* or C_4^*.*

Proof (by induction on n). For $n \leqslant 3$ the statement is easily verified. So assume that $n \geqslant 4$. We consider first the case where D has a vertex x which dominates only one vertex, say y. Then x is dominated by all vertices of $D-x$. Now the digraph D_1 obtained from $D-x$ by adding all arcs from $D-x-y$ to y is not even, and it satisfies the assumption of Theorem 2.7. Hence D_1 is a subdigraph of an extended caterpillar whose basic path has y as its first or second vertex. Hence also D is a subdigraph of an extended caterpillar whose first and second vertex are x and y, respectively.

We consider next the case where D has a vertex x such that $D-x$ is not strong, i.e., the vertex set of $D-x$ has a decomposition $A \cup B$ such that no vertex in A dominates any vertex in B. Suppose without loss of generality that $|A| \leqslant |B|$. Then any vertex of A is dominated by some vertex of B (since all vertices have degree at least n). Let D' denote the digraph obtained from D by adding all arcs from x to A. As in the proof of Theorem 2.5, we see that

D' is not even. Now the subdigraph D_2 of D' induced by $\{x\} \cup A$ satisfies the assumption of Theorem 2.7 and is therefore a subdigraph of an extended caterpillar. Hence D_2 has a vertex distinct from x, which dominates only one vertex in D_2 (and hence also in D). We have already disposed of that case.

So we can assume that D is 2-connected. If D has no cycle $x_1x_2 \cdots x_{n-1}x_1$ missing precisely one vertex x, then, by a result of Häggkvist and Thomassen [5], n is even and D is isomorphic to the symmetric complete bipartite digraph $K^*_{n/2,\, n/2}$, which for $n = 4$ is isomorphic to C_4^* and which for $n \geqslant 6$ contains a subdivision of C_3^*, contrary to the assumption that D is not even. So assume that the cycle $x_1x_2 \cdots x_{n-1}x_1$ exists. We show that this leads to a contradiction. Since D contains no subdivision of C_3^*, x is incident with only one 2-cycle. So x is adjacent to all vertices of $D - x$, and we can assume that D contains the 2-cycle $xx_{n-1}x$. If there exist natural numbers i, j such that $1 \leqslant i < j \leqslant n-2$ and the arcs xx_i, x_jx are present, then D contains a subdivision of the second digraph in Figure 1 and hence D is even, a contradiction. So there exists a k, $1 \leqslant k \leqslant n-3$, such that x is dominated by $x_{n-1}, x_1, x_2, \ldots, x_k$ and no other vertex, and x dominates $x_{k+1}, \ldots, x_{n-1}$ and no other vertex. We now define the digraph D' as follows: If x_{n-1} has degree at least $n-1$ in $D - x$, we put $D' = D - x$. Otherwise, there exists a vertex x_i, $1 \leqslant i \leqslant k$, such that x_{n-1} is not dominated by x_i, or there exists a vertex x_j, $k+1 \leqslant j \leqslant n-2$, such that x_{n-1} does not dominate x_j. Then D' is obtained from $D - x$ by adding either x_ix_{n-1} or $x_{n-1}x_j$. Clearly, D' satisfies the assumption of the theorem, and hence $D' \simeq C_4^*$ or D' is a subdigraph of an extended caterpillar. Since D is not even, we can assume that the second alternative holds. Let $z_1z_2 \cdots z_m$ be the basic path of D'. Since D is 2-connected, x is dominated by z_1 and dominates z_m. Since an extended caterpillar has only one Hamiltonian cycle, namely $x_1x_2 \cdots x_{n-1}x_1$, we conclude that $z_1z_2 \cdots z_m$ is a segment of that cycle. In particular, x_{n-1} is not in the path $z_2z_3 \cdots z_{m-1}$. We can assume that $x_{n-1} \neq z_1$ (otherwise, we consider the reverse digraph of D instead of D). Now we let D'' be obtained from $D - x$ by adding all arcs z_1y such that y is dominated by x in D. Clearly, D'' is not even. Since x_{n-1} is dominated by x and $x_{n-1} \neq z_1$ it follows that x_{n-1} has degree at least $n-1$ in D'' and hence, by the induction hypothesis, D'' is a subdigraph of an extended caterpillar. But this is impossible. For in D', z_1 dominates z_2 only and since z_1 has degree at least n in D, z_1 is dominated in $D - x$ by all vertices other than z_1. In particular, all vertices other than z_2 have outdegree at least 2 in $D - x$. So the basic path in D'' must start at z_2 and since z_1 has degree at least n in D'', the basic path in D'' must include z_1 because all vertices outside the basic path have degree $n-1$ in an extended caterpillar. Hence the basic path in D'' ends at z_1, i.e., z_1 has indegree 1 in D''. Since z_1 is dominated by all vertices of $D - x - z_1$ we have obtained a contradiction which proves the theorem. ■

3. COMBINATORIALLY SYMMETRIC L-MATRICES AND EVEN SYMMETRIC DIGRAPHS

In Section 2 we considered digraphs (namely the extended caterpillars) which are not even and which are dense in the sense that they have large degrees or many arcs (or both). However, they all have vertices of outdegree or indegree 1. [The corresponding L-matrices contain a row (or column), respectively with only two nonzero entries.] In this section we consider 2-connected digraphs which are not even. We say that an *undirected graph G is even* if and only if the symmetric digraph G^* is even. If G is not even and xy is an edge of G, then it is easy to verify that we obtain a noneven graph by adding a path xx_1x_2y of length 3. In fact, two of the six arcs corresponding to the path xx_1x_2y can be weighted at random. Clearly, C_4 is noneven, and we define a *C_4-cockade* as a graph which can be obtained from C_4 by repeated use of the 3-path operation above.

Clearly, a C_4-cockade is 2-connected and has an even number n of vertices, and it has $\frac{3}{2}n - 2$ edges. To any C_4-cockade G we associate the graph $T(G)$ whose vertices are the 4-cycles in G such that two vertices are adjacent in $T(G)$ iff the corresponding 4-cycles have an edge in common. It is easy to see that $T(G)$ is a tree. An end vertex of $T(G)$ corresponds to a 4-cycle in G which has two adjacent vertices of degree 2 in G. If $T(G)$ is a path, as is the case if G is the second or third graph in Figure 2, then G has a unique Hamiltonian cycle C. For any edge e which is not in C, G is partitioned into two subgraphs each of which is a C_4-cockade and such that they have precisely e and the ends of e in common.

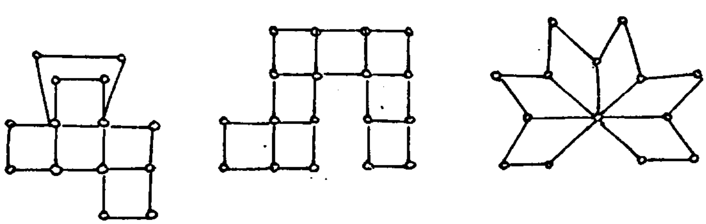

FIG. 2. C_4-cockades.

It is easy to see that the arcs of the symmetric digraph associated with a C_4-cockade with n vertices can be weighted in precisely 2^{n-1} ways such that there is no cycle of even weight. When weights are associated to the arcs of a symmetric digraph whose underlying undirected graph has only blocks that are subgraphs of C_4-cockades (in such a way that there is no cycle of even weight) then that gives rise to a symmetric L-matrix. The next result shows that every symmetric L-matrix can be obtained in that way.

Theorem 3.1. *Let G be a 2-connected undirected graph. Then the following statements are equivalent*:

(i) G is even.

(ii) $G \simeq C_n$, n odd, or G contains two vertices which are joined by three internally disjoint paths one of which has even length.

(iii) G is not a subgraph of a C_4-cockade.

Proof. Clearly (ii) implies (i) because the three paths in (ii) correspond to a subdigraph in G^* containing a subdivision of the double cycle C^*_{m+1}, where m is the even length occurring in (ii). Also (i) implies (iii) because a C_4-cockade is not even. So in order to complete the proof we show (by induction on the total number of vertices and edges of the graph) that if G does not satisfy (ii), then it is a subgraph of a C_4-cockade. This is clearly true for graphs with at most 4 vertices, so assume that G has at least 5 vertices.

We consider first the case where G has two adjacent vertices x, y, each of degree 2. Let x' (respectively y') be the other neighbor of x (respectively y). Then $x' \neq y'$, because G is 2-connected. Let G' be obtained from $G - \{x, y\}$ by adding the edge $x'y'$ if it is not already present. By the induction hypothesis, G' is a subgraph of a C_4-cockade and so is G.

So assume that G has no two adjacent vertices of degree 2. We shall obtain a contradiction from that. Let H be a maximal proper 2-connected subgraph of G. It is easy to see that G arises from H by adding a path $z_1 z_2 \cdots z_m$ where each of $z_2, z_3, \ldots, z_{m-1}$ has degree 2 in G. By the above assumption, $m \leqslant 3$. Since G does not satisfy (ii), m is even, i.e., $m = 2$. By the induction hypothesis, H is a subgraph of a C_4-cockade H'. Since G has no two adjacent vertices of degree 2, the tree $T(H')$ has only two end vertices [i.e., $T(H')$ is a path] and z_1 and z_2 belong to 4-cycles corresponding to the two end vertices of $T(H')$. By a previous remark, H' has a unique Hamiltonian cycle C, and since H is 2-connected, H contains C. Let x be the neighbor of z_1 which has degree 2 in H' (and hence also in H). The neighbor y of x distinct from z_1 has degree at least 3 in H. Let v be a neighbor of y such that yv is a chord of C in H. Then yv partitions H' into two C_4-cockades H_1 and H_2 such that H_1, say, contains x and z_1. Since $T(H')$ is a path, z_2 is in $H_2 - \{y, v\}$ and now H contains three internally disjoint yz_1 paths one of which is yxz_1. This contradiction proves the theorem. ■

The equivalence of (i) and (ii) in Theorem 3.1 was found independently by Harary et al. [4] and by Manber et al. [9].

4. ALGORITHMIC ASPECTS OF THE EVEN CYCLE PROBLEM

Klee et al. [6] and the author [14] independently showed that it is NP-complete to decide whether or not a digraph has an even cycle through a given arc. This might indicate that the problem of finding an even cycle in a digraph is difficult. On the other hand, the problem of finding an odd cycle through a given arc is equivalent (from an algorithmic point of view) to the above problem, and yet it is easy to decide if a digraph has an odd cycle. Indeed, if D is a digraph and A is its adjacency matrix (i.e. $A = \{a_{ij}\}$, where $a_{ij} = 1$ if vertex i dominates vertex j and zero otherwise), then D has an odd cycle iff for some odd natural number k, A^k has a nonzero main diagonal and the smallest odd k for which this holds is the length of a shortest odd cycle in D. The problem of deciding whether or not a real matrix is an L-matrix was also shown to be NP-complete by Klee et al. [6].

By [15, Theorem 4.1] a digraph D contains a vertex meeting all cycles in D if and only if D does not contain two disjoint cycles or a subdivision of any of the digraphs is Figure 1. This gives a good (i.e. polynomially bounded) algorithm for finding an even cycle in a digraph D with no two disjoint cycles: For each vertex x, we investigate if $D - x$ has a cycle. If this is the case for each vertex x, then D has a cycle of even length. On the other hand, if $D - x$ is acyclic for some x, then we can find the cycle length distribution of D (i.e., the number of 2-cycles, the number of 3-cycles etc.) in polynomial time. More generally, we have

THEOREM 4.1. *Let k be a fixed natural number. If D is a digraph containing a set S of at most k vertices such that $D - S$ is acyclic, then the cycle lengths in D can be found in polynomial time.*

Proof. Any cycle C in D contains a vertex in S and can therefore be described as the union of paths

$$P_1 \cup L_1 \cup Q_1 \cup R_1 \cup P_2 \cup L_2 \cup Q_2 \cup R_2 \cup \cdots \cup P_m \cup L_m \cup Q_m \cup R_m,$$

where the paths $P_1, P_2, \dots, P_m$ are pairwise disjoint paths with vertices in S;

$Q_1, Q_2, \ldots, Q_m$ are pairwise disjoint paths in $D - S$; and for $i = 1,2,\ldots,m$, L_i is an arc from the end of P_i to the starting vertex of Q_i, and R_i is an arc from the end of Q_i to the starting vertex of P_{i+1} (where $P_{m+1} = P_1$). The number of possibilities for the ordered sequence $P_1, P_2, \ldots, P_m$ is less than $2^k k!$, and the number of possibilities for $L_1, L_2, \ldots, L_m, R_1, R_2, \ldots, R_m$ is less than n^{2k}. So it is sufficient to show that, for fixed $P_1, L_1, R_1, \ldots, P_m, L_m, R_m$, the possible values of the total lengths of the paths $Q_1, Q_2, \ldots, Q_m$ can be found in polynomial time. For this we use an idea of Fortune, Hopcroft, and Wyllie [3]. We consider the digraph D^* whose vertices are all m-tuples consisting of distinct vertices of $D - S$ such that there is an arc in D^* from $Z = (z_1, z_2, \ldots, z_m)$ to $Z' = (z'_1, z'_2, \ldots, z'_m)$ if there is a $j \in \{1,2,\ldots,m\}$ such that $z_i = z'_i$ for $i \in \{1,2,\ldots,k\} \setminus \{j\}$ and $D - S$ contains the arc $z_j z'_j$ and contains no path from z_i to z_j whenever $i \neq j$. It is easy to see that a path of length q in D^* corresponds to a path system of total length q in $D - S$. Also, D^* is acyclic, and hence the number of paths of a given length from a given vertex Z to another vertex Z' can be calculated in polynomial time (for example, by using powers of the adjacency matrix). If we let Z (respectively, Z') consist of the ends of $L_1, L_2, \ldots, L_m$ (respectively, the starting vertices of $R_1, R_2, \ldots, R_m$), we complete the proof. ∎

The digraphs satisfying the condition of Theorem 4.1 correspond to those $n \times n$ matrices $A = \{a_{ij}\}$ which (after appropriate row and column permutations) satisfy the condition $a_{ij} = 0$ whenever $i \geqslant j > k$. Thus we can check, in polynomial time, if such a matrix is an L-matrix.

Consider a strong weighted digraph D containing a vertex z such that $D - z$ is not strong, i.e., $D - z$ consists of two disjoint digraphs D_1 and D_2 and possibly some arcs from D_1 to D_2. We now consider two digraphs D'_1 and D'_2 with vertex sets $V(D_1) \cup \{z\}$ and $V(D_2) \cup \{z\}$, respectively. For each arc xy with $x \in V(D_1)$ and $y \in V(D_2)$ we consider a cycle C_{xy} containing xy. If C_{xy} has odd weight, we add the arcs xz and zy and assign weights to these new arcs such that $C_{xy} \cup \{xz, zy\}$ has no cycle of even weight. Then we delete the arc xy. With this notation we have

LEMMA 4.2. *D has a cycle of even weight if and only if either one of the cycles C_{xy} has even weight or either D'_1 or D'_2 has a cycle of even weight.*

Proof. If one of C_{xy} has even weight, we have finished, so assume that each C_{xy} has odd weight. If one of D'_1, D'_2 has a cycle of even weight, it is easy to find a cycle of even weight in D (because of the way weights have been assigned to the new arcs). Suppose conversely that D has a cycle C of even weight, and assume that this is not in D'_1 or D'_2. Then C contains precisely one arc xy such that $x \in V(D_1)$ and $y \in V(D_2)$. Since C_{xy} has odd weight, we can assume that the segments of C_{xy} and C from z to x have the same parity and that the segments of C_{xy} and C from y to z have different parity. But then the cycle in D_2' consisting of the arc zy and the segment of C from y to z has even weight. ■

Lemma 4.2 reduces the problem of finding a cycle of even weight in a digraph D to the case where D is 2-connected. For if D is not 2-connected we form D_1' and D_2' as above. If one (or both) of these is (are) not 2-connected we perform the same operation on D_1' or D_2'. Continuing like this, we obtain in polynomial time a sequence of at most $n = |V(D)|$ 2-connected digraphs each of order at most n such that D has a cycle of even weight if and only if one of the 2-connected digraphs in the sequence has a cycle of even weight. We can even go a step further.

THEOREM 4.3. *If D is a digraph of order n, then we can construct, in polynomial time, a sequence of at most n digraphs $D_1, D_2, \dots$ each of order at most n such that D has a cycle of even weight if and only if one of $D_1, D_2, \dots$ has such a cycle and, moreover, each of $D_1, D_2, \dots$ is 2-connected and has an underlying undirected graph which is either 3-connected or isomorphic to a cycle of length at most 5 or obtained from a 3-connected graph by subdividing some edges once.*

Proof. By the reasoning preceding Theorem 4.3 we can assume that D is 2-connected. If the underlying undirected graph of D does not have the structure described in Theorem 4.3, then D is the union of two induced subdigraphs $D = D_1' \cup D_2'$ such that $V(D_1') \cap V(D_2') = \{x, y\}$ and each of D_1' and D_2' has at least four vertices. Since D is 2-connected, D_1' contains an xy path P_1 and a yx path P_2. If the weights of P_1 and P_2 have different parity, then we denote by D_2'' the digraph obtained from D_2 by adding arcs xy and yx whose weights have the same parity as the weights of P_1 and P_2, respectively. Otherwise, D_2'' will denote the digraph obtained from D_2' by adding a new vertex z and the new arcs zx, xz, yz, zy such that the weights of the paths xzy and yzx have the same parity as the weight of P_1 and the cycles zxz and zyz have odd weights. We construct D_1'' analogously. It is easy to see that D has a cycle of even weight if and only if one of D_1'', D_2'' has such a cycle. If D_1'' and D_2'' do not satisfy the conclusion of the theorem, then we iterate the above construction. This completes the proof. ■

If we apply the reductions in Theorem 4.3 on the noneven digraphs in this paper and also on those of large outdegrees in [14], then we end up with double cycles of length 2 or 4. Other "irreducible" digraphs are shown in Figure 3.

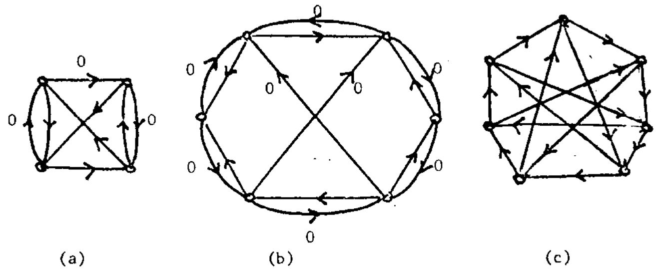

FIG. 3. Examples of 2-connected noneven digraphs.

The digraph of Figure 3(c) was found by Sylvia Boyd (private communication) and is the only known 2-connected digraph with no cycle of even length. This raises the following question:

QUESTION 1. Are there infinitely many 2-connected digraphs with no cycle of even length?

As we have seen, there are infinitely many weighted symmetric 2-connected digraphs with no cycle of even weight. But perhaps those which cannot be reduced by Theorem 4.3 have such a simple structure that it can be checked in polynomial time whether or not a given digraph has that structure. If so, we would have a polynomial algorithm for finding a cycle of even weight in a digraph. This suggests the following question:

QUESTION 2. Is every 3-connected digraph even?

In case Question 2 has a negative answer, one can ask the same question with 10^{10} instead of 3. The corresponding even length problem was raised by Lovász [7].

Theorem 4.3 and Question 2 may be a first step towards explaining why a given digraph is noneven. It is also of interest to give a good characterization of the even digraphs. We have previously observed that any odd double cycle and any digraph which can be obtained from such a digraph by splitting vertices and subdividing arcs is even. This raises the question whether any even digraph must contain such a digraph. Theorem 3.1 and the proofs of Theorems 2.5, 2.7 show that this holds for symmetric digraphs, semicomplete digraphs, and digraphs with n vertices and minimum degree at least n. In a forthcoming paper Seymour and the author [13] answer the above question in the affirmative.

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Received 16 July 1984; revised 28 December 1984