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The Singular Submodule Splits Off*

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INTRODUCTION

In the usual torsion theory over a commutative integral domain R, a significant place is occupied by the question of when "the torsion submodule $t(M)$ of an R-module M is a direct summand of M" (hereafter referred to as "the condition"). Kaplansky [*10*] has shown that if the condition is satisfied by every finitely generated R-module, then R is a Prüfer domain; the converse is well known [*2*], Chapt. VII. Prop. 11.1. If the condition is satisfied by every R-module M, whose torsion submodule is of bounded order, Chase [*3*], Theorem 4.3, has shown that R is, then, a Dedekind domain; the converse has been shown by Kaplansky [*11*], Theorem 5. Finally, if the condition is satisfied by every R-module, Rotman [*16*] has shown that R is, then, a field.

A well-known extension of the notion of the torsion submodule for modules over arbitrary rings is the notion of the singular submodule of a module [*5*, *7*, *9*]. It is the purpose of this paper to investigate "the condition" of the preceding paragraph, in the context of the singular theory over a nonsingular commutative ring R. In short, if M is an R-module and $Z(M)$ its singular submodule, we study the condition: "$Z(M)$ is a direct summand of M". It is worthwhile to note that the commutative nonsingular rings are precisely the commutative semi-prime rings [*12*], Ex. 1, p. 108.

Throughout this paper, unless otherwise indicated, a ring R is a commutative ring with identity; all modules are unitary. For all homological notions, used in this paper, the reader is referred to [*2*].

1. PRELIMINARIES

Let R be a ring and M an R-module.

DEFINITION 1.1. We say M splits if $Z(M)$ is a direct summand of M.

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The notion of large ideal (submodule) is well known, e.g. [*17*].

DEFINITION 1.2. We say M is of bounded order if there exists a large ideal I of R such that $MI = (0)$.

DEFINITION 1.3. (a) R has FGSP if every finitely generated R-module splits.

(b) R has BSP if every R-module, whose singular submodule is of bounded order, splits.

(c) R has SP if every R-module splits.

The notion of closed ideal (submodule) is used here in the sense of Goldie [*6*, *7*, *18*]; it is defined over a not necessarily commutative ring R.

DEFINITION 1.4. A submodule B_R of a right R-module A_R is closed in A if B has no proper essential extension in A, i.e., if C is a submodule containing B as a large submodule, then $B = C$; equivalently if T_R is a submodule of A_R such that T is maximal with respect to the property that $B \cap T = (0)$, then B is maximal with respect to the property that $B \cap T = (0)$.

Commutativity is not needed in the following; for any right R-module M_R, $L(M_R)$ denotes the lattice of large submodules of M_R.

LEMMA 1.5. *Let* $M_R \xrightarrow{f} N_R \to 0$ *be an exact sequence of right R-modules, such that* $\ker f = K_R$ *is closed in* M_R. *If* $A_R \in L(M_R)$, *then* $f(A_R) \in L(N_R)$.

Proof. It is clearly sufficient to show the lemma in case $N = M/K$; we may further assume that $K \subset A$. Let T_R be a submodule of M_R, maximal with respect to the property that $K \cap T = (0)$ and let $\overline{W} = W/K$ be a submodule of M/K such that $A/K \cap W/K = (\bar{0})$; equivalently $A \cap W \subseteq K$ and thus $A \cap W \cap T \subsetneq K \cap T = (0)$. Since $A_R \in L(M_R)$, we have $W \cap T = (0)$. It follows now by the maximality of K (Definition 1.4) that $W = K$ or $\overline{W} = (\bar{0})$ and hence A/K is a large submodule of M/K. Q.E.D.

COROLLARY. *Let R be a (not necessarily commutative) ring and I a two-sided ideal of R, closed as a right R-submodule of R. If J is a large right ideal of R, then* $\nu(J)$ *is a large right ideal of* R/I. (ν *is the natural epimorphism* $R \to R/I$).

Proof. Let $r \in I$ such that $r \not\equiv 0 \pmod{I}$. By Lemma 1.5, $\nu(J)$ is a large right R-submodule of R/I, so there exists $t \in R$ such that $rt \in J$ and $rt \not\equiv 0 \pmod{I}$. We thus have $(r + I)(t + I) = \overline{rt} \in \nu(J)$ with $\overline{rt} \neq \bar{0}$, hence $\nu(J)$ is a large right ideal of R/I. Q.E.D.

PROPOSITION 1.6. *If R is a ring with $Z(R) = (0)$ and M any finitely generated R-module, then $Z(M)$ is of bounded order.*

Proof. If $Z(M) = M$, then M is clearly of bounded order. In any case there exists closed submodule T of M such that $Z(M) \cap T = (0)$ and $Z(M) \oplus T \in L(M)$. It follows from Lemma 1.5 that $Z(M) \oplus T/_T \in L(M/T)$ and since $Z(M) \oplus T/_T \subseteq Z(M/T)$ we further have that $Z(M/T) = M/T$, [7], Prop. 2.3. Now $Z(M)$ is isomorphic to a submodule of M/T, it is, hence, of bounded order since M/T is. Q.E.D.

Remark. A consequence of the above proposition is that if R with $Z(R) = (0)$ has BSP, R, then, has FGSP.

We use the well-known, e.g. [*17*], concept of right quotient ring of a not necessarily commutative ring R as follows:

DEFINITION 1.7. A ring S containing a ring R is a right quotient ring of R if $R_R \in L(S_R)$.

Let S be a right quotient ring of R. Observe that if $A_S \in L(S_S)$, then $(R \cap A)_R \in L(R_R)$ and if $I_R \in L(R_R)$, then $IS \in L(S_S)$. Now if M_R is any right R-module, then $M \otimes_R S$ is a right R- and S-module. It follows from the above observation that $Z((M \otimes_R S)_R) = Z((M \otimes_R S)_S)$ and hereafter we write $Z(M \otimes S)$. Furthermore, for any left R-module ${}_RN$ we write $M \otimes N$ for $M \otimes_R N$, if no ambiguity arises.

If a (commutative) ring R has any of the properties of Definition 1.3, then certain quotient rings of R inherit them, and we deal with this problem in Proposition 1.9. But first a more general result is needed:

PROPOSITION 1.8. *Let R be a ring with $Z(R_R) = (0)$ and S a right quotient ring of R satisfying*: (i) *S is flat as a left R-module, and* (ii) *$S \otimes_R S \cong S$ (by the canonical map $\sum_i s_i \otimes t_i \to \sum s_i t_i$). The following statements are, then, true*:

(a) For any right R-module M_R with $Z(M_R) = (0)$, we have $Z(M \otimes S) = (0)$.

(b) For any right S-module A_S and R-submodule B_R of A, we have $B \otimes S \cong BS$.

Proof. (a) We consider M_R as an R-submodule of $M \otimes S$ [*17*], Prop. 2.2, and note that if $\nu : M \otimes S \to M \otimes S/_{Z(M \otimes S)}$ is the natural epimorphism (of R-modules) and $\bar{\nu} = \nu \mid M$ (i.e., the restriction of ν to M), then $0 \to M \xrightarrow{\bar{\nu}} M \otimes S/_{Z(M \otimes S)}$ is exact since $M \cap Z(M \otimes S) = (0)$. Now the sequence $0 \to M \otimes S \to (M \otimes S/_{Z(M \otimes S)}) \otimes_R S$ is exact by (i). By (ii) and the associativity of the tensor product [*2*] Chapt. *II*, Prop. 5.1, we have, clearly that $(M \otimes S/_{Z(M\otimes S)}) \otimes_R S \cong M \otimes S/_{Z(M\otimes S)}$. Thus, $Z(M \otimes S) = (0)$ since $Z(M \otimes S/_{Z(M\otimes S)}) = (0)$, [7], Prop. 2.3.

(b) From the inclusion map $B_R \xrightarrow{i} A_S$ and properties (i) and (ii) of S, we have : $0 \to B \otimes S \to A \otimes_R S$ is exact and $A \otimes_R S \cong A_S$ by $\sum_i a_i \otimes s_i \to \sum a_i s_i$. It follows now that $B \otimes S \cong BS$, by the above map. Q.E.D.

PROPOSITION 1.9. *Let R be a ring with $Z(R) = (0)$ and S a quotient ring of R, such that S is R-flat and $S \otimes_R S \cong S$ (canonically). The following statements are, then, true:*

(a) If R has FGSP, then S has FGSP.

(b) If R has BSP, then S has BSP.

(c) If R has SP, then S has SP.

Proof. (a) Let $A = a_1S + \cdots + a_nS$ be any finitely generated S-module and set $A^* = a_1R + \cdots + a_nR \subseteq A$. If R has FGSP, then $A_R^* = Z(A_R^*) \oplus B$ and this gives $A^* \otimes S \cong (Z(A_R^*) \otimes S) \oplus (B \otimes S)$. By Prop. 1.8 we have $A^* \otimes S \cong A^*S = A$, $Z(A^*) \otimes S \cong Z(A^*)\, S$ and $B \otimes S \cong BS$; in particular $Z(BS) = (0)$ and $Z(Z(A^*)\, S) = Z(A^*)\, S$, so the above direct sum reduces to $A = Z(A^*)\, S \oplus BS$. Thus S has FGSP if R does.

(b) Let A be an S-module such that $Z(A_S)$ is of bounded order. It is clear that $Z(A_R) = Z(A_S)$, hence if R has BSP, then $A_R = Z(A_R) \oplus B$. As in (a), it follows by Prop. 1.8 that $A = Z(A) \oplus BS$ and S has BSP.

(c) The proof of (c) is the obvious modification of the proof of (b). Q.E.D.

An exact sequence $0 \to A \to B \to C \to 0$ of R-modules is called an extension of A by C. The following lemma is well known [2], Chapt. XIV:

LEMMA 1.10. *If A and C are R-modules, then* $\mathrm{Ext}_R{}^1(C, A) = (0)$ *if and only if every extension of A by C splits* [2], p. 5.

PROPOSITION 1.11. *For any (commutative) ring R, the following statements are equivalent:*

(a) *R has* FGSP.

(b) $Z(R) = (0)$ and $\mathrm{Ext}_R{}^1(M, S) = (0)$ for every FGNS R-module M and singular R-module S, where FGNS means "finitely generated nonsingular".

Proof. (a) implies (b). $Z(R)$ contains no idempotents $\neq 0$ so (a) implies that $Z(R) = (0)$. If $0 \to S \to X \to M \to 0$ is any extension of S by M where $Z(S) = S$ and M is FGNS, we see that $Z(X) = S$ since $S \subseteq Z(X)$

and $Z(X/S) \simeq Z(M) = (0)$. Furthermore since X/S is finitely generated, there exists finitely generated R-module $B \subseteq X$ such that $X = B + S$. It follows from (a) that $B = Z(B) \oplus C$ and we see that $X = S + C$. Since $C \cap Z(B) = (0)$ the (last) sum is direct and the extension splits. It follows from Lemma 1.10 that $\mathrm{Ext}_R{}^1(M, S) = (0)$.

(b) implies (a). It follows from (b) that the sequence

$$0 \to Z(M) \to M \to M/Z(M) \to 0$$

splits for any finitely generated R-module M, hence R has FGSP. Q.E.D.

The following corollary extends Kaplansky's result on Prüfer domains (see Introduction).

COROLLARY. *If R has* FGSP, *then R is semi-hereditary.*

Proof. It is sufficient to show that every torsionless R-module is flat [3], Theorem 4.1. Since a torsionless R-module over a nonsingular ring R is clearly nonsingular, it follows, by a standard direct limit argument, that we shall have the corollary if we show that every "finitely generated nonsingular" (FGNS) R-module is flat.

Let Z be the ring of integers and C any divisible Abelian group. For any large ideal I of R and R-module M, we have

$$\mathrm{Ext}_R{}^1(M, \mathrm{Hom}_Z(R/I, C)) \simeq \mathrm{Hom}_Z(\mathrm{Tor}_1{}^R(R/I, M), C)$$

by [2], Chapt. VI, Prop. 5.1. Let M be a FGNS R-module. Since $\mathrm{Hom}_Z(R/I, C)$ is obviously a singular (of bounded order I, in fact) R-module, it follows by Prop. 1.11 and the identity above that $\mathrm{Tor}_1{}^R(R/I, M) = (0)$ for every large ideal I. The module M is hence R-flat [12], Ex. 1, p. 135, and R is semi-hereditary. Q.E.D.

Remark. The converse of the above corollary is not in general true. There exists a commutative regular ring R, which does not have FGSP [15], p. 97: 22.6.

We complete what we started in Prop. 1.11, with the following homological characterization of BSP and SP. The proof is an easy consequence of Lemma 1.10 and it is omitted.

PROPOSITION 1.12. *For a ring R with $Z(R) = (0)$ the following statements hold:*

(a) *R has* BSP *if and only if* $\mathrm{Ext}_R{}^1(M, S) = (0)$ *for every R-module M such that $Z(M) = (0)$ and R-module S of bounded order.*

(b) *R has* SP *if and only if* $\mathrm{Ext}_R{}^1(M, S) = (0)$ *for every R-module M such that $Z(M) = (0)$ and R-module S such that $Z(S) = S$.*

COROLLARY. *Let R be a ring with $Z(R) = (0)$, I a large ideal of R and N an R-module such that $Z(N) = (0)$. If R has* BSP, *then $N \otimes R/I \cong N/NI$ is R/I-projective.*

Proof. R is semi-hereditary by Prop. 1.6 (remark) and the Corollary to Prop. 1.11. Now the torsion submodule of N, $t(N) = \{x \in N / xd = 0$ for some nonzero divisor d of $R\}$, is contained in $Z(N)$, hence N is torsion free and thus R-flat [4], Theorem 5. If S is any R/I-module we have by [2], Chapt. VI, Prop. 4.1.3, the isomorphism

$$\mathrm{Ext}^1_{R/I}(N \otimes R/I, S) \cong \mathrm{Ext}^1_R(N, S). \tag{1}$$

Since S is an R-module of bounded order I, the corollary follows from Proposition 1.12(a) and [2], Chapt. VI, Corollary 2.2. Q.E.D.

Remark. It is clear from the above proof and Prop. 1.11 that if R has FGSP, then $M \otimes R/I$ is R/I-projective for any FGNS R-module M and large ideal I of R.

In this paper a ring R is regular in the sense of Von Neumann [20].

If R is a (not necessarily commutative) regular ring and A a right ideal of R, then A is generated by its idempotents and in particular $A^2 = A$. This implies that if B is any right ideal containing A, then $AB = A$. We now, easily, have:

PROPOSITION 1.13. *Let R be a regular (commutative) ring and I a large ideal of R. If R has* BSP, *then R/I is a hereditary ring.*

Proof. Let J/I be an ideal of R/I; by the preceding paragraph $J/I = J/JI$ and, hence, J/I is R/I-projective by the Corollary to Prop. 1.2. R/I is, by definition [2], a hereditary ring. Q.E.D.

We close this section with a modification of Chase's Theorem 3.1 [3], p. 464. Commutativity is not needed in the following.

DEFINITION 1.14. Let R be a ring, A a left R-module and B a submodule of A. B will be called a pure submodule of A if $B \cap rA = rB$ for all $r \in R$.

DEFINITION 1.15. Let R be a ring and A a left R-module. Let $\{C_\beta\}$ be a family of left R-modules (where β traces some index set) and let $f_\beta \in \mathrm{Hom}_R(A, C_\beta)$. The family $\{f_\beta\}$ will be called a Φ-family of homomorphisms if the following conditions are satisfied for any $x \neq 0$ in A:

(a) $f_\beta(x) = 0$ for almost all β.

(b) $f_\beta(x) \neq 0$ for some β.

THEOREM 1.16. *Let R be a ring and J an infinite set of cardinality ζ where $\zeta \geqslant \operatorname{card} R$. Set $A = \prod_{\alpha \in J} R^{(\alpha)}$ where $R^{(\alpha)} = R$ as a left R-module. Let I be a two-sided ideal of R such that A/IA is a pure submodule of a left R/I-module of the form $\sum \oplus_\beta C_\beta$, where each C_β is generated by a set of cardinality less than or equal to ζ. Then any descending chain of principal right ideals of R all containing I, must terminate.*

Proof. Chase's proof can be used with slight modifications and we indicate these here. The notation used is that of Chase.

Let $f_\beta \in \operatorname{Hom}_R(A/IA, C_\beta)$ be the restriction to A/IA of the projection of C onto C_β; $\{f_\beta\}$ is easily seen to be a Φ-family.

Now suppose that the theorem is false and hence let

$$R = a_0 R \underset{\neq}{\supset} a_1 R \underset{\neq}{\supset} a_2 R \underset{\neq}{\supset} \cdots$$

be a nonterminating strictly descending chain of principal right ideals of R, all containing I. Let $\nu : A \to A/IA$ be the natural epimorphism (of left R-modules) and set $g_\beta = f_\beta \nu$; $\{g_\beta\}$ is not necessarily a Φ-family but it satisfies property (a) of a Φ-family and this is the property that is contradicted in Chase's proof. Each g_β induces a Z-homomorphism $g_{\beta k} : A_{0k} \to C_{\beta k}$; it is easily shown that, for a fixed k, $\{g_{\beta k}\}$ is a Φ-family of Z-homomorphisms. With no difficulty Chase's argument now gives his crucial condition:

($*$) For any n, $k \geqslant 0$ and any $\beta_1, \ldots, \beta_r$ there exists $\bar{x} \in A_{nk}$ and $\beta \neq \beta_1, \ldots, \beta_r$ such that $g_{\beta k}(\bar{x}) \neq 0$.

Since for each k, $\{g_{\beta k}\}$ is a Φ-family, Chase's inductive argument goes through to give a sequence $x_0, x_1, x_2, \ldots$ of elements of A and a sequence $g_{\beta_0}, g_{\beta_1}, g_{\beta_2}, \ldots$ selected from $\{g_\beta\}$ such that :

(i) $x_n \in a_n A_n$.

(ii) $g_{\beta_n}(x_n) \not\equiv 0 \pmod{a_{n+1} C_{\beta_n}}$.

(iii) $g_{\beta_n}(x_k) = 0$ for $k < n$.

Chase's construction now gives an element x of A with the property that $g_\beta(x) \neq 0$ for infinitely many β and we have the desired contradiction. Q.E.D.

An immediate application of Theorem 1.16 is the following much needed result (we resume commutativity of R):

PROPOSITION 1.17. *Let R be a ring with $Z(R) = (0)$ and I a large ideal of R. If R has* BSP, *then any descending chain of principal ideals of R all containing I must terminate.*

Proof. Let J be an index set with card $J = R$. Let $A = \prod_{\alpha \in J} R^{(\alpha)}$ where $R^{(\alpha)} = R$ and let $0 \to K \to F \to A \to 0$ be an exact sequence of R-modules with F R-free. It follows from the Corollary to Prop. 1.12 that the sequence $0 \to K \otimes R/I \to F \otimes R/I \to A \otimes R/I \to 0$ is split exact since $Z(A) = (0)$. The hypotheses of Theorem 1.16 are now fulfilled with $C = F \otimes R/I$ and the proposition follows. Q.E.D.

2. Rings with SP

The main result of this section is the following characterization of rings with SP. Semi-simple means semi-simple with d.c.c.

THEOREM 2.1. *For a ring R the following are equivalent:*

(a) *R has* SP.

(b) *R is regular and has* BSP.

(c) *$Z(R) = (0)$ and for every large ideal I of R, the ring R/I is semi-simple.*

(d) *Every R-module M with $Z(M) = M$ is R-injective.*

In particular if R has SP, *then R is hereditary.*

We give a circular proof in the order: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a). Several of the results needed are of interest in themselves.

(a) implies (b). It suffices to show that R is regular and this constitutes

PROPOSITION 2.2. *If R has* SP, *then R is regular.*

Proof. R has FGSP and is, hence, semi-hereditary by the Corollary to Prop. 1.11. It follows that the total quotient ring K of R is regular [*4*], Theorem 2. Thus it suffices to show that every nonzero divisor of R is invertible in R. To show this we use a variation of the argument used by Rotman in [*16*].

Let p be a nonzero divisor of R and assume that $p^{-1} \notin R$. We, then, obtain a strictly descending chain of principal ideals of $R: R \underset{\neq}{\supset} pR \underset{\neq}{\supset} p^2R \underset{\neq}{\supset} \ldots$. Set $M = \prod_{n=1}^{\infty} R/p^nR$ and $1_n = 1 + p^nR \in R/p^nR$. If x is in M, we write $x = [x_n 1_n]$ where $x_n \in R$ for each n.

An element x of M is said to have infinite p-height, if for every $n \geqslant 1$ there exists an element y of M such that $p^n y = x$.

M has no (nonzero) elements of infinite height. Indeed suppose x and y are in M such that $p^n y = x$. It follows that $p^n y_k - x_k \in p^k R$ for each k, hence $x_k \in p^k R + p^n R$. If $n \geqslant k$, then $p^k R \supseteq p^n R$ so that $x_k \in p^k R$. Thus if x has infinite p-height, then $x = 0$.

We show next that $M/Z(M)$ does have elements of infinite p-height and this will complete the proof. Observe that $\sum_{n=1}^{\infty} \oplus R/p^n R \subseteq Z(M)$. Let

$$x = (1_1, 1_2, p1_3, p1_4, \ldots, p^n 1_{2n+1}, p^n 1_{2n+2}, \ldots).$$

We claim that $x \not\equiv 0 \pmod{Z(M)}$. If $x \in Z(M)$, then there exists $J \in L(R)$ such that $Jx = (0)$. In particular $Jp^n \subseteq p^{2n+1}R$ for each n. This implies $J \subseteq p^{n+1}R$ for each n, a contradiction to BSP by the Corollary to Theorem 1.16.

Now for any n, let $y_{(n)} = (0, 0,..., 1_{2n+1}, 1_{2n+2}, p1_{2n+3}, p1_{2n+4},...)$. As in the case of x above, we have $y_{(n)} \not\equiv 0 \pmod{Z(M)}$. Now for each $n \geqslant 1$

$$\begin{aligned}\bar{x} = x + Z(M) &= (1_1, 1_2,..., p^{n-1}1_{2(n-1)+1}, p^{n-1}1_{2(n-1)+2}, 0, 0,...)\\ &\quad + p^n y_{(n)} + Z(M) = p^n y_{(n)} + Z(M) = p^n \bar{y}_{(n)}\end{aligned}$$

and thus $\bar{x}$ is of infinite p-height. Q.E.D.

Remark. Since a regular integral domain is a field, Rotman's result [*16*] is now a consequence of the above proposition.

(b) implies (c). Clearly $Z(R) = (0)$ as R is regular.

In the following sequence of results we show that R/I is semi-simple by showing that R/I has no infinite set of orthogonal idempotents. (Corollary to Prop. 2. 5).

We start with a result of Tarski [*19*]:

THEOREM T. (Tarski). *If Γ is a countably infinite set, there exists a class K of subsets of Γ such that*:

(a) card $K = 2^{\aleph_0}$

(b) card $A = \aleph_0$ *for every* $A \in K$

(c) card $A \cap B < \infty$ *for all* $A, B \in K$, $(A \neq B)$.

In the following application of Theorem T, R need not be commutative.

PROPOSITION 2. 3. *Let R be a ring (with 1) and $\{A^{(n)} : n \in \Gamma\}$ a countably infinite family of nonzero right R-modules. Then $M_R = \prod_n A^{(n)}/\sum_n \oplus A^{(n)}$ contains a submodule which is the direct sum of $2^{\aleph_0}$ submodules of M.*

Proof. Let $K = \{X_\alpha : \alpha \in \Lambda\}$ be a class of subsets of Γ with card $\Lambda = 2^{\aleph_0}$, card $X_\alpha = \aleph_0$ and card $X_\alpha \cap X_\beta < \infty$ $(\alpha \neq \beta)$, by Theorem T. If x is an element of $\prod_n A^{(n)}$ we write $x = [x(n)]$ with $x(n) \in A^{(n)}$. For each $\alpha \in \Lambda$, let $m_\alpha = [m_\alpha(n)]$, where $0 \neq m_\alpha(n) \in A^{(n)}$ if $n \in X_\alpha$, $m_\alpha(n) = 0$ if $n \notin X_\alpha$. Clearly $m_\alpha \not\equiv 0 \pmod{\sum_n \oplus A^{(n)}}$.

Set $N = \sum_\alpha \bar{m}_\alpha R$, $\bar{m}_\alpha = m_\alpha + \sum_n \oplus A^{(n)}$; obviously $N \subseteq M$. We claim the sum $\sum_\alpha \bar{m}_\alpha R$ is direct. Suppose $\bar{x} \in \bar{m}_\alpha R \cap \sum_{\beta \neq \alpha} \bar{m}_\beta R$; we have $\bar{x} = \bar{m}_\alpha r = \bar{m}_{\beta_1} r_1 + \cdots + \bar{m}_{\beta_k} r_k$ where $r_i \in R$, $\alpha \neq \beta_i$. This implies that $[m_\alpha(n)\, r] - \{[m_{\beta_1}(n)\, r_1] + \cdots + [m_{\beta_k}(n)\, r_k] \in \sum_n \oplus A^{(n)}$. Now $\mathrm{card}(X_\alpha - \bigcup_{i=1}^k X_{\beta_i}) = \infty$ since $X_\alpha - \bigcup_{i=1}^k X_{\beta_i} = X_\alpha - \bigcup_{i=1}^k (X_\alpha \cap X_{\beta_i})$

and $\mathrm{card}(\bigcup_{i=1}^{k}(X_\alpha \cap X_{\beta_i})) < \infty$ by property (c), Theorem T. From this it follows that $m_\alpha(n)\, r = 0$ for all but a finite number of the indices n, hence $x \equiv 0 (\mathrm{mod} \sum_n \oplus A^{(n)})$. Q.E.D.

The following is a generalization of a result by Sandomierski [18], Theorem 2.1; we supply the proof for completeness. Commutativity of R is not needed.

THEOREM 2.4. *Let R be a ring with $Z(R_R) = (0)$ and P_R a projective module. If P_R contains a large submodule B_R, which has a set of generators $G = \{k_i : i \in I\}$ with* card $G = \infty$, *then P_R has a set of generators G' with* card $G' \leqslant$ card G.

Proof. By [2], Chapt. VII, Prop. 3.1, there exists a family $\{x_\alpha\}$ of elements of P_R and a family $\{f_\alpha\} \subseteq \mathrm{Hom}_R(P_R, R_R)$ such that for all $x \in P$ we have $x = \sum x_\alpha f_\alpha(x)$, where $f_\alpha(x) = 0$ for all but a finite number of the α.

It suffices to show that f_α is the zero map for all but a set A of indices α, with card $A \leqslant$ card G.

Let $A = \{\alpha / f_\alpha(k_i) \neq 0$ for some $i \in I\}$; clearly card $A \leqslant$ card G and for all $\alpha \notin A$ we have $f_\alpha(B_R) = (0)$. Let x be any element of P; the ideal $I = \{r \in R / xr \in B\}$ is, then, large by [17], Prop. 1.2. Furthermore let $\alpha \notin A$; for any $r \in I$ we have $0 = f_\alpha(xr) = f_\alpha(x)\, r$, hence $f_\alpha(x)\, I = (0)$. Thus $f_\alpha(x) = 0$, as $Z(R_R) = (0)$, hence $f_\alpha = 0$.

Let $G' = \{x_\alpha : \alpha \in A\}$. Q.E.D.

LEMMA 2.5. *If R is a (commutative) regular ring, then idempotents can be lifted modulo any ideal I of R.*

Proof. Let $\bar{p} = p + I$ be an idempotent in R/I; we have $p = eu$ for some idempotent e and unit u in R [4], Theorem 1. Furthermore $eu^2 - eu \in I$ so that $(eu^2 - eu)\, u^{-1} = eu - e \in I$ and, thus, $\bar{p} = \bar{e}$, $e^2 = e \in R$. Q.E.D.

PROPOSITION 2.6. *Let R be a ring and I a large ideal. If R is regular and has* BSP, *then R/I does not contain a countably infinite set $\{\bar{e}_n : n \in \Gamma\}$ of orthogonal idempotents with the property that $\sum_n \oplus \bar{e}_n R/I$ is a large ideal of R/I.*

Proof. Suppose the proposition is false. There exists, then, a countably infinite set $\{e_n : n \in \Gamma\}$ of idempotents in R such that $\{\bar{e}_n : n \in \Gamma\}$ is a countably infinite set of orthogonal idempotents in R/I, with the property that $A = \sum_n \oplus \bar{e}_n R/I$ is R/I-large in R/I. By a well-known argument we may further assume that the idempotents $\{e_n\} \subseteq R$ are orthogonal.

Now consider the exact sequence

$$0 \to \sum_n \oplus e_n R \xrightarrow{i} \prod_n e_n R \xrightarrow{\nu} \prod_n e_n R / \sum_n \oplus e_n R \to 0 \tag{1}$$

where i is the canonical embedding of $\sum \oplus e_nR$ in $\prod e_nR$ and ν is the natural epimorphism. Every R-module is flat [4], Prp. 10 and in particular R/I, so that the following sequence is exact:

$$0 \to \left(\sum \oplus e_nR\right) \otimes R/I \to \left(\prod e_nR\right) \otimes R/I \to \left(\prod e_nR/_{\sum \oplus e_nR}\right) \otimes R/I \to 0. \tag{2}$$

We now show the following:

Claim. Every R/I-submodule of $\prod e_nR \otimes R/I$ containing $(\sum \oplus e_nR) \otimes R/I$ is countably generated.

By the Corollary to Prop. 1.12, $\prod e_nR \otimes R/I$ is R/I-projective; since R/I is hereditary, by Prop. 1.13, every R/I-submodule of $\prod e_nR \otimes R/I$ is R/I-projective [2], Chapt. I, Theorem 5.4. The claim will now follow from Theorem 2.4 if we show that $(\sum \oplus e_nR) \otimes R/I$ is R/I-large in $\prod e_nR \otimes R/I$. Thus, let $y = \sum_{i=1}^{k} x_i \otimes \bar{1} \neq 0$, where $x_i \in \prod e_nR$. Now $\prod e_nR \otimes R/I$ is R/I-nonsingular since it is R/I-projective and R/I is a nonsingular ring [5], p. 426. It follows that $yA \neq (0)$ so there exists $\bar{e}_l$ such that $0 \neq y\bar{e}_l = (\sum x_i \otimes \bar{1})\, \bar{e}_l = \sum x_i e_l \otimes \bar{1}$. Clearly $x_i e_l \in \sum \oplus e_nR$, $i = 1,\dots, k$ so $0 \neq y\bar{e}_l \in (\sum \oplus e_nR) \otimes R/I$ and the proof of the claim is complete.

From this claim and sequence (2) we have:

(*) every R/I-submodule of $(\prod e_nR/_{\sum \oplus e_nR}) \otimes R/I$ is countably generated.

We now use Prop. 2.3 to construct an R/I-submodule N of

$$(\prod e_nR/\sum \oplus e_nR) \otimes R/I,$$

such that N cannot be generated by fewer than $2^{\aleph_0}$ generators; this contradicts (*) and the proof shall be complete.

Using the notation of Prop. 2.3, we let $r_\alpha = [r_\alpha(n)]$, $\alpha \in A$ where $r_\alpha(n) = e_n$ if $n \in X_\alpha$, $r_\alpha(n) = 0$ if $n \notin X_\alpha$. We set $N = \sum_{\alpha \in A} \oplus \bar{r}_\alpha R \subseteq \prod e_nR/\sum \oplus e_nR$ where card $A = 2^{\aleph_0}$. The sequence $0 \to N \otimes R/I \to (\prod e_nR/\sum \oplus e_nR) \otimes R/I$ is exact and $N \otimes R/I \simeq \sum_{\alpha \in A} \oplus (\bar{r}_\alpha R \otimes R/I)$. We shall have the contradiction to (*) if we show that $\bar{r}_\alpha R \otimes R/I \neq (0)$ for each $\alpha \in A$. If $\bar{r}_\alpha R \otimes R/I = (0)$, then $\bar{r}_\alpha = \bar{r}_\alpha t$ for some $t \in I$; from this we have $[r_\alpha(n)] - [r_\alpha(n)\, t] \in \sum \oplus e_nR$, an impossibility by the definition of r_α and the properties of $\{X_\alpha : \alpha \in A\}$. Hence $N \otimes R/I$ cannot be generated by fewer than $2^{\aleph_0}$ generators, a contradiction of (*): the proposition is true. Q.E.D.

COROLLARY (Hypotheses as in Prop. 2.6). *R/I contains no infinite sets of orthogonal idempotents.*

Proof. Suppose the corollary is false and let $\{\bar{e}_n : n \in \Gamma\}$ be a countably infinite set of orthogonal idempotents in R/I. Let $\nu : R \to R/I$ be the canonical epimorphism and set $\bar{J} = \sum_n \oplus \bar{e}_n R/I$ where $J = \nu^{-1}(\bar{J})$. Let $\bar{K}$, where $K = \nu^{-1}(\bar{K})$, be an ideal of R/I, maximal with respect to the property that $\bar{J} \cap \bar{K} = (\bar{0})$. In particular $\bar{J} \oplus \bar{K}$ is a large ideal of R/I. Let $\bar{\nu} : R/I \to R/K$ be the epimorphism $x + I \to x + K$. It is easy to see that $\bar{\nu}(\bar{e}_n) \neq 0 \in R/K$ and $\{\bar{\nu}(\bar{e}_n) : n \in \Gamma\}$ is a set of orthogonal idempotents in R/K. Furthermore $\bar{K} = K/I$ is a closed ideal of R/I so that $\bar{\nu}(\bar{J} \oplus \bar{K})$ is a large ideal of $R/K \cong R/I/K/I$ by the Corollary to Lemma 1.5. But

$$\bar{\nu}(\bar{J} \oplus \bar{K}) = \textstyle\sum_n \oplus \bar{\nu}(\bar{e}_n)\, R/K$$

and since K is a large ideal of R, the last statement contradicts Proposition 2.6.
Hence, R/I contains no infinite set of orthogonal idempotents. Q.E.D.

We now complete the proof (b) implies (c): R/I is a regular ring, since R is, and contains no infinite sets of orthogonal idempotents by the above corollary. This is well known to imply that R/I is a semi-simple ring.

(c) implies (d). Let M be an R-module such that $Z(M) = M$. To show "M is R-injective" it is sufficient to show that for every large ideal J of R and $f \in \mathrm{Hom}_R(J, M)$ there exists $f^* \in \mathrm{Hom}_R(R, M)$ such that $f^* \mid J = f$. Observe that $I_R = \ker f$ is large in J_R, hence I is a large ideal of R. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "i"] \arrow[d, "\mu"'] & R \arrow[d, "\nu"] \\
0 \arrow[r] & J/I \arrow[r, "j"] \arrow[d, "\bar{f}"'] & R/I \arrow[dl, dashed, "g"] \\
 & M &
\end{tikzcd}$$

where μ, ν are natural epimorphisms, i, j are inclusion maps and $\bar{f}$ is induced by f; we have $\bar{f}\mu = f$ and $\nu i = j\mu$. From (c) it follows that J/I is a direct summand of R/I, hence there exists $g \in \mathrm{Hom}_R(R/I, M)$ such that $gj = \bar{f}$. Let $f^* = g\nu$; we have $f^*i = (g\nu)\, i = gj\mu = \bar{f}\mu = f$. Q.E.D.

(d) implies (a). Tivial.

We complete the proof of the theorem by showing that if R has SP, then R is hereditary. Let M be any R-module and E its R-injective hull. In the exact sequence $0 \to M \to E \to E/M \to 0$, E/M is R-injective by (d). It follows that inj. $\dim_R M \leqslant 1$ for every R-module M. R is hereditary, follows now from [2], Chapt. VI, Prop. 2.8.

Remark. It is now clear (e.g., from (b)) that if R has SP, then every homomorphic image of R has SP.

Osofsky has shown [*14*] that a right hereditary ring which is right self-injective is a semi-simple ring. From this and Theorem 2.1 we obtain immediately the following:

THEOREM 2.7. *For any commutative ring R the following are equivalent*:

(a) *R is semi-simple.*

(b) *R is self-injective and has* SP.

Remark. A ring which has SP need not be semi-simple. Let K be a field and A an infinite index set. Let $Q = \prod_{\alpha \in A} K^{(\alpha)}$, where $K^{(\alpha)} = K$, and $R = \sum_{\alpha \in A} \oplus K^{(\alpha)} + 1 \cdot K \subseteq Q$, $1 \in Q$. R is easily seen to have only one large ideal, namely $I = \sum_{\alpha \in A} \oplus K^{(\alpha)}$ and I is, of course, maximal. Since R is clearly regular, R has SP by Theorem 2.1 (c). R is not semi-simple.

The fact that the ring in the above example has only one large ideal is not totally unrelated to SP. We show below (Theorem 2.9) that a semi-hereditary ring with finitely many large ideals, has SP.

Left perfect rings have been studied by Bass [*1*]; the following theorem, contained in Bass' Theorem P, records all the information we need here:

THEOREM B (Bass). *For any ring R, the following statements are equivalent*:

(a) *R is left perfect.*

(b) *R satisfies the descending chain condition on principal right ideals.*

(c) *Every flat left R-module is projective.*

LEMMA 2.8. *Let R be a ring and I any* (*two-sided*) *ideal of R. If A is a flat right R-module then $A \otimes R/I \cong A/AI$ is a flat right R/I-module.*

Proof. For any left R/I-module C we have

$$\operatorname{Tor}_n^R(A, C) \cong \operatorname{Tor}_n^{R/I}(A \otimes R/I, C) \qquad n > 0$$

by [*2*], Chapt. VI, Prop. 4.1.1. Q.E.D.

THEOREM 2.9. *If R is a* (*commutative*) *semi-hereditary ring with finitely many large ideals $A_1, \ldots, A_n$, then R has* SP.

Proof. Let M and N be R-modules such that $Z(M) = M$ and $Z(N) = 0$. Set $I = \bigcap_{i=1}^n A_i$; it follows by [*17*], Prop. 1.2, that I is large and it is clear that $MI = (0)$. Furthermore N is R-flat [*4*], Theorem 5, and since R/I clearly has d.c.c. (in fact finitely many ideals) it follows from Lemma 2.8

and Theorem B (c) that N/NI is R/I-projective. R, now, has SP by (1), Corollary to Prop. 1.12 and Prop. 1.12(b). Q.E.D.

Remark. We do not know whether the converse of Theorem 2.9 also holds.

3. Rings with BSP

In this section we establish the following characterization:

THEOREM 3.1. *For a ring R with $Z(R) = (0)$, the following statements are equivalent:*

(a) *R has* BSP.

(b) *R is semi-hereditary and for every large ideal I of R, the ring R/I has* d.c.c.

The proof of (b) implies (a) is essentially contained in the proof of Theorem 2.9; observe that it suffices to show that $\mathrm{Ext}_R^1(N, M) = (0)$ for every pair of modules M and N such that $Z(N) = (0)$ and M is of bounded order, say I (Prop. 1.12(a)).

We postpone the proof of (a) implies (b) until some of the ideas involved have been sufficiently developed below.

Let T denote the set of nonzero divisors in R and K the total quotient ring of R [*4*].

DEFINITION 3.2. For any ideal I of R, let

$$I' = \{r \in R/rd \in I, \text{ for some } d \in T\}.$$

If M is an R-module we let $t(M) = \{m \in M/md = 0, \text{ for some } d \in T\}$, the usual torsion submodule of M. It follows easily from Definition 3.2 that $t(R/I) = I'/I$ for any ideal I.

LEMMA 3.3. *Let R be a ring and I an ideal of R. The following statements are, then, true:*

(a) *If R is semi-hereditary and $I = I'$, then $IJ = I \cap J$ for any other ideal J of R.*

(b) *If $I = I'$, then the sequence $0 \to R/I \to K/IK$ (canonical map) is exact and K/IK is an R/I-essential extension of R/I.*

(c) *$IK = I'K$.*

Proof. (a) R/I is torsion-free as an R-module, hence it is R-flat [*4*], Theorem 5. The sequence $0 \to R/I \otimes J \to R/I$, induced by the inclusion map $J \subseteq R$, is thus exact and since $R/I \otimes J \cong J/IJ$ we have $IJ = I \cap J$.

(b) It is well known (e.g. [*13*], Prop. 1.5) that if M is any torsion-free R-module, then $MK = \{md^{-1}/m \in M, d \in T\}$. Thus, let $x \in IK \cap R$; we have $x = ad^{-1}$, $a \in I$, $d \in T$ and from this $xd = a \in I$, hence $x \in I' = I$. We have the first assertion of (b). To show the second part, observe that if $ad^{-1} \in K - IK$, then $a \notin I$ and $d \notin I$; we thus have $(ad^{-1} + IK)(d + I) = (a + IK) \in Im(R/I \to K/IK) \cong R/I$ and $a \not\equiv 0 \pmod{I}$.

(c) Clearly $IK \subseteq I'K$. Let $xd^{-1} \in I'K$; there exists $t \in T$ such that $xt = a \in I$. Thus $x = at^{-1}$ and $xd^{-1} = a(td)^{-1} \in IK$. Q.E.D.

PROPOSITION 3.4. *Let R be a ring with $Z(R) = (0)$ and* BSP. *Let I be a large ideal of R. The following statements are, then, true*:

(a) *K has* SP.

(b) *If $I = I'$, then $R/I \cong K/IK$; in particular R/I has* d.c.c.

(c) *If $I \cap T \neq \emptyset$, then R/I has* d.c.c.

Proof. (a) R is semi-hereditary by Prop. 1.6. (remark) and the Corollary to Prop. 1.9; it follows from this that K is regular [*4*], 2. Now K satisfies the conditions of Prop. 1.9, it, hence, has BSP and by Theorem 2.1 (b) it has SP. In particular K/IK is a semi-simple ring.

(b) If J/I is any ideal of R/I, it follows from Lemma 3.3 (a) that $J/I \cong J \otimes R/I$; from this and the Corollary to Prop. 1.12 we see that J/I is R/I-projective, hence R/I is hereditary.

Now by Lemma 3.3 (b) and (a) above, K/IK is the maximal quotient ring of R/I; in particular R/I is a finite dimensional ring [*17*], Theorem 1.6. By a result of Hattori [7], Lemma 3, p. 156, R/I is a finite direct sum of Dedekind domains, say $R/I = D_1 \oplus \cdots \oplus D_n$ and thus $K/IK \cong Q_1 \oplus \cdots \oplus Q_n$, Q_i the quotient field of D_i. By the Corollary to Prop. 1.12, K/IK is R/I-projective and this clearly implies that Q_i is D_i-projective. The last condition is well known to imply that $D_i = Q_i$ for each i, hence $R/I \cong K/IK$. In particular R/I has d.c.c.

(c) Let $d \in I \cap T$; it follows by an argument due to Chase [*3*], Theorem 4.3 that R/dR has d.c.c. and this clearly implies that R/I has d.c.c. Q.E.D.

We can now prove:

(a) implies (b). Let I be a large ideal of R and $f: R/I \to K/IK$ the homomorphism defined by $f: r + I \to r + IK$. We claim that f is an epimorphism; this follows from the fact that f is the composition $R/I \xrightarrow{\nu} R/I' \xrightarrow{\eta} K/I'K$ where η is an isomorphism by Prop. 3.4 (b), ν is the natural epimorphism and $K/I'K = K/IK$ by Lemma 3.3 (c). Note that $\ker f = I'/I$ and K/IK is R/I-projective; it follows that the exact sequence $0 \to I'/I \to R/I \to K/IK \to 0$ splits. In particular I'/I is a cyclic R/I-module with the property that $t(I'/I) = I'/I$. This implies that $I'/I \cong R/J$ for some ideal J of R such that $J \cap T \neq \emptyset$. Now R/I has d.c.c. since both R/J and K/IK do, by Prop. 3.4. Q.E.D.

The proof of the theorem is now complete.

4. Direct Products of Hereditary Rings

An easy consequence of the Corollary to Lemma 1.5 and [*17*], Prop. 1.2 (4), is that if M is an R/I-module, where I is a closed ideal of R, then $Z(M_R) = Z(M_{R/I})$. It follows easily from this that a finite direct sum of rings has FGSP, BSP, SP if and only if each of the summands, correspondingly, has these properties. An infinite direct product of rings, however, does not preserve the last two. More generally we have the following:

Theorem 4.1. *Let $\{R_\alpha : \alpha \in \Gamma\}$ be an infinite collection of right hereditary rings R_α (with identity). Then, the direct product $\prod_{\alpha\in\Gamma} R_\alpha$ is not a right hereditary ring.*

Proof. It is sufficient to show the theorem in case card $\Gamma = \aleph_0$, since the direct product of countably many of the rings R_α is a direct summand of $\prod_{\alpha\in\Gamma} R_\alpha$ in an obvious manner. Set $\Lambda = \prod R_\alpha$ and $I = \sum_\alpha \oplus R_\alpha$; I is obviously a large (2-sided) ideal of Λ and it is countably generated over Λ.

Assume the theorem is false, hence Λ is hereditary. It follows from Theorem 2.4, that every right ideal of Λ/I is countably generated. By Tarski's Theorem T Λ has $2^{\aleph_0}$ idempotents $\{e_i\}$ whose image $\{\bar{e}_i\}$ in Λ/I is a set of $2^{\aleph_0}$ (distinct) orthogonal idempotents. The right ideal $\sum \bar{e}_i\Lambda$ of Λ/I is obviously not countably generated.

The theorem is, hence, true. Q.E.D.

We see from the proof of Theorem 4.1 that Λ/I does not have d.c.c., where the ideal I is large in Λ. From this and Theorem 3.1 we have easily:

Theorem 4.2. *Let $\{R_\alpha : \alpha \in \Gamma\}$ be an infinite collection of (commutative) rings R_α satisfying $Z(R_\alpha) = (0)$ and* BSP *for each $\alpha \in \Gamma$. Then the direct product $\prod_\alpha R_\alpha$ does not have* BSP.

In the case of FGSP an analogous theorem does not hold. Any infinite direct product of self-injective nonsingular rings has FGSP [*17*], Theorem 2.7.

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