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Derived equivalence classification for cluster-tilted algebras of type A_n ☆

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Abstract

In this paper we give the derived equivalence classification of cluster-tilted algebras of type A_n. We show that the bounded derived category of such an algebra depends only on the number of 3-cycles in the quiver of the algebra.

Keywords: Cluster-tilted algebra; Tilting module; Derived category

Introduction

Cluster categories were introduced in [BMRRT] as a framework for a categorification of Fomin–Zelevinsky cluster algebras [FZ1]. In [CCS1], a category was introduced independently for type A, which was shown to be equivalent to the cluster category. For any finite-dimensional hereditary algebra H over a field k, the cluster category $\mathcal{C}_H$ is the quotient of the bounded derived category $\mathcal{D}_H = D^b$ (mod H) by the functor $F = \tau^{-1}[1]$, where τ denotes the AR-translation. $\mathcal{C}_H$ is canonically triangulated [K], and it has AR-triangles induced by the AR-triangles in $\mathcal{D}_H$.

In a cluster category $\mathcal{C}_H$, tilting objects are defined as objects which have no self-extensions, and are maximal with respect to this property. The endomorphism rings of such objects are called *cluster-tilted algebras* [BMR1]. These algebras are of finite representation type if and only if H is the path algebra of a simply-laced Dynkin quiver.

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Cluster-tilted algebras have several interesting properties. In particular, by [BMR1] their representation theory can be completely understood in terms of the representation theory of the corresponding hereditary algebra H. Furthermore, their relationship to tilted algebras is well understood by [ABS1,ABS2], see also [Rin].

Homologically, they are very different from hereditary and tilted algebras, since they have in general infinite global dimension. In fact they are Gorenstein of dimension 1 and in particular they have finitistic dimension 1, by [KR]. Cluster-tilted algebras also play a role in the construction of cluster algebras from cluster categories [CK1,CK2], see also [BMRT].

The purpose of this paper is to describe when two cluster-tilted algebras from the cluster category $\mathcal{C}_H$ have equivalent derived categories, where H is the path algebra of a quiver whose underlying graph is A_n. We will get an exact description of the quivers of such algebras, and their relations are given by [CCS1]. The main result is the following.

Theorem. *Two cluster-tilted algebras of type A_n are derived equivalent if and only if their quivers have the same number of* 3*-cycles.*

For this, we show that if we have an almost complete cluster-tilting object $\overline{T}$ in $\mathcal{C}_H$ with complements T_i and T_i^* such that the cluster-tilted algebras given by $\Gamma = \operatorname{End}_{\mathcal{C}_H}(\overline{T} \amalg T_i)^{\mathrm{op}}$ and $\Gamma' = \operatorname{End}_{\mathcal{C}_H}(\overline{T} \amalg T_i^*)^{\mathrm{op}}$ have quivers with the same number of 3-cycles, then Γ' is in a natural way isomorphic to the endomorphism ring of a tilting module over Γ. Then it is well known that Γ and Γ' are derived equivalent, see [Ha,CPS].

The outline of the paper is as follows: After some basic notions, we describe the *mutation class* of A_n, that is, the quivers of cluster-tilted algebras of A_n-type. In Section 4 we give a simple proof of a special case of a result by Holm [Ho], which is a formula for the determinant of the Cartan matrices of the cluster-tilted algebras of A_n-type. We use this to distinguish between algebras of this type which are not derived equivalent. In Section 5 we prove the main result.

For notions and basic results about finite-dimensional algebras, we refer the reader to [ASS] or [ARS].

1. Preliminaries

We will now review some basic notions concerning cluster-tilted algebras. This theory is developed in [BMRRT,BMR1], and in the Dynkin case there is an independent approach in [CCS1,CCS2].

Throughout, H will denote the path algebra $k\vec{A}_n$ of a quiver $\vec{A}_n$ with underlying graph A_n. By mod H we will mean the category of finitely generated left H-modules. Then the AR-quiver of the derived category $\mathcal{D} = D^b$ (mod H) is isomorphic to the stable translation quiver $\mathbb{Z}A_n$ (see e.g. [Ha]). $\mathcal{D}$ does not depend on the orientation of $\vec{A}_n$.

If τ is the AR-translation in $\mathcal{D}$, we consider the functor $F = \tau^{-1}[1]$ and the orbit category $\mathcal{C} = \mathcal{D}/F$. Then $\mathcal{C}$ is called the *cluster category* of type A_n. This is a Krull–Schmidt category, and it follows from [K] that it has a triangulated structure inherited from $\mathcal{D}$.

A (*cluster*) *tilting object* in $\mathcal{C}$ is an object T with n non-isomorphic indecomposable direct summands such that $\operatorname{Ext}^1_{\mathcal{C}}(T, T) = 0$. An object in $\mathcal{C}$ with $n-1$ non-isomorphic direct summands satisfying the same Ext-condition will be called an *almost complete* (*cluster*) *tilting object*. An indecomposable object M such that $\overline{T} \amalg M$ is a tilting object is said to be a *complement* of $\overline{T}$.

We will use the following result, which is one of the main results in [BMRRT], and which uses the notion of *approximations* from [AS]:

Theorem 1.1. *An almost complete tilting object $\overline{T}$ in $\mathcal{C}$ has exactly two complements M and M^*. These are related by unique triangles*

$$M \to B \to M^* \to$$

and

$$M^* \to B' \to M \to$$

where the maps $M \to B$ and $M^* \to B'$ are minimal left $\operatorname{add}\overline{T}$*-approximations and the maps $B \to M^*$ and $B' \to M$ are minimal right* $\operatorname{add}\overline{T}$*-approximations.*

For a tilting object T in $\mathcal{C}$, we call the endomorphism ring $\Gamma_T = \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ a *cluster-tilted algebra*. There is a close connection between the module category of Γ_T and $\mathcal{C}$, from [BMR1]:

Theorem 1.2. *With Γ_T as above, the functor $G = \operatorname{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \to \operatorname{mod} \Gamma_T$ is full and dense and induces an equivalence*

$$\overline{G} : \mathcal{C}/\operatorname{add}(\tau T) \to \operatorname{mod} \Gamma_T.$$

By [BMR2], the cluster-tilted algebras of type A_n are exactly the algebras given by quivers obtained from A_n-quivers by *mutation*, an operation which will be described in Section 2, with certain relations determined by the quiver [BMR3].

2. The mutation class of A_n

In this section we will provide an explicit description of the mutation class of A_n-quivers. The ideas underlying our presentation can be found already in [CCS1], where a geometric interpretation of mutation of A_n-quivers is given. The mutation class is implicit in [CCS1], see also [S] for an explicit, but slightly differently formulated description. The technical Lemma 2.3 will be crucial in the proof of our main theorem in Section 5.

Quiver mutation was introduced by Fomin and Zelevinsky [FZ1] as a generalisation of the sink/source reflections used in connection with BGP functors [BGP]. Any quiver Q with no loops and no cycles of length two, can be mutated at vertex i to a new quiver Q^* by the following rules:

- The vertex i is removed and replaced by a vertex i^*, all other vertices are kept.
- For any arrow $i \to j$ in Q there is an arrow $j \to i^*$ in Q^*.
- For any arrow $j \to i$ in Q there is an arrow $i^* \to j$ in Q^*.
- If there are $r > 0$ arrows $j_1 \to i$, $s > 0$ arrows $i \to j_2$ and t arrows $j_2 \to j_1$ in Q, there are $t - rs$ arrows $j_2 \to j_1$ in Q^*. (Here, a negative number of arrows means arrows in the opposite direction.)
- All other arrows are kept.

Note that if we mutate Q at vertex i, and then mutate Q^* at i^*, the resulting quiver is isomorphic to (and will be identified with) Q. We want to describe the class of quivers which can be obtained by iterated mutation on a quiver of type A_n. Such quivers are said to be *mutation equivalent* to A_n, as iterated mutation produces an equivalence relation.

The following lemma is a well-known fact:

Lemma 2.1. *All orientations of A_n are mutation equivalent.*

From now on, let $\mathcal{Q}_n$ be the class of quivers with n vertices which satisfy the following:

- all non-trivial cycles are oriented and of length 3,
- a vertex has at most four neighbours,
- if a vertex has four neighbours, then two of its adjacent arrows belong to one 3-cycle, and the other two belong to another 3-cycle,
- if a vertex has exactly three neighbours, then two of its adjacent arrows belong to a 3-cycle, and the third arrow does not belong to any 3-cycle.

Note that by a *cycle* in the first condition we mean a cycle in the underlying graph, not passing through the same edge twice. In particular, this condition excludes multiple arrows. We will show that $\mathcal{Q}_n$ is the mutation class of A_n.

Lemma 2.2. *$\mathcal{Q}_n$ is closed under quiver mutation.*

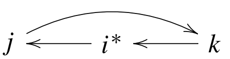
Proof. Let $Q \in \mathcal{Q}_n$. We will see what happens locally when we mutate.

If we mutate at a vertex i which is a source or a sink, then the arrows to or from i changes direction, and everything else is left unchanged. Thus the new quiver Q^* will also satisfy the conditions in the description of $\mathcal{Q}_n$.

Next we consider the case where i is the source of exactly one arrow and the target of exactly one arrow:

$$j \longrightarrow i \longrightarrow k$$

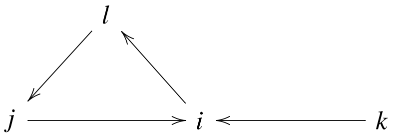
Two cases can occur. Suppose first that there is no arrow from k to j in Q. Then there is an arrow from j to k in Q^*:



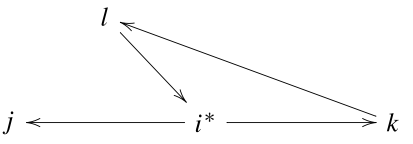
Thus the numbers of neighbours for j and k increase by 1. It is impossible that j or k has four neighbours in Q, since then the arrow to or from i would be part of a 3-cycle in Q, and i would have a third neighbour. Thus j and k have $\leqslant 4$ neighbours in Q^* as well. There are no other (non-oriented) paths between j and k in Q^* than the two pictured in the diagram above, so the other conditions are also satisfied: If j or k has four neighbours in Q^*, then the last two arrows will be part of a 3-cycle in both Q and Q^*.

In the other case, there is an arrow $k \to j$ in Q. Then this is removed in passing to Q^*. The numbers of neighbours of j and k decrease by 1, and cannot be larger than 3. If, say, j has three neighbours in Q^*, then it must have had four neighbours in Q, and the two arrows not involving i or k are part of a 3-cycle in both Q and Q^*. The arrow $i^* \to j$ is not part of a 3-cycle, since the only arrow with i^* as target comes from k, and there is no arrow $j \to k$.

We use similar arguments for the other cases, and just point out how the mutations work. Now let i be a vertex of Q with three neighbours. Suppose first that the one arrow to or from i which is not on a 3-cycle has i as the target:

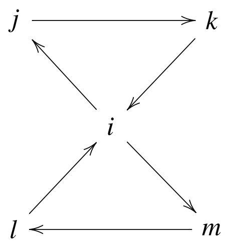


Then the mutation will remove the lj-arrow and produce a new triangle i^*kl:

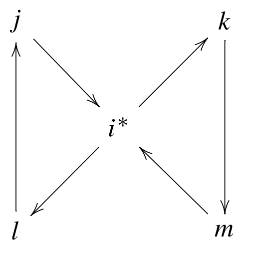


Similarly for the case where the third arrow has i as the source.

Finally, let i be a vertex with four neighbours:



Mutate:

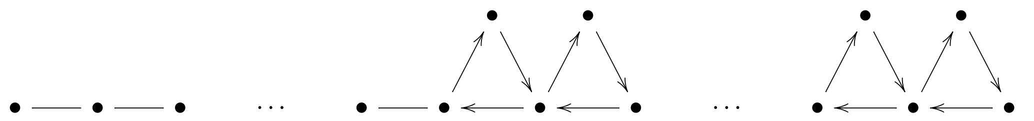


So for the cases where i has three or four neighbours, we see that neither in Q nor in Q^* are there other paths between j, k, l and m than those passing through the diagrams. By similar arguments as above, Q^* also satisfies the conditions in the description of $\mathcal{Q}_n$. □

We will need the following lemma for the proof of the main result in Section 5.

Lemma 2.3. *If Q_1 and Q_2 are quivers in $\mathcal{Q}_n$, and Q_1 and Q_2 have the same number of* 3*-cycles, then Q_2 can be obtained from Q_1 by iterated mutation where all the intermediate quivers also have the same number of* 3*-cycles.*

Proof. It is enough to show that all quivers in $\mathcal{Q}_n$ can be mutated without changing the number of 3-cycles to a quiver looking like this:



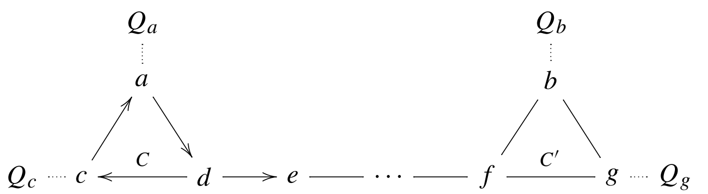
In this process we are only allowed to mutate in sinks, sources and vertices of valency three and four, as these are the mutations which will not change the number of 3-cycles for quivers in $\mathcal{Q}_n$.

For the purposes of this proof, we introduce a distance function on the set of 3-cycles in quivers in $\mathcal{Q}_n$. For each pair C, C' of *different* 3-cycles in Q, we define $d_Q(C, C')$ to be the length of the unique minimal (perhaps non-oriented) path between C and C', i.e. the number of arrows in this path.

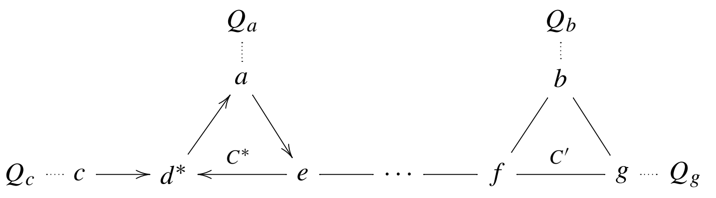
Let Q be a quiver in $\mathcal{Q}_n$, and suppose that the underlying graph of Q is not A_n. We now define a total order on a subset $\mathcal{S}_Q$ of the set of 3-cycles of Q. This subset is not uniquely defined. Q must contain a 3-cycle which is only connected to other 3-cycles through (at most) one of its vertices. Choose one such 3-cycle and call it C_1. If there are more 3-cycles, let C_2 be the unique 3-cycle which minimises $d_Q(C_1, -)$. If there are more 3-cycles, let C_3 be one of the at most two which minimise $d_Q(C_2, -)$ among the 3-cycles not equal to C_1.

If C_i is defined for some $i \geqslant 3$, and there exists one or more 3-cycles C such that $d_Q(C_i, C) < d_Q(C_j, C)$ for $j < i$, let C_{i+1} be one of the at most two which minimise $d_Q(C_i, -)$ among 3-cycles with this property. Continue in this way until C_s is defined, but C_{s+1} cannot be defined. Let $\mathcal{S}_Q = \{C_1, \dots, C_s\}$ be our totally ordered set of 3-cycles.

Next, we will see that we have a procedure for moving 3-cycles in the quiver closer together. Let C and C' be a pair of neighbouring 3-cycles in Q (i.e. no edge in the path between them is part of a 3-cycle) such that $d_Q(C, C') \geqslant 1$. We want to move C and C' closer together by mutation. Up to orientation on the arrow from d to e, it looks like the following diagram. The other orientation gives a similar situation.



(In the diagram, the Q_i are subquivers.) Mutating at d will produce a quiver Q^* which looks like this:

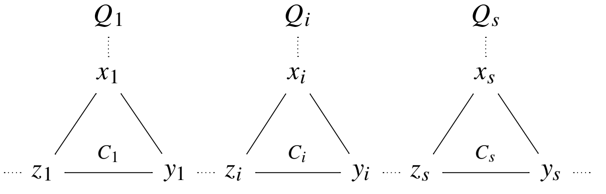


The only differences between Q and Q^* are that $d_{Q^*}(C^*, C') = d_Q(C, C') - 1$, and there is after the mutation a path of length 1 between C^* and Q_c.

This is the kind of mutation we use for moving 3-cycles closer together.

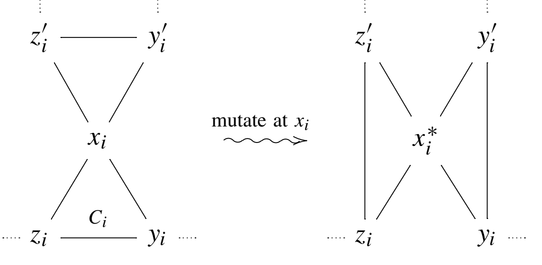
Suppose that there is a 3-cycle C in Q which is not in our sequence $\mathcal{S}_Q$. We will now use the procedure of moving 3-cycles to produce a new quiver Q^* with a sequence $\mathcal{S}_{Q^*}$ of 3-cycles such that the size of $\mathcal{S}_{Q^*}$ equals the size of $\mathcal{S}_Q$ plus one.

The quiver Q, with its sequence $\mathcal{S}_Q$, looks like this:



(1)

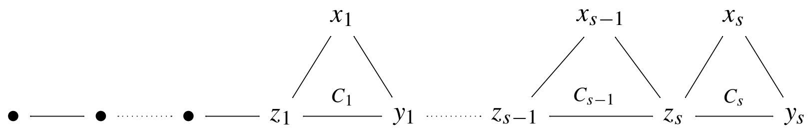
where the Q_i are subquivers, and C is in Q_i for some $i = 2, 3, \ldots, s-1$. (This follows from the definition of C_1 and s.) Without loss of generality, we may assume that C is the 3-cycle in Q_i which is closest to C_i. C may be moved towards x_i using the procedure above. So we may perform this procedure until C and C_i share the vertex x_i, and x_i has four neighbours. We then mutate at the vertex x_i:



Call the resulting quiver Q^*. After a suitable labelling, we now have a sequence $C_1^*, \ldots, C_{s+1}^*$ of 3-cycles in the quiver Q^*, where $C_j^* = C_j$ for $j < i$ and $C_j^* = C_{j-1}$ for $j > i+1$. This may serve as a sequence $\mathcal{S}_{Q^*}$.

Enlarging our totally ordered set like this the necessary number of times will give a quiver where *all* the 3-cycles are in a sequence $C_1, \ldots, C_s$ as in diagram (1) for some s, and the subquivers $Q_1, \ldots, Q_s$ are just (non-directed) paths.

If y_s in diagram (1) has valency 3, we now move C_s to the right by mutating at y_s and continuing in the same way. When we reach a diagram as in (1) above where y_s has only two neighbours (x_s and z_s), we shrink Q_s by mutating at x_s and continuing until the new x_s has only y_s and z_s as neighbours. By suitably orienting Q_s beforehand as in Lemma 2.1, we can do this in such a way that y_s still only has two neighbours, and C_s is connected to the rest of the quiver only through z_s. Successively doing this to $C_{s-1}, \ldots, C_1$ will give a quiver consisting of a sequence of 3-cycles with $d_Q(C_i, C_{i+1}) = 0$ for neighbouring C_i and C_{i+1}, and possibly with some non-directed path connected to it:



(2)

The orientation of C_s does not matter, since we can just flip it in the diagram. If i is the biggest number $< s$ such that C_i is not oriented in the clockwise direction, we mutate at $y_i = z_{i+1}$ and get a similar diagram where the new C_{i+1} is oriented in the anticlockwise direction, and the new C_i is oriented clockwise. Doing this the necessary number of times, we get the quiver we want. $\square$

It should be remarked that the following proposition follows from Lemma 2.2 and the fact that quivers in $\mathcal{Q}_n$ are 2-finite [FZ2], see also [S]. However, we can now give an independent argument:

Proposition 2.4. *A quiver Q is mutation equivalent to A_n if and only if $Q \in \mathcal{Q}_n$.*

Proof. Obviously, all orientations of A_n are in $\mathcal{Q}_n$.

It follows from the proof of Lemma 2.3 that all members of $\mathcal{Q}_n$ can be reached by iterated mutation on an A_n-quiver, since mutating the quiver in (2) in all the x_i will give a quiver with underlying graph A_n, and we can reverse the procedure in the proof to come to any $Q \in \mathcal{Q}_n$. $\square$

3. Relations

In this section we give the relations on the quivers of cluster-tilted algebras of type A_n, which are given in [CCS1] and have been generalised in [CCS2] and [BMR3]. This gives the complete description of this class of algebras, and we use it to establish that these algebras are gentle.

Proposition 3.1. *The cluster-tilted algebras of type A_n are exactly the algebras kQ/I where Q is a quiver in $\mathcal{Q}_n$ and I is the ideal generated by the directed paths of length 2 which are part of a 3-cycle.*

Given such a quiver Q, we will sometimes denote the corresponding cluster-tilted algebra kQ/I by Γ_Q.

If Q is a finite quiver and I is an ideal in the path algebra kQ, then kQ/I is *special biserial* [SkW] if it satisfies

- for every vertex p in Q, there are at most two arrows starting in p and at most two arrows ending in p.
- For every arrow β in Q, there is at most one arrow α_1 in Q with $\beta\alpha_1 \notin I$ and at most one arrow γ_1 in Q with $\gamma_1\beta \notin I$.

A special biserial algebra kQ/I is *gentle* [AsSk] if it also satisfies

- I is generated by paths of length 2.
- For every arrow β in Q there is at most one arrow α_2 such that $\beta\alpha_2$ is a path and $\beta\alpha_2 \in I$, and at most one arrow γ_2 such that $\gamma_2\beta$ is a path and $\gamma_2\beta \in I$.

Corollary 3.2. *Cluster-tilted algebras of type A_n are gentle.*

4. Cartan determinants

The *Cartan matrix* (C_{ij}) of a finite-dimensional k-algebra Λ is by definition the matrix with ijth entry $C_{ij} = \dim_k e_i \Lambda e_j$, that is, the columns are the dimension vectors of the indecomposable projectives. The determinant of the Cartan matrix is invariant under derived equivalence. (See [BoSk] for a proof.)

Since cluster-tilted algebras of type A_n are gentle, the following result is a special case of a result by Holm [Ho]. We include the proof, which is a lot simpler than in the general case.

Proposition 4.1. *If Γ is a cluster-tilted algebra of type A_n, then the determinant of the Cartan matrix of Γ is 2^t, where t is the number of 3-cycles in the quiver of Γ.*

Proof. Let Q be any quiver with relations R, and let $\Lambda_{Q,R}$ be the corresponding algebra. We will study the behaviour of the Cartan determinant of the algebra under two types of enlargements of Q and R. All quivers in the mutation class of A_n can be built by successive enlargements of this kind, and this will yield the result.

Let k denote the number of vertices in Q.

The first type of enlargement goes as follows. Construct Q' from Q by adding one new vertex labelled $k+1$ and one arrow α between Q and $k+1$. We assume that $k+1$ is the target of α. (The proof is similar in the other case.) Let the relations R' on Q' be R, i.e. Q' just inherits the relations from Q. Then the Cartan matrix $C(\Lambda_{Q',R'})$ is

$$\begin{pmatrix} * & \cdots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & 0 \\ * & \cdots & * & 1 \end{pmatrix}$$

where the Cartan matrix $C(\Lambda_{Q,R})$ sits in the top left corner. We see that the determinant of $C(\Lambda_{Q',R'})$ equals the determinant of $C(\Lambda_{Q,R})$, so this construction does not change the Cartan determinant.

We now turn to the second type of enlargement. Construct a quiver Q'' from Q by adding two vertices $k+1$ and $k+2$, and three arrows α, β, γ such that $\gamma\beta\alpha$ is a 3-cycle running through $k+1$ and $k+2$. We may assume that the third vertex in this 3-cycle is labelled k. Let the relations on Q'' be given by $R'' = R \cup \{\beta\alpha, \gamma\beta, \alpha\gamma\}$.

Since the minimal relations involving α, β and γ do not involve the relations in R, we have, for $i \leqslant k$,

$$c_{k+1,i} = \dim_k e_{k+1}(\Lambda_{Q'',R''})e_i = \dim_k e_k(\Lambda_{Q,R})e_i = c_{k,i},$$

$$c_{i,k+2} = \dim_k e_i(\Lambda_{Q'',R''})e_{k+2} = \dim_k e_i(\Lambda_{Q,R})e_k = c_{i,k}.$$

This gives the following $(k+2)\times(k+2)$ Cartan matrix of $C(\Lambda_{Q'',R''})$,

$$\begin{pmatrix} * & \cdots & * & | & 0 & | \\ \vdots & & \vdots & | & \vdots & | \\ * & \cdots & * & | & 0 & | \\ - & - & - & 1 & 0 & 1 \\ - & - & - & 1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Here, rows k and $k+1$ are equal except for the two rightmost entries, and similarly for columns k and $k+2$. Again, we find $C(\Lambda_{Q,R})$ in the top left corner. Expanding the determinant along the bottom row, we get

$$\left|C(\Lambda_{Q'',R''})\right| = M_{k+2,k+2} - M_{k+2,k+1}$$

where M_{ij} denotes the ijth minor. We see that $M_{k+2,k+2} = |C(\Lambda_{Q,R})|$, while

$$M_{k+2,k+1} = \begin{vmatrix} * & \cdots & * & | & | \\ \vdots & & \vdots & | & | \\ * & \cdots & * & | & | \\ - & - & - & 1 & 1 \\ - & - & - & 1 & 0 \end{vmatrix}$$

where rows k and $k+1$ are equal except for the last entry, and similarly for columns k and $k+1$. The top left part is still $C(\Lambda_{Q,R})$. Upon subtracting row $k+1$ from row k and then expanding the determinant along row k, we find that

$$M_{k+2,k+1} = -\left|C(\Lambda_{Q,R})\right|$$

and thus

$$\left|C(\Lambda_{Q'',R''})\right| = 2\cdot\left|C(\Lambda_{Q,R})\right|.$$

The statement in the proposition now follows from the following observation: Given any cluster-tilted algebra Γ of type A_n, we may build the quiver of Γ with the appropriate relations by starting with A_1 and performing the above types of enlargements sufficiently many times. The determinant of the Cartan matrix is multiplied by two for each 3-cycle added. □

5. Derived equivalence

We now prove the main result, namely that the cluster-tilted algebras of type A_n which have the same Cartan determinant are also derived equivalent.

Theorem 5.1. *Two cluster-tilted algebras of type A_n are derived equivalent if and only if their quivers have the same number of 3-cycles.*

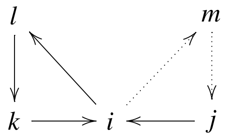
Proof. Let Γ and Γ' be two cluster-tilted algebras of type A_n.

If the quivers of Γ and Γ' do not have the same number of 3-cycles, then by Proposition 4.1 the determinants of their Cartan matrices are not equal, and thus they are not derived equivalent.

By the results in Section 2, and in particular Lemma 2.3, it is enough to show that if Γ and Γ' are two cluster-tilted algebras of type A_n, and the quiver of one of them can be obtained by mutating the quiver of the other in one vertex without changing the number of 3-cycles, then Γ and Γ' are derived equivalent. The strategy is to show that replacing a certain direct summand of the tilting object T in $\mathcal{C}$ with $\Gamma = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, we find a tilting Γ-module whose endomorphism ring is isomorphic to Γ'.

If the vertex i for the necessary mutation is a source or a sink, the mutation just corresponds to APR-tilting, so this is a well-known case [APR].

We consider the case where i is a vertex of the quiver of Γ with two arrows ending there. There might be one or two arrows with i as the initial vertex. (Having proved the result for this case, we need not do it for the case where i is a vertex with two arrows out and one arrow in, since this is just the reverse operation of what we have done.)



Let $T = \overline{T} \amalg T_i$ be the tilting object in the cluster category $\mathcal{C}$ which gives rise to $\Gamma = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, and suppose T_i is the indecomposable summand which through the functor $\mathrm{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \to \mathrm{mod}\ \Gamma$ corresponds to the vertex where we must mutate to get the quiver of Γ' from the quiver of Γ. Denote by T_i^* the unique second indecomposable object which completes $\overline{T}$ to a tilting object in $\mathcal{C}$. Then

$$\Gamma' = \mathrm{End}_{\mathcal{C}}\big(\overline{T} \amalg T_i^*\big)^{\mathrm{op}}.$$

By Theorem 1.2, the functor $G = \mathrm{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \to \mathrm{mod}\ \Gamma$ is full and dense and induces an equivalence

$$\overline{G} : \mathcal{C}/\mathrm{add}(\tau T) \to \mathrm{mod}\ \Gamma.$$

Let $T_1, \ldots, T_n$ be the indecomposable summands of T. The images of these objects under $\mathrm{Hom}_{\mathcal{C}}(T, -)$ are the indecomposable projective Γ-modules:

$$P_i = G(T_i) = \mathrm{Hom}_{\mathcal{C}}(T, T_i).$$

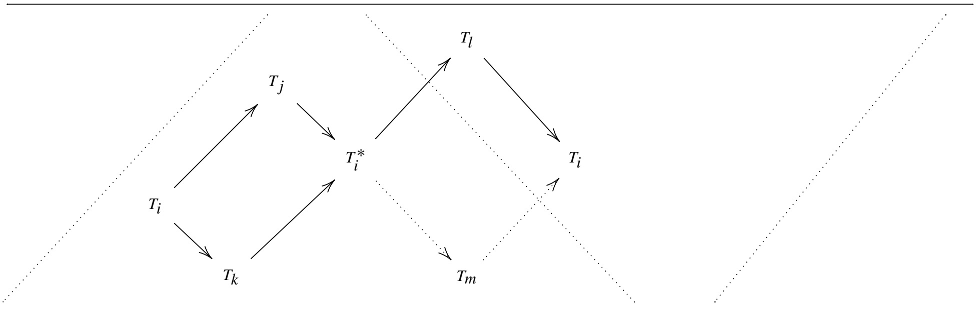
Denote by P_i^* the image of T_i^*:

$$P_i^* = \mathrm{Hom}_{\mathcal{C}}\big(T, T_i^*\big).$$

T_i^* is found by completing a minimal left $\mathrm{add}\,\overline{T}$-approximation $T_i \to B$ into a triangle in $\mathcal{C}$ (cf. Theorem 1.1):

$$T_i \to B \to T_i^* \to . \qquad (3)$$

We see that B in this triangle is $T_j \amalg T_k$, where j and k are the labels on the vertices which have arrows to i in the quiver of Γ. Recall that the AR-quiver of $\mathcal{C}$ is a Möbius band with mod $H \vee H[1]$ as a fundamental domain for the functor $F = \tau^{-1}[1]$, which takes the $\mathbb{Z}A_n$ to the Möbius band [BMRRT]. The AR-quiver is drawn in the following diagram, where the dotted lines indicate a choice of fundamental domain for F.



We must show that $P_i^* \neq 0$, which is the same as showing that T_i^* is not in $\operatorname{add}(\tau T)$. Since T_i^* is indecomposable, this would mean that $T_i^* = \tau T_q$ for some $q = 1, 2, \dots, n$. But by Serre duality this would lead to

$$\operatorname{Ext}^1_{\mathcal{C}}(T_q, T_j) \simeq D \operatorname{Hom}_{\mathcal{C}}(T_j, \tau T_q) = D \operatorname{Hom}_{\mathcal{C}}\bigl(T_j, T_i^*\bigr) \neq 0,$$

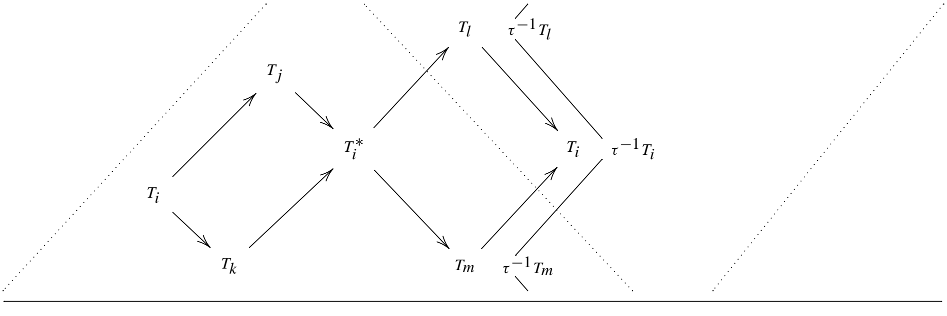
which is absurd, since T is a tilting object in $\mathcal{C}$. Thus we conclude that $P_i^* \neq 0$.

We will now show that no non-zero endomorphisms of $\overline{T} \amalg T_i^*$ factor through $\operatorname{add}(\tau T)$. Then G and the induced functor $\overline{G}$ will give the isomorphism

$$\begin{aligned} \Gamma' = \operatorname{End}_{\mathcal{C}}\bigl(\overline{T} \amalg T_i^*\bigr) &\simeq \operatorname{End}_{\Gamma}\bigl(\overline{P} \amalg P_i^*\bigr), \\ f &\mapsto \operatorname{Hom}_{\mathcal{C}}(T, f). \end{aligned} \tag{4}$$

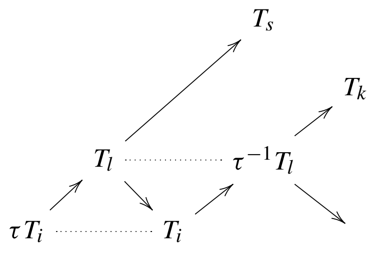
It is only necessary to consider maps involving T_i^*. Also, we see that if a non-zero map factors $T_i^* \to \tau T_q \to T_s$, then $q = i$, for this implies the existence of an extension between T_i^* and T_q. We find the maps in $\operatorname{End}_{\mathcal{C}}(\overline{T} \amalg T_i^*)$ from the AR-quiver of $\mathcal{C}$, and we recall from [BMRRT] that the k-dimension of $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is at most 1 when X and Y are summands of a tilting object.

We consider first the situation where i is the initial vertex of two arrows:

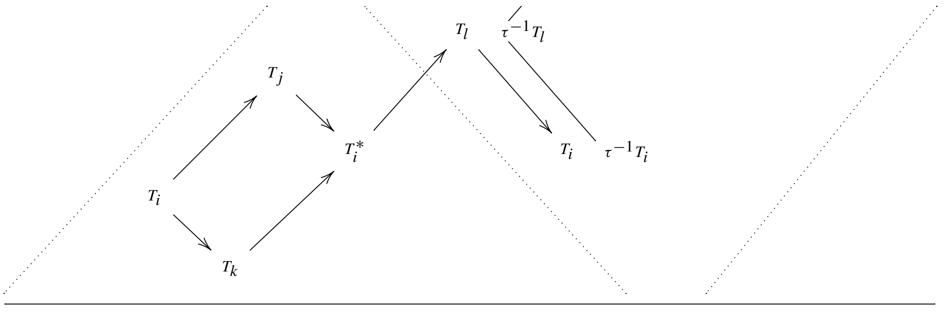


Suppose there is a non-zero map $\phi : T_i^* \to T_s$ for some s. Then ϕ factors through either T_l or T_m, since the map $T_i^* \to T_l \amalg T_m$ is a minimal left $\mathrm{add}(\overline{T})$-approximation. Also, ϕ cannot factor through $\tau^{-1}T_l$ or $\tau^{-1}T_m$ since $\overline{T}$ is exceptional. Therefore T_s must be on one of the rays starting in either T_l or T_m. Since there are arrows in the quiver of Γ from i to l and m, T_s cannot be on the rays from T_l and T_m to T_i in the AR-quiver, so it must be on one of the rays starting in T_i^*, after T_l or T_m.

So the only way ϕ can factor through τT_i is if one of the maps $T_l \to T_i$ and $T_m \to T_i$ is irreducible. But this is impossible, since this would imply that there is a non-zero map $\tau^{-1}T_l \to T_k$, i.e. an extension between T_l and T_k, or similarly for T_m and T_j.



Now consider the case where i is the initial vertex of only one arrow.



Again, a non-zero map $\phi : T_i^* \to T_s$ must factor through T_l, since $T_i^* \to T_l$ is a minimal left $\mathrm{add}(\overline{T})$-approximation. For the same reasons as above, it cannot factor through $\tau^{-1}T_l$, so T_s must be on one of the rays starting in T_l, and it cannot be between T_l and T_i. The possibility that it is on the same ray as T_i is then ruled out by the fact that the map $T_i^* \to T_l \to T_i$ is a composition of two maps in a triangle and therefore zero. This forces us to the situation described above with an irreducible map $T_l \to T_i$, which is again impossible. Summarising, we have that no non-zero morphisms $T_i^* \to T_s$ can factor through a τT_q.

Next note that a non-zero morphism $T_s \to T_i^*$ factoring through a τT_q would provide an extension between T_q and T_s, which is not possible. Obviously, there cannot be any endomorphisms of T_i^* factoring through other indecomposable objects. Hence, the isomorphism (4) is established.

Our goal is now to show that $\overline{P} \amalg P_i^*$ is a tilting module over Γ. Then the isomorphism (4) will imply that the derived categories $D^b(\mathrm{mod}\,\Gamma)$ and $D^b(\mathrm{mod}\,\Gamma')$ are equivalent, by Happel's theorem [Ha,CPS].

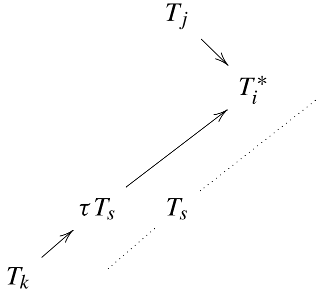
We now claim that in mod Γ the triangle (3) gives us a projective resolution of P_i^*:

$$0 \to P_i \to P_j \amalg P_k \to P_i^* \to 0. \tag{5}$$

So the Γ-module P_i^* has projective dimension 1. Indeed, the triangle (3) provides the long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Hom}_{\mathcal{C}}\big(T, \tau^{-1}T_j \amalg \tau^{-1}T_k\big) \to \mathrm{Hom}_{\mathcal{C}}\big(T, \tau^{-1}T_i^*\big) \\ \to \mathrm{Hom}_{\mathcal{C}}(T, T_i) \to \mathrm{Hom}_{\mathcal{C}}(T, T_j \amalg T_k) \to \cdots \end{aligned} \tag{6}$$

and the map $P_i \to P_j \amalg P_k$ in (5) is mono if and only if the first map in (6) is epi. To see that this is the case, we consider maps $\psi : \tau T_s \to T_i^*$, $1 \leqslant s \leqslant n$, and show that they factor through $T_j \amalg T_k$. This is easily seen when $s = i$. If ψ factors through T_s, it will also factor through $T_j \amalg T_k \to T_i^*$, since this is an add $\overline{T}$-approximation. If ψ does not factor through T_s, τT_s must be on one of the rays pointing to T_i^* in the AR-quiver. Without loss of generality, assume that this is the ray which T_k lies on. τT_s cannot be between T_k and T_i^*, since this would imply an extension between T_k and T_s:



In the case where T_k is between τT_s and T_i^*, ψ factors through T_k, which was what we wanted to show. Thus the first map in (6) is epi, and (5) is the projective resolution of P_i^*.

We need to check that $\mathrm{Ext}^a_\Gamma(P_i^*, P_l) = 0$ for $a = 1, 2, 3, \ldots$ and all indecomposable projectives $P_l \neq P_i$. Since P_i^* has projective dimension 1, the $a \geqslant 2$ case is trivial. Passing to the derived category $\mathcal{D}_\Gamma = D^b$ (mod Γ), we identify P_i^* with the deleted projective resolution

$$\cdots \to 0 \to P_i \to P_j \amalg P_k \to 0 \to \cdots$$

where $P_j \amalg P_k$ sits in degree 0. This derived category is equivalent to the homotopy category $K^{-,b}(\mathrm{proj}\,\Gamma)$ of upper bounded complexes of projective modules with non-zero homology only in a finite number of positions. Since

$$\mathrm{Ext}^1_\Gamma(X, Y) \simeq \mathrm{Hom}_{\mathcal{D}_\Gamma}\big(X, Y[1]\big)$$

for objects X and Y of $\mathcal{D}_\Gamma$, we need to show that there are no non-zero (up to homotopy) morphisms of complexes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & P_i & \longrightarrow & P_j \amalg P_k & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P_q & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

But all morphisms $P_i \to P_q$ where P_q is an indecomposable projective factor through $P_i \to P_j \amalg P_k$, since this map is a minimal left add $\overline{P}$-approximation, so all morphisms of complexes as above are null-homotopic. Thus all the Ext-groups vanish.

Since the number of indecomposables in $\overline{P} \amalg P_i^*$ equals n, the number of simples over Γ, and $\overline{P} \amalg P_i^*$ has projective dimension 1, it is a tilting module, and we are done. $\square$

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