Precise asymptotics in the self-normalized law of the iterated logarithm

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Received 5 March 2007
Available online 29 September 2007
Submitted by M. Ledoux

Abstract

Let $X, X_1, X_2, \ldots$ be i.i.d. nondegenerate random variables with zero means, $S_n = \sum_{j=1}^{n} X_j$ and $V_n^2 = \sum_{j=1}^{n} X_j^2$. We investigate the precise asymptotics in the law of the iterated logarithm for self-normalized sums, $S_n/V_n$, also for the maximum of self-normalized sums, $\max_{1 \leq k \leq n} |S_k|/V_n$, when $X$ belongs to the domain of attraction of the normal law.

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Keywords: Precise asymptotics; Law of the iterated logarithm; Self-normalized sums

1. Introduction and main results

Throughout this paper, we let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. nondegenerate random variables with common distribution function $F$, and set $S_n = \sum_{j=1}^{n} X_j$ for $n \geq 1$, $\log x = \ln(x \vee e)$ and $\log \log x = \log(\log x)$. Hsu and Robbins [15] and Erdős [10] established the well-known complete convergence

$$\sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) < \infty$$

holds for all $\varepsilon > 0$ if and only if $EX = 0$ and $EX^2 < \infty$. Baum and Katz [1] extended this result and proved the following theorem.

**Theorem A.** Let $1 \leq p < 2$ and $r \geq p$. Then

$$\sum_{n=1}^{\infty} n^{r-2} P(|S_n| \geq \varepsilon n^{1/p}) < \infty \quad (1.1)$$

holds for all $\varepsilon > 0$ if and only if $EX = 0$ and $EX^{r/p} < \infty$. 

*T.-X. Pang’s research is supported by the National Natural Science Foundation of China (No. 10671176), and L.-X. Zhang and J.-F. Wang’s research is supported by the National Natural Science Foundation of China (No. 10471126).

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doi:10.1016/j.jmaa.2007.09.054
Many authors considered various extensions of the results of Hsu–Robbins–Erdős and Baum–Katz. Some of them studied the precise asymptotics of the infinite sums as \( \varepsilon \to 0 \) (cf. Heyde [14], Chen [5] and Spătaru [21]). But, this kind of result does not hold for \( p = 2 \). However, by replacing \( n^{1/p} \) by \( \sqrt{n \log \log n} \), Gut and Spătaru [13] established the following results called the precise asymptotics of the law of the iterated logarithm.

**Theorem B.** Suppose that \( EX = 0 \), \( EX^2 = \sigma^2 < \infty \) and \( EX^2 (\log \log |X|)^{1+\delta} < \infty \) for some \( \delta > 0 \), and let \( a_n = O(\sqrt{n}/(\log \log n)^\gamma) \) for some \( \gamma > 1/2 \). Then

\[
\lim_{\varepsilon \searrow 1} \sqrt{\varepsilon^2 - 1} \sum_{n=1}^\infty \frac{1}{n} P(|S_n| \geq \varepsilon \sigma \sqrt{2n \log \log n} + a_n) = 1. \tag{1.2}
\]

**Theorem C.** Suppose that \( EX = 0 \) and \( EX^2 = \sigma^2 < \infty \). Then

\[
\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^\infty \frac{1}{n \log n} P(|S_n| \geq \varepsilon \sqrt{n \log \log n}) = \sigma^2. \tag{1.3}
\]

It is well known that the so-called self-normalized limit theorems put a totally new countenance upon classical limit theorems. We refer to Griffin and Kuelbs [12] for the law of the iterated logarithm, Giné, Götze and Mason [11] for the necessary and sufficient condition for the asymptotic normality, Csörgő, Szyszkowicz and Wang [7,8] for the Darling–Erdős theorem and Donsker’s theorem. Other self-normalized results can be found in Bentkus and Götze [2] as well as Wang and Jing [23] for Berry–Esseen inequalities, Shao [18,20] as well as Jing, Shao and Wang [16] for Cramér type large deviations. For a survey on recent developments in this area, we refer to Shao [19] or Csörgő, Szyszkowicz and Wang [9]. The purpose of this paper is to develop precise asymptotics in the law of the iterated logarithm for so-called self-normalized sums.

Write \( M_n = \max_{k \leq n} |S_k|, V_n^2 = \sum_{j=1}^n X_j^2 \) and \( l(x) = EX^2 I(|X| \leq x) \). The following two theorems are the main results.

**Theorem 1.1.** Let \( \{X, X_n; n \geq 1\} \) be a sequence of nondegenerate i.i.d. symmetric random variables with \( EX = 0 \) and \( l(x) \) be a slowly varying function at \( \infty \), satisfying \( l(x) \leq c_1 \exp(c_2 (\log x)^{\beta}) \) for some \( c_1 > 0, c_2 > 0 \) and \( 0 \leq \beta < 1 \). Let \( a > -1 \) and \( b > -1/2 \). Assume that \( \alpha_n(\varepsilon) \) is a nonnegative function of \( \varepsilon \) such that

\[
\alpha_n(\varepsilon) \log \log n \to \tau \quad \text{as} \quad n \to \infty \quad \text{and} \quad \varepsilon \searrow \sqrt{1 + a}. \tag{1.4}
\]

Then

\[
\lim_{\varepsilon \searrow 1 + a} (\varepsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^\infty \frac{(\log n)^a (\log \log n)^b}{n} P\left( |S_n| \geq \sqrt{2V_n^2 \log \log n} (\varepsilon + \alpha_n(\varepsilon)) \right) = \frac{1}{\pi(1 + a)} \exp(-2\tau \sqrt{1 + a}) \Gamma(b + 1/2), \tag{1.5}
\]

where \( \tau \) is a finite constant and \( \Gamma(\cdot) \) is the gamma function.

**Theorem 1.2.** Let \( \{X, X_n; n \geq 1\} \) be a sequence of nondegenerate i.i.d. random variables with \( EX = 0 \) and \( l(x) \) be a slowly varying function at \( \infty \), satisfying \( l(x) \leq c_1 \exp(c_2 (\log x)^{\beta}) \) for some \( c_1 > 0, c_2 > 0 \) and \( 0 \leq \beta < 1 \). Assume that \( \alpha_n = O(1/\log \log n) \). Then, for \( d > -1 \), we have

\[
\lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^\infty \frac{(\log n)^d}{n \log n} P\left( |M_n| \geq \sqrt{2V_n^2 \log \log n} (\varepsilon + \alpha_n) \right) = \frac{2}{\sqrt{\pi(1 + d)}} \Gamma(d + 3/2) \sum_{k=0}^\infty \frac{(-1)^k}{(2k + 1)^{2d + 2}}, \tag{1.6}
\]

and

\[
\lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^\infty \frac{(\log n)^d}{n \log n} P\left( |S_n| \geq \sqrt{2V_n^2 \log \log n} (\varepsilon + \alpha_n) \right) = \frac{1}{\sqrt{\pi(1 + d)}} \Gamma(d + 3/2). \tag{1.7}
\]
Remark 1.1. Note that $X$ belonging to the domain of attraction of the normal law is well known to be equivalent to $l(x)$ being a slowly varying function at $\infty$. We note also that $l(x) \leq c_1 \exp(c_2 (\log x)^\beta)$ is a weak enough assumption, which is satisfied by a large class of slowly varying function such as $(\log \log x)^\tau$ and $(\log x)^\tau$, for some $0 < \tau < \infty$.

Remark 1.2. If $-1 < a < 0$, $b > -1/2$ or $a = 0$, $-1/2 < b < 0$, similar to the proof of Theorem 1.2 below, (1.5) still holds without symmetry assumption. So, (1.5) might hold without symmetry assumption for all $a > -1$ and $b > -1/2$. To get such an improvement of the result, we think that a different approach is necessary.

Remark 1.3. If $0 < 1 + a < 1$, then $P(|S_n| \geq \sqrt{2V_n^2 \log n \log n (\epsilon + a_n(\epsilon))}) = 1$ as $n$ large enough in view of the law of the iterated logarithm for self-normalized sums (cf. Griffin and Kuelbs [12]), which implies that the infinite series in the left-hand side of (1.5) will approach infinity. Hence, (1.5) is a precise result about the trade-off between large $n$ and $\epsilon$. The similar remark is applicable to the case of $1 + a \geq 1$ in (1.5) and (1.7).

Throughout this paper, we let $A$ denote a positive constant, whose values can differ in different places. $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$.

2. The proof of Theorem 1.1

We start with some notations. Put $c = \inf \{x \geq 1: l(x) > 0\}$ and

$$
\eta_n = \inf \{s: s \geq c + 1, \frac{l(s)}{s^2} \leq \frac{(\log \log n)^2}{n}\}.
$$

Furthermore, for each $n$ and $1 \leq i \leq n$, we let

$$
\Xi_{ni} = X_i I\{\left|X_i\right| \leq \eta_n\}, \quad \Xi_n = \sum_{i=1}^{n} \Xi_{ni}, \quad \Xi_n^2 = \sum_{i=1}^{n} \Xi_{ni}^2,
$$

$$
\Sigma_n^i = S_n - X_i, \quad \Sigma_n^2 = (\Sigma_n^2 - \Xi_n^2)^{1/2},
$$

$$
\Sigma_n^i = \Xi_n - \Xi_{ni}, \quad \Sigma_n^2 = (\Sigma_n^2 - \Xi_n^2)^{1/2}.
$$

Noting that $nl(\eta_n) \sim n(\log \log n)^2$ by the definition of $\eta_n$. Moreover, we have

$$
l(\eta_n) \leq c_1 \exp(c_2 (\log \eta_n)^\beta) \leq c_1 \exp(c_2 (\log n)^\beta)
$$

for large $n$. We also have that $l(\eta_n)$ and $c_1 \exp(c_2 (\log \eta_n)^\beta)/l(\eta_n)$ are slowly varying functions at $\infty$ (see Proposition 1.3.6 in [4, p. 16]). Using these facts, it follows easily that

$$
c_1 \exp(c_2 (\log j)^\beta)/l(\eta_j) \leq \frac{1}{2c_1 \exp(c_2 (\log k)^\beta)/l(\eta_k)}
$$

for all $j \geq k$ and $k$ large enough, which implies

$$
\frac{\exp(c_2 (\log k)^\beta)}{2l(\eta_k)} \sum_{j=k}^{\infty} \frac{1}{j \exp(c_2 (\log j)^\beta)(\log j)(\log \log j)^2} \leq \sum_{j=k}^{\infty} \frac{1}{j l(\eta_j)(\log j)(\log \log j)^2}.
$$

(2.1)

It is easily seen that

$$
\frac{1}{j \exp(c_2 (\log j)^\beta)(\log j)(\log \log j)^2}
$$

is a decreasing function of $j$, which leads to

$$
\int_k^{\infty} \frac{1}{x \exp(c_2 (\log x)^\beta)(\log x)(\log \log x)^2} dx \leq \sum_{j=k}^{\infty} \frac{1}{j \exp(c_2 (\log j)^\beta)(\log j)(\log \log j)^2}
$$

$$
\leq \int_{k-1}^{\infty} \frac{1}{x \exp(c_2 (\log x)^\beta)(\log x)(\log \log x)^2} dx
$$

(2.2)
for large \( k \). On the other hand, we have the following fact:

\[
\int_{k}^{\infty} \frac{1}{x \exp(c_2(\log x)\beta)(\log \log x)^2} \, dx \sim \frac{1}{\exp(c_2(\log k)\beta)} \int_{k}^{\infty} \frac{1}{x(\log x)(\log \log x)^2} \, dx
\]

\[
= \frac{1}{\exp(c_2(\log k)\beta)(\log \log k)}
\]

as \( k \) large enough. To this end, we denote

\[ f( x) = \frac{1}{x(\log x)(\log \log x)^2} \quad \text{and} \quad L(x) = \frac{1}{\exp(c_2(\log x)\beta)} \]

for simplicity. It is easily seen that for any small \( \varepsilon > 0 \),

\[
\left| \int_{k}^{k+\varepsilon} f( x) \left( \frac{L(x)}{L(k)} - 1 \right) \, dx \right| \leq \int_{k}^{k+\varepsilon} f( x) \frac{L(x)}{L(k)} \, dx - \int_{k+\varepsilon}^{\infty} f( x) \, dx + \int_{k}^{k+\varepsilon} f( x) \, dx - \int_{k}^{k+\varepsilon} f( x) \, dx
\]

\[
\leq \int_{k}^{k+\varepsilon} f( x) \frac{L(x)}{L(k)} \, dx - 1 \, dx + \int_{k+\varepsilon}^{\infty} f( x) \frac{L(x)}{L(k)} \, dx + \int_{k+\varepsilon}^{\infty} f( x) \, dx
\]

\[
\leq \int_{k}^{k+\varepsilon} f( x) \frac{L(x)}{L(k)} \, dx - 1 \, dx + 2 \int_{k+\varepsilon}^{\infty} f( x) \, dx \to 0 \quad (k \to \infty)
\]

by noting that \( f_{a}^{\infty} f( x) < \infty \) for any fixed \( a > 0 \) and

\[
\sup_{x \in [k, k+\varepsilon]} \left| \frac{L(x)}{L(k)} - 1 \right| = \left| \frac{L(k+\varepsilon)}{L(k)} - 1 \right| \to 0 \quad (k \to \infty).
\]

Therefore, we have

\[
\int_{k}^{\infty} f( x) L(x) \, dx \sim L(k) \int_{k}^{\infty} f( x) \, dx
\]

as \( k \) large enough, which reduces to (2.3). Combining (2.1), (2.2) with (2.3) leads to

\[
\frac{A}{l(\eta_k)(\log k)^\beta(\log \log k)^2} \leq \exp(c_2(\log k)\beta) \sum_{j=k}^{\infty} \frac{1}{j \exp(c_2(\log j)\beta)(\log j)(\log \log j)^2}
\]

\[
\leq \sum_{j=k}^{\infty} \frac{1}{jl(\eta_j)(\log j)(\log \log j)^2}
\]

(2.4)

as \( k \) large enough (see also Wang [22]).

We first will prove Theorem 1.1 in the case that \( X, X_1, X_2, \ldots \) are normal random variables. Let \( N \) be a standard normal variable, we have the following proposition.

**Proposition 2.1.** Let \( a > -1 \), \( b > -1/2 \) and \( \alpha_n(\varepsilon) \) be a nonnegative function of \( \varepsilon \) satisfying (1.4). Then

\[
\lim_{\varepsilon \to \sqrt{1+a}} \left( \varepsilon^2 + a - 1 \right)^{b+1/2} \sum_{n=1}^{\infty} \frac{\log n^a(\log \log n)^b}{n} \cdot \mathbb{P}( |N| \geq \sqrt{2 \log \log n (\varepsilon^2 + \alpha_n(\varepsilon))} )
\]

\[
= \sqrt{\frac{1}{\pi (1+a)}} \exp(-2\tau \sqrt{1+a}) \Gamma(b+1/2).
\]

(2.5)
Proof. First, note that the limit in (2.5) does not depend on any finite terms of the infinite series. Secondly, we have
\[ P(|N| \geq x) \sim \frac{2}{\sqrt{2\pi x}} e^{-x^2/2} \quad \text{as} \quad x \to +\infty. \]
Hence, by the condition (1.4) we have
\[ P(|N| \geq \sqrt{2\log n} (\varepsilon + \alpha_n(\varepsilon))) \sim \frac{2}{\sqrt{2\pi(\varepsilon + \alpha_n(\varepsilon))}\sqrt{2\log n}} \exp\left(-\left(\varepsilon + \alpha_n(\varepsilon)\right)^2 \log \log n\right) \]
\[ \sim \frac{1}{\sqrt{\pi\varepsilon}} \frac{1}{\sqrt{\log n}} \exp\left(-\varepsilon^2 \log \log n\right) \exp\left(-2\varepsilon \alpha_n(\varepsilon) \log \log n\right) \]
as \( n \to \infty \), uniformly in \( \varepsilon \in (\sqrt{1+a}, \sqrt{1+a+\delta}) \) for some \( \delta > 0 \). So, for any \( 0 < \theta < 1 \), there exist \( \delta > 0 \) and \( n_0 \) such that for all \( n \geq n_0 \) and \( \varepsilon \in (\sqrt{1+a}, \sqrt{1+a+\delta}) \),
\[ \frac{1}{\sqrt{\pi(1+a)\sqrt{\log n}}} \exp\left(-\varepsilon^2 \log \log n\right) \exp\left(-2\varepsilon \sqrt{1+a} - \theta\right) \leq P(|N| \geq \sqrt{2\log n} (\varepsilon + \alpha_n(\varepsilon))) \]
\[ \leq \frac{1}{\sqrt{\pi(1+a)\sqrt{\log n}}} \exp\left(-\varepsilon^2 \log \log n\right) \exp\left(-2\varepsilon \sqrt{1+a} + \theta\right), \]
by the condition (1.4) again. Since
\[ \frac{(\log n)^a (\log \log n)^b}{n} \frac{1}{\sqrt{\log n}} \exp\left(-\varepsilon^2 \log \log n\right) \]
is a decreasing function of \( n \), we have
\[ \int_{[e^x]}^{\infty} \frac{(\log x)^a (\log \log x)^{b-1/2}}{x} \exp\left(-\varepsilon^2 \log \log x\right) dx \leq \sum_{n=[e^x]}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \frac{1}{\sqrt{\log n}} \exp\left(-\varepsilon^2 \log \log n\right) \]
\[ \leq \frac{\infty}{[e^x]^{-1}} \int_{[e^x]}^{\infty} \frac{(\log x)^a (\log \log x)^{b-1/2}}{x} \exp\left(-\varepsilon^2 \log \log x\right) dx, \]
where \([x]\) denotes the integer part of \( x \). Hence,
\[ \lim_{\varepsilon \downarrow \sqrt{1+a}} \left(\varepsilon^2 - a - 1\right)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \frac{1}{\sqrt{\log n}} \exp\left(-\varepsilon^2 \log \log n\right) \]
\[ = \lim_{\varepsilon \downarrow \sqrt{1+a}} \left(\varepsilon^2 - a - 1\right)^{b+1/2} \int_{e^x}^{\infty} \frac{(\log x)^a (\log \log x)^{b-1/2}}{x} \exp\left(-\varepsilon^2 \log \log x\right) dx \]
\[ = \lim_{\varepsilon \downarrow \sqrt{1+a}} \left(\varepsilon^2 - a - 1\right)^{b+1/2} \int_{1}^{\infty} y^{b-1/2} \exp\left(-y(\varepsilon^2 - 1 - a)\right) dy \]
\[ = \lim_{\varepsilon \downarrow \sqrt{1+a}} \int_{e^x}^{\infty} y^{b-1/2} e^{-y} dy = \int_{0}^{\infty} y^{b-1/2} e^{-y} dy = \Gamma(b + 1/2). \]
The proposition is proved. \( \square \)

Secondly, we will prove Theorem 1.1 in general case via the nonuniform Berry–Esseen bound for self-normalized sums. The following two lemmas will be used in the following proof.
Lemma 2.1. Let $X_1, X_2, \ldots, X_n$ be independent symmetric random variables. Then for any $x \geq 0$ and $n \geq 1$, we have
\[
P(S_n \geq x V_n) \leq \exp \left( -\frac{x^2}{2} \right).
\]

Proof. See Lemma 4.3 in Wang and Jing [23].

Lemma 2.2. Let $X, X_1, \ldots, X_n$ be i.i.d. symmetric random variables with $EX = 0$ and $EX^3 < \infty$. Then for all $n \geq 1$ and $x \in R$, we have
\[
\left| P(S_n/V_n < x) - \Phi(x) \right| \leq A(1 + |x|^3) \exp \left( -\frac{x^2}{2} \right) \frac{E|X|^3}{n^{1/2} \sigma^3},
\]
where $\sigma^2 = \text{Var}(X)$ and $\Phi(\cdot)$ is the distribution function of the standard normal variable.

Proof. See Corollary 2.1 in Wang and Jing [23].

Proof of Theorem 1.1. By Proposition 2.1, it suffices to prove that
\[
\lim_{\epsilon \searrow 1+a} \sum_{n=1}^{\infty} \frac{(\log n)^a(\log \log n)^b}{n} \left| P(|S_n| \geq 2V_n^2 \log \log n(\epsilon + \alpha_n(\epsilon))) \right|
- \left| \mathbb{P}(|N| \geq 2\log \log n(\epsilon + \alpha_n(\epsilon))) \right| = 0.
\]

Since
\[
\left| P(|S_n| \geq 2V_n^2 \log \log n(\epsilon + \alpha_n(\epsilon))) \right| - \left| \mathbb{P}(|N| \geq 2\log \log n(\epsilon + \alpha_n(\epsilon))) \right|
\leq \left| \mathbb{P}(|S_n| \geq 2V_n^2 \log \log n(\epsilon + \alpha_n(\epsilon))) \right| - \left| \mathbb{P}(|N| \geq 2\log \log n(\epsilon + \alpha_n(\epsilon))) \right|
+ \left| \mathbb{P}(|S_n| \geq 2V_n^2 \log \log n(\epsilon + \alpha_n(\epsilon))) \right| - \left| \mathbb{P}(|N| \geq 2\log \log n(\epsilon + \alpha_n(\epsilon))) \right|
\]
\[
\leq \left| \mathbb{P}(S_n \geq 2V_n^2 \log \log n(\epsilon + \alpha_n(\epsilon))) \right| - \left| \mathbb{P}(N \geq 2V_n^2 \log \log n(\epsilon + \alpha_n(\epsilon))) \right|
+ \left| \mathbb{P}(|S_n| \geq 2V_n^2 \log \log n(\epsilon + \alpha_n(\epsilon))) \right| - \left| \mathbb{P}(|N| \geq 2V_n^2 \log \log n(\epsilon + \alpha_n(\epsilon))) \right|
+ \left| \mathbb{P}(|S_n| \geq 2V_n^2 \log \log n(\epsilon + \alpha_n(\epsilon))) \right| - \left| \mathbb{P}(|N| \geq 2\log \log n(\epsilon + \alpha_n(\epsilon))) \right|
\]
\[
= I_1 + I_2 + I_3.
\]

Thus, to prove (2.6), it suffices to show that
\[
\lim_{\epsilon \searrow 1+a} \sum_{n=1}^{\infty} \frac{(\log n)^a(\log \log n)^b}{n} I_i = 0, \quad i = 1, 2, 3.
\] (2.7)

Noting that for any $s, t \in R, c \geq 0$ and $x \geq 1$,
\[
x \sqrt{c + t^2} = \sqrt{(x^2 - 1)c + t^2 + c + (x^2 - 1)t^2}
\geq \sqrt{(x^2 - 1)c + t^2 + 2t(\sqrt{x^2 - 1}c)}
= t + \sqrt{(x^2 - 1)c},
\]
we have
\[
\{s + t \geq x \sqrt{c + t^2} \} \subset \{s \geq (x^2 - 1)^{1/2} \sqrt{c} \}.
\]
Hence,
Notice that $\eta_n \sim n l(\eta_n)/(\log \log n)^2$, by Lemma 2.1, (2.4) and that $l(\eta_n)$ is a slowly varying function at $\infty$, for some $0 \leq \beta < 1$, we have

$$\lim_{\epsilon \to \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} (\log n)^a (\log \log n)^b \frac{1}{n} \cdot \sum_{i=1}^{n} \mathbb{P}(S_n(i) \geq (2(\epsilon + \alpha_n(\epsilon))^2 \log \log n - 1)^{1/2} V_n^{(i)} \mid |X_i| > \eta_n)$$

$$\leq \epsilon^{1/2} \lim_{\epsilon \to \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} (\log n)^a (\log \log n)^b \mathbb{P}(|X| > \eta_n)$$

$$\leq A \lim_{\epsilon \to \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{k=1}^{\infty} \eta_k^{-2} \text{EX}^2 I \{ \eta_k < |X| \leq \eta_{k+1} \} \sum_{n=1}^{k} (\log n)^{-1} (\log \log n)^b$$

$$\leq A \lim_{\epsilon \to \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{k=1}^{\infty} \frac{(\log \log k)^2}{k l(\eta_k)} \text{EX}^2 I \{ \eta_k < |X| \leq \eta_{k+1} \} \frac{k (\log \log k)^b}{\log k}$$

$$\leq A \lim_{\epsilon \to \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{k=1}^{\infty} \frac{1}{l(\eta_k) (\log \log k)^2} \text{EX}^2 I \{ \eta_k < |X| \leq \eta_{k+1} \}$$

$$\leq A \lim_{\epsilon \to \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j l(\eta_j) (\log j) (\log \log j)^2} \text{EX}^2 I \{ \eta_k < |X| \leq \eta_{k+1} \}$$

$$\leq A \lim_{\epsilon \to \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{j=1}^{\infty} \frac{\text{EX}^2 I \{ |X| \leq \eta_{j+1} \}}{j l(\eta_j) (\log j) (\log \log j)^2} = 0. \quad (2.9)$$

Similarly,

$$\lim_{\epsilon \to \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} (\log n)^a (\log \log n)^b \frac{1}{n} \cdot \sum_{i=1}^{n} \mathbb{P}(S_n(i) \geq (2(\epsilon + \alpha_n(\epsilon))^2 \log \log n - 1)^{1/2} V_n^{(i)} \mid |X_i| > \eta_n) = 0. \quad (2.10)$$
It follows from (2.8)–(2.10) that
\[
\lim_{\varepsilon \to 0} \left( e^{2} - a - 1 \right)^{b+1/2} \frac{\sum_{n=1}^{\infty} (\log n)^a (\log \log n)^b}{n} I_1 = 0.
\] (2.11)

The similar argument results in
\[
\lim_{\varepsilon \to 0} \left( e^{2} - a - 1 \right)^{b+1/2} \frac{\sum_{n=1}^{\infty} (\log n)^a (\log \log n)^b}{n} I_2 = 0.
\] (2.12)

Notice that \( \eta_n^2 \sim n l(\eta_n)/(\log \log n)^2 \) again, by Lemma 2.2, (2.4) and the argument similar to (2.6), for some \( 0 \leq \beta < 1 \),
\[
\lim_{\varepsilon \to 0} \left( e^{2} - a - 1 \right)^{b+1/2} \sum_{n=1}^{\infty} (\log n)^a (\log \log n)^b I_3 \\
\leq A \lim_{\varepsilon \to 0} \left( e^{2} - a - 1 \right)^{b+1/2} \sum_{n=1}^{\infty} (\log n)^a (e + \alpha_n(\varepsilon))^2 (\log \log n)^{b+3/2} \mathbb{E}[|X|^3 | |X| \leq \eta_n] \frac{n^{3/2} (l(\eta_n))^{3/2}}{n^{3/2} (l(\eta_n))^{3/2}} \\
\leq A \lim_{\varepsilon \to 0} \left( e^{2} - a - 1 \right)^{b+1/2} \sum_{n=1}^{\infty} \eta_k \mathbb{E}[X^2 I \{ \eta_{k-1} < |X| \leq \eta_k \}] \sum_{n=k}^{\infty} (\log n)^{b+3/2} \frac{n^{3/2} (\log \log n)^{3/2}}{n^{3/2} (\log \log n)^{3/2}} \\
\leq A \lim_{\varepsilon \to 0} \left( e^{2} - a - 1 \right)^{b+1/2} \sum_{n=1}^{\infty} \frac{l(\eta_k)}{(\log k)^{\beta}} (\log \log k)^{2} \mathbb{E}[X^2 I \{ \eta_{k-1} < |X| \leq \eta_k \}] \\
\leq A \lim_{\varepsilon \to 0} \left( e^{2} - a - 1 \right)^{b+1/2} \sum_{j=1}^{\infty} \frac{\mathbb{E}[X^2 I \{ |X| \leq \eta_j \}]}{j l(\eta_j) (\log j) (\log \log j)^2} = 0.
\] (2.13)

Thus, (2.7) follows from (2.11)–(2.13). The proof is now completed. \(\Box\)

3. The proof of Theorem 1.2

We first will prove Theorem 1.2 in the case that \( X, X_1, X_2, \ldots \) are normal random variables. Let \( N \) be a standard normal variable and \( \{ W(t); \ t \geq 0 \} \) be a standard Wiener process, we have the following proposition.

**Proposition 3.1.** Let \( d > -1 \) and \( \alpha_n = O(1/\log \log n) \). Then
\[
\lim_{\varepsilon \to 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log n)^d}{n \log n} \mathbb{P} \left( \sup_{0 \leq t \leq 1} |W(x)| \geq \sqrt{2 \log \log n(\varepsilon + \alpha_n)} \right) \\
= \frac{2}{(d + 1) \sqrt{\pi}} \Gamma(d + 3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^{2d+2}}
\] (3.1)

and
\[
\lim_{\varepsilon \to 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log n)^d}{n \log n} \mathbb{P} (|N| \geq \sqrt{2 \log \log n(\varepsilon + \alpha_n)}) = \frac{1}{(d + 1) \sqrt{\pi}} \Gamma(d + 3/2).
\] (3.2)

The following lemma will be used in the proof of Proposition 3.1.

**Lemma 3.1.** Let \( \{ W(t); \ t \geq 0 \} \) be a standard Wiener process. Then for all \( x > 0 \),
\[
\mathbb{P} \left( \sup_{0 \leq t \leq 1} |W(x)| \geq x \right) = 1 - \sum_{k=-\infty}^{\infty} (-1)^k \mathbb{P} ((2k - 1)x \leq N \leq (2k + 1)x)
\]
\[= 4 \sum_{k=0}^{\infty} (-1)^k P(N \geq (2k + 1)x)\]

\[= 2 \sum_{k=0}^{\infty} (-1)^k P(|N| \geq (2k + 1)x).\]

In particular,

\[P\left( \sup_{0 \leq s \leq 1} |W(s)| \geq x \right) \sim 2P(|N| \geq x) \sim \frac{4}{\sqrt{2\pi x}} e^{-x^2/2} \text{ as } x \to +\infty.\]

Proof. It is well known. See Billingsley [3, pp. 79–80]. \(\square\)

Now, we turn to prove Proposition 3.1.

Proof of Proposition 3.1. Noting that

\[P(|N| \geq x) = 2P(N \geq x), \quad \forall x > 0,\]

and for any \(m \geq 1\) and \(x > 0,\)

\[4 \sum_{k=0}^{2m+1} (-1)^k P(N \geq (2k + 1)x) \leq P\left( \sup_{0 \leq s \leq 1} |W(s)| \geq x \right) \leq 4 \sum_{k=0}^{2m} (-1)^k P(N \geq (2k + 1)x).\]

It is sufficient to show that for any \(q > 0,\)

\[\lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{\log \log n}{n \log n} P(N \geq q(\varepsilon + \alpha_n)\sqrt{2\log \log n}) = q^{-2(d+1)} \frac{1}{2(d+1)} \Gamma(d+3/2).\]

Obviously,

\[\lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{\log \log n}{n \log n} P(N \geq q(\varepsilon + \alpha_n)\sqrt{2\log \log n}) = q^{-2(d+1)} \lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{\log \log n}{n \log n} P(N \geq (\varepsilon + q\alpha_n)\sqrt{2\log \log n}).\]

So, it is sufficient to show that

\[\lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{\log \log n}{n \log n} P(N \geq (\varepsilon + \alpha_n)\sqrt{2\log \log n}) = \frac{1}{2(d+1)} \Gamma(d+3/2).\]

Without loss of generality, we assume that \(|\alpha_n| \leq A/\log \log n.\) Notice that

\[\left| P(N \geq (\varepsilon + \alpha_n)\sqrt{2\log \log n}) - P(N \geq \varepsilon\sqrt{2\log \log n}) \right| \leq \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{2\log \log n (\varepsilon - A/\log \log n)^2}{2} \right) |\alpha_n|\sqrt{2\log \log n} \leq \frac{A}{\sqrt{\log \log n}} \exp(-\varepsilon^2 \log \log n + 2\varepsilon A)\]

and
\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} \frac{1}{\sqrt{\log \log n}} \exp(-\varepsilon^2 \log \log n)
\]

\[
= \lim_{\varepsilon \downarrow 0} \varepsilon^{2(d+1)} \int_{e^\varepsilon}^\infty \frac{(\log \log x)^{d-1/2}}{x \log x} \exp(-\varepsilon^2 \log x) \, dx
\]

\[
= \lim_{\varepsilon \downarrow 0} \varepsilon^{2(d+1)} \int_1^\infty x^{d-1/2} \exp(-\varepsilon^2 x) \, dy = \lim_{\varepsilon \downarrow 0} \varepsilon \int_{\varepsilon^2}^\infty x^{d-1/2} e^{-y} \, dy
\]

\[
= \lim_{\varepsilon \downarrow 0} \int_{\varepsilon^2}^{1} x^{d-1/2} e^{-y} \, dy + \lim_{\varepsilon \downarrow 0} \varepsilon \int_{1}^{\infty} x^{d-1/2} e^{-y} \, dy
\]

\[
\leq \lim_{\varepsilon \downarrow 0} \int_{\varepsilon^2}^{1} x^{d-1/2} \, dy = 0.
\]

Thus, it follows that

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} \cdot P(N \geq (\varepsilon + \alpha_n) \sqrt{2 \log \log n})
\]

\[
= \lim_{\varepsilon \downarrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} \cdot P(N \geq \varepsilon \sqrt{2 \log \log n})
\]

\[
= \lim_{\varepsilon \downarrow 0} \varepsilon^{2(d+1)} \int_{e^\varepsilon}^\infty \frac{(\log \log x)^d}{x \log x} \cdot P(N \geq \varepsilon \sqrt{2 \log \log x}) \, dx
\]

\[
= \lim_{\varepsilon \downarrow 0} \int_{\varepsilon^2}^{\infty} x^{d} \cdot P(N \geq \sqrt{2y}) \, dy = \frac{1}{d+1} \int_{0}^{\infty} P(N \geq \sqrt{2y}) \, dy^d
\]

\[
= \frac{1}{d+1} \cdot P(N \geq \sqrt{2y}) \cdot y^{d+1} \bigg|_{0}^{\infty} + \frac{1}{2(d+1) \sqrt{\pi}} \int_{0}^{\infty} y^{d+1/2} e^{-y} \, dy
\]

\[
= \frac{1}{2(d+1) \sqrt{\pi}} \Gamma(d+3/2).
\]

The proposition is now proved. \(\square\)

Next, we will use the strong approximation method to show the probability in (1.6) for \(M_n/V_n\) can be approximated by that for \(\sup_{0 \leq s \leq 1} |W(s)|\) and the probability in (1.7) for \(S_n/V_n\) can be approximated by that for \(N\). To this end, for each \(n\) and \(1 \leq i \leq n\), we define \(\eta_n\) and \(\overline{X}_{ni}\) as in Section 2, and let

\[
X^*_ni = \overline{X}_{ni} - \mathbb{E}\overline{X}_{ni}, \quad S^*_ni = \sum_{j=1}^{i} X^*_nj,
\]

\[
M^*_n = \max_{k \leq n} |S^*_nk|, \quad B^*_n = \sum_{j=1}^{n} \text{Var}(\overline{X}_{nj}).
\]

It follows easily that

\[
B^*_n \sim \sum_{j=1}^{n} \mathbb{E}\overline{X}^2_{nj} \sim n\mathbb{E} \sim \eta_n^2 (\log \log n)^2.
\]
Proposition 3.2. Let $d > -1$ and $2 \geq p > 1/2$. Then there exists a sequence of positive number $\{q_n\}$ such that
\[
P\left( \sup_{0 \leq s \leq 1} |W(s)| \geq x + 3/(\log \log n)^p \right) - q_n \leq \mathbb{P}(V_n \geq xB_n) \leq \mathbb{P}\left( \sup_{0 \leq s \leq 1} |W(s)| \geq x - 3/(\log \log n)^p \right) + q_n, \quad \forall x \geq 0, \quad (3.3)
\]
and
\[
P(|N| \geq x + 3/(\log \log n)^p) - q_n \leq \mathbb{P}(|S_n| \geq xB_n) \leq \mathbb{P}(|N| \geq x - 3/(\log \log n)^p) + q_n, \quad \forall x \geq 0, \quad (3.4)
\]
where $q_n$ satisfies
\[
\sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} q_n = O(1). \quad (3.5)
\]

We give two lemmas which will be used in the proof of Proposition 3.2.

Lemma 3.2. For any sequence of independent random variables $\{\xi_n; \ n \geq 1\}$ with mean zero and finite variance, there exists a sequence of independent normal variables $\{Y_n; \ n \geq 1\}$ with $EY_n = 0$ and $EY^2_n = E\xi^2_n$ such that, for all $Q > 2$, $\xi$ and $y > 0$,
\[
P\left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \xi_i - \sum_{i=1}^{k} Y_i \right| \geq y \right) \leq (CQ)^Q y^{-Q} \sum_{i=1}^{n} E|\xi_i|^Q,
\]
whenever $E|\xi_i|^Q < \infty, i = 1, \ldots, n$. Here, $C$ is an universal constant.

Proof. See Sakhanenko [17, p. 783]. \hfill \Box

Lemma 3.3. Let $Q > 2$, $\xi_1, \xi_2, \ldots, \xi_n$ be independent random variables with $E\xi_k = 0$ and $E|\xi_k|^Q < \infty$, $k = 1, \ldots, n$. Then for all $y > 0$,
\[
P\left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \xi_i \right| \geq y \right) \leq 2 \exp\left( -\frac{y^2}{8 \sum_{k=1}^{n} \text{Var}(\xi_k)} \right) + (2CQ)^Q y^{-Q} \sum_{i=1}^{n} E|\xi_i|^Q,
\]
where $C$ is an universal constant as in Lemma 3.2.

Proof. It follows from Lemma 3.2 easily. \hfill \Box

Proof of Proposition 3.2. We show (3.3) only, since (3.4) can be proved in the same way. By Lemma 3.2, there exists an universal constant $C > 0$ and a sequence of standard Wiener processes $\{W_n(\cdot)\}$ such that for all $Q > 2$,
\[
P\left( \max_{1 \leq n \leq \eta_n} \left| S^*_n - W_n\left( \frac{k}{n}B_n^2 \right) \right| \geq \frac{1}{2} B_n/(\log \log n)^p \right) \leq (CQ)^Q \left( \frac{2(\log \log n)^p}{B_n} \right)^Q \sum_{k=1}^{n} E|X^*_nk|^Q
\]
\[
\leq An \left( \frac{\log \log n)^p}{\sqrt{n!(n\eta_n)}} \right)^Q E|X|^Q I\{|X| \leq \eta_n\}.
\]

On the other hand, by Lemma 1.1.1 of Csörgő and Révész [6],
\[
P\left( \sup_{0 \leq s \leq 1} \left| W_n(sB_n^2) - W_n\left( \frac{\lfloor ns \rfloor}{n} B_n^2 \right) \right| \geq \frac{1}{2} B_n/(\log \log n)^p \right)
\]
\[ p_n := \mathbb{P} \left( \sup_{0 \leq s \leq 1} \left| \frac{S_{[ns]} - W_n(s B_n^2)}{B_n} \right| \geq \frac{1}{(\log \log n)^{\beta}} \right). \] (3.7)

Then \( q_n \) satisfies (3.3) and (3.4), since \( \{ W_n(t B_n^2)/B_n; \ t \geq 0 \} \overset{D}{=} \{ W_n(t); \ t \geq 0 \} \) for each \( n \). And also,

\[ q_n \leq \mathbb{P}(\Delta_n \geq B_n/(\log \log n)^{\beta}) + p_n \] (3.8)

and

\[ p_n \leq A \left( \frac{(\log \log n)^{\beta}}{\sqrt{n l(\eta_n)}} \right)^{Q} \mathbb{E}[|X| Q I\{|X| \leq \eta_n\}] + A \exp \left( -\frac{1}{12} n/(\log \log n)^{2\beta} \right). \]

By \( \eta_n^2 \sim n l(\eta_n)/(\log \log n)^2 \) and (2.4), we have

\[
\sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} p_n \leq A + A \sum_{n=1}^{\infty} \frac{(\log \log n)^{d+Q}}{n^{Q/2}(\log n)(l(\eta_n))^{Q/2}} \mathbb{E}[|X|^Q I\{|X| \leq \eta_n\}]
\leq A + A \sum_{n=1}^{\infty} \eta_n^{Q-2} \mathbb{E} X^2 I\{|X| \leq \eta_n\} \sum_{n=1}^{\infty} \frac{(\log \log n)^{d+Q}}{n^{Q/2}(\log n)(l(\eta_n))^{Q/2}}
\leq A + A \sum_{n=1}^{\infty} \frac{1}{l(\eta_n)(\log j)(\log \log j)^2} \mathbb{E} X^2 I\{|X| \leq \eta_j\}
\leq A + A \sum_{n=1}^{\infty} \frac{\eta_j^{-1} \mathbb{E} X^2 I\{|X| \leq \eta_j\}}{j l(\eta_j)(\log j)(\log \log j)^2} = O(1). \] (3.9)

Moreover, let \( \beta_n := n \mathbb{E}[|X| I\{|X| > \eta_n\}] \) and \( \mathcal{L} := \{ n: \beta_n \leq \frac{1}{8} B_n/(\log \log n)^2 \} \). We have

\[ \{ \Delta_n \geq B_n/(\log \log n)^{\beta} \} \subset \bigcup_{j=1}^{n} \{ X_j \neq \bar{X}_j \}, \quad n \in \mathcal{L}. \]

Hence, similar to the proof of (2.9), by (2.4), for \( n \in \mathcal{L} \) we have

\[
\sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} \mathbb{P}(\Delta_n \geq B_n/(\log \log n)^{\beta}) \leq \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} \mathbb{P}(|X| > \eta_n) = O(1). \] (3.10)

If \( n \in \bar{\mathcal{L}} \), by \( B_n^2 \sim n l(\eta_n)/(\log \log n)^2 \) and (2.4) again, we have

\[
\sum_{n \in \bar{\mathcal{L}}} \frac{(\log \log n)^d}{n \log n} \mathbb{P}(\Delta_n \geq B_n/(\log \log n)^{\beta}) \leq A + 8 \sum_{n=1}^{\infty} \frac{(\log \log n)^{d+2}}{n^{3/2}(\log n)(l(\eta_n))^{1/2}} \beta_n
\leq A + A \sum_{k=1}^{\infty} \eta_k^{-1} \mathbb{E} X^2 I\{|X| \leq \eta_{k+1}\} \sum_{n=1}^{k} \frac{(\log \log n)^{d+2}}{n^{1/2}(\log n)(l(\eta_n))^{1/2}}
\]
and
\[ A + A \sum_{j=1}^{\infty} \frac{1}{j (\log j)^2} = O(1). \]

So, \( q_{\delta} \) satisfies (3.5) by (3.8)–(3.11). The proof is completed. \( \square \)

**Proof of Theorem 1.2.** We show (1.6) only, since the proof of (1.7) is similar. Let \( \delta > 0 \) be small enough. Using Bernstein inequality and (3.10), there exists a constant \( \theta := \theta(\delta) > 0 \) such that

\[
\sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} P(V_n^2 > (1 + \delta/2)nl(\eta_n))
\leq \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} P\left(\sum_{j=1}^{n} X_{nj}^2 > (1 + \delta/2)nl(\eta_n)\right) + \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} P(|X| > \eta_n)
\leq A + A \sum_{n=1}^{\infty} n^{-1}(\log n)^{-1-\theta} = O(1)
\]

and

\[
\sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} P(V_n^2 < (1 - \delta)nl(\eta_n)) \leq \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} P\left(\sum_{j=1}^{n} X_{nj}^2 < (1 - \delta)nl(\eta_n)\right)
\leq A \sum_{n=1}^{\infty} n^{-1}(\log n)^{-1-\theta} = O(1).
\]

At the same time, by \( B_n^2 \sim nl(\eta_n) \) and Proposition 3.2, we have for large \( n \) and any \( \varepsilon > 0 \),

\[
P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq (\varepsilon \sqrt{1 + \delta} + 2|\alpha_n| + 3/(\sqrt{2} \log \log n)) \sqrt{2 \log \log n} \right) - q_n - P(V_n^2 > (1 + \delta/2)nl(\eta_n))
\leq P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq (\varepsilon + \alpha_n)\sqrt{2(1 + \delta)\log \log n + 3/(\log \log n)^\theta} \right) - q_n - P(V_n^2 > (1 + \delta)B_n^2)
\leq P(M_n \geq (\varepsilon + \alpha_n)\sqrt{2(1 + \delta)B_n^2 \log \log n}) - P(V_n^2 > (1 + \delta)B_n^2)
\leq P(M_n \geq (\varepsilon + \alpha_n)\sqrt{2V_n^2 \log \log n})
\leq P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq (\varepsilon + \alpha_n)\sqrt{2(1 - \delta)B_n^2 \log \log n} \right) + q_n + P(V_n^2 < (1 - \delta)B_n^2)
\leq P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq (\varepsilon \sqrt{1 - \delta} - |\alpha_n| - 3/(\sqrt{2} \log \log n)) \sqrt{2 \log \log n} \right) + q_n + P(V_n^2 < (1 - \delta)nl(\eta_n)).
\]

Thus, it follows from Proposition 3.1, (3.5), (3.12) and (3.13) that

\[
\frac{2(1 + \delta)^{-d-1}}{(d + 1)\sqrt{\pi}} \Gamma(d + 3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^{2d+2}}
\leq \liminf_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} P(M_n \geq (\varepsilon + \alpha_n)\sqrt{2V_n^2 \log \log n}).
\]
\[ \limsup_{\varepsilon \downarrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \left( \frac{\log \log n}{n \log n} \right)^d \mathbb{P}(M_n \geq (\varepsilon + \alpha_n) \sqrt{2V_n^2 \log \log n}) \leq \frac{2(1-\delta)^{-d-1}}{(d+1)\sqrt{\pi}} \Gamma(d + 3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2d+2}}. \]

Letting \( \delta \to 0 \), the proof is completed. \( \square \)

Acknowledgment

The authors would like to express their gratitude to an anonymous referee for his/her constructive comments which led to an improved presentation of the paper.

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