View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

Available online at www.sciencedirect.com

ScienceDirect

Journal of Algebra 320 (2008) 1311–1325

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Combinatorial Gelfand models

Ron M. Adin [a,1], Alexander Postnikov [b], Yuval Roichman [a,*,1]

[a] *Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel*

[b] *Department of Applied Mathematics, Massachusetts Institute of Technology, MA 02139, USA*

Received 30 October 2007

Available online 9 May 2008

Communicated by Vera Serganova

Abstract

A combinatorial construction of a Gelfand model for the symmetric group and its Iwahori–Hecke algebra is presented.

© 2008 Elsevier Inc. All rights reserved.

Keywords: Symmetric group; Iwahori–Hecke algebra; Descents; Inversions; Character formulas; Gelfand model

1. Introduction

A complex representation of a group or an algebra A is called a *Gelfand model* for A, or simply a *model*, if it is equivalent to the multiplicity free direct sum of **all** A-irreducible representations.

Models (for compact Lie groups) were first constructed by Bernstein, Gelfand and Gelfand [7]. Constructions of models for the symmetric group, using induced representations from centralizers, were found by Klyachko [12,13] and by Inglis, Richardson and Saxl [10]; see also [2–5,19]. Our goal is to determine an explicit and simple combinatorial action which gives a model for the symmetric group and its Iwahori–Hecke algebra.

* Corresponding author.

E-mail addresses: radin@math.biu.ac.il (R.M. Adin), apost@math.mit.edu (A. Postnikov), yuvalr@math.biu.ac.il (Y. Roichman).

1 First and third authors supported in part by the Israel Science Foundation, founded by the Israel Academy of Sciences and Humanities.

0021-8693/$ – see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2008.03.030

1.1. Signed conjugation

Let S_n be the symmetric group on n letters, $S = \{s_1, \dots, s_{n-1}\}$ its set of simple reflections, $I_n = \{\pi \in S_n \mid \pi^2 = id\}$ its set of involutions, and $V_n := \mathrm{span}_{\mathbb{Q}}\{C_w \mid w \in I_n\}$ a vector space over $\mathbb{Q}$ formally spanned by the involutions.

Recall the standard length function on the symmetric group

$$\ell(\pi) := \min\bigl\{\ell \bigm| \pi = s_{i_1}s_{i_2}\cdots s_{i_\ell},\ s_{i_j} \in S\ (\forall j)\bigr\},$$

the descent set

$$\mathrm{Des}(\pi) := \bigl\{s \in S \bigm| \ell(\pi s) < \ell(\pi)\bigr\},$$

and the descent number $\mathrm{des}(\pi) := \#\mathrm{Des}(\pi)$.

Define a map $\rho : S \to GL(V_n)$ by

$$\rho(s)C_w := \mathrm{sign}(s; w) \cdot C_{sws} \quad (\forall s \in S,\ w \in I_n) \tag{1}$$

where

$$\mathrm{sign}(s; w) := \begin{cases} -1, & \text{if } sws = w \text{ and } s \in \mathrm{Des}(w); \\ 1, & \text{otherwise.} \end{cases} \tag{2}$$

Theorem 1.1. *ρ determines an S_n-representation.*

Theorem 1.2. *ρ determines a Gelfand model for S_n.*

1.2. Hecke algebra action

Consider $H_n(q)$, the Hecke algebra of the symmetric group S_n (say over the field $\mathbb{Q}(q^{1/2})$), with set of generators $\{T_i \mid 1 \leqslant i < n\}$ and defining relations

$$\begin{aligned} (T_i + q)(T_i - 1) &= 0 \quad (\forall i), \\ T_iT_j &= T_jT_i \quad \text{if } |i - j| > 1, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad (1 \leqslant i < n - 1). \end{aligned}$$

Note that some authors use a slightly different notation, with T_i consistently replaced by $-T_i$.

In order to construct an extended signed conjugation providing a model for $H_n(q)$, we extend the standard notions of length and weak order. Recall that the (right) weak order on S_n is the reflexive and transitive closure of the relation: $w \prec_R ws$ if $s \in S$ and $\ell(ws) = \ell(w) + 1$.

Definition 1.3. Define the *involutive length* of an involution $w \in I_n$ of cycle type 2^k1^{n-2k} as

$$\hat{\ell}(w) := \min\bigl\{\ell(v) \bigm| w = vs_1s_3\cdots s_{2k-1}v^{-1},\ v \in S_n\bigr\},$$

where $\ell(v)$ is the standard length of $v \in S_n$.

Define the *involutive weak order* $\leqslant_I$ on I_n as the reflexive and transitive closure of the relation: $w \prec_I sws$ if $s \in S$ and $\hat{\ell}(sws) = \hat{\ell}(w) + 1$.

Now define a map $\rho_q : S \to GL(V_n)$ by

$$\rho_q(T_s)C_w := \begin{cases} -qC_w, & \text{if } sws = w \text{ and } s \in \mathrm{Des}(w); \\ C_w, & \text{if } sws = w \text{ and } s \notin \mathrm{Des}(w); \\ (1-q)C_w + qC_{sws}, & \text{if } w <_I sws; \\ C_{sws}, & \text{if } sws <_I w. \end{cases} \tag{3}$$

Theorem 1.4. ρ_q *is a Gelfand model for* $H_n(q)$ (q *indeterminate*); *namely,*

(1) ρ_q *is an* $H_n(q)$*-representation.*
(2) ρ_q *is equivalent to the multiplicity free sum of all irreducible* $H_n(q)$*-representations.*

The proof involves Lusztig's version of Tits' deformation theorem [15]. For other versions of this theorem see [8, §4], [9, §68.A] and [6].

Let $\mu = (\mu_1, \mu_2, \dots, \mu_t)$ be a partition of n and let $a_j := \sum_{i=1}^{j} \mu_i$ $(0 \leqslant j \leqslant t)$. A permutation $\pi \in S_n$ is *μ-unimodal* if for every $0 \leqslant j < t$ there exists $1 \leqslant d_j \leqslant \mu_{j+1}$ such that

$$\pi_{a_j+1} < \pi_{a_j+2} < \cdots < \pi_{a_j+d_j} > \pi_{a_j+d_j+1} > \cdots > \pi_{a_{j+1}}.$$

The character of ρ_q may be expressed as a generating function for the descent number over μ-unimodal involutions.

Proposition 1.5.

$$\mathrm{Tr}\big(\rho_q(T_\mu)\big) = \sum_{\{w \in I_n \mid w \text{ is } \mu\text{-unimodal}\}} (-q)^{\mathrm{des}(w)}$$

where

$$T_\mu := T_1 T_2 \cdots T_{\mu_1 - 1} T_{\mu_1 + 1} \cdots T_{\mu_1 + \cdots + \mu_t - 1}$$

is the subproduct of $T_1 T_2 \cdots T_{n-1}$ *obtained by omitting* $T_{\mu_1 + \cdots + \mu_i}$ *for all* $1 \leqslant i < t$.

2. Proof of Theorem 1.1

2.1. First proof

This proof relies on a variant of the inversion number, which is introduced in this section. Recall the definition of the inversion set of a permutation $\pi \in S_n$,

$$\mathrm{Inv}(\pi) := \big\{\{i, j\} \bigm| (j - i) \cdot \big(\pi(j) - \pi(i)\big) < 0\big\}.$$

Definition 2.1. For an involution $w \in I_n$ let $\mathrm{Pair}(w)$ be the set of 2-cycles of w (considered as unordered 2-sets). For a permutation $\pi \in S_n$ and an involution $w \in I_n$ let

$$\mathrm{Inv}_w(\pi) := \mathrm{Inv}(\pi) \cap \mathrm{Pair}(w)$$

and

$$\mathrm{inv}_w(\pi) := \#\,\mathrm{Inv}_w(\pi).$$

Now redefine $\rho : S_n \to GL(V_n)$ by

$$\rho(\pi)C_w := (-1)^{\mathrm{inv}_w(\pi)} \cdot C_{\pi w \pi^{-1}} \quad (\forall \pi \in S_n,\ w \in I_n). \tag{4}$$

Note that for every Coxeter generator $s = (i, i+1) \in S$ and every involution $w \in I_n$,

$$\begin{aligned} \mathrm{inv}_w(s) &= \begin{cases} 1, & \text{if } w(i) = i+1; \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } sws = w \text{ and } s \in \mathrm{Des}(w); \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, definition (4) of ρ coincides on the Coxeter generators with the original definition (1). In order to prove that ρ is an S_n-representation it suffices to prove that ρ is a group homomorphism.

Indeed, for every pair of permutations $\sigma, \pi \in S_n$, every involution $w \in I_n$, and every $1 \leqslant i < j \leqslant n$,

$$\chi\big[\{i,j\} \in \mathrm{Inv}_w(\sigma\pi)\big] = \chi\big[\{i,j\} \in \mathrm{Inv}_w(\pi)\big] \cdot \chi\big[\big\{\pi(i), \pi(j)\big\} \in \mathrm{Inv}_{\pi w \pi^{-1}}(\sigma)\big],$$

where $\chi[\text{event}] := -1$ if the event holds and 1 otherwise. Hence, for every pair of permutations $\sigma, \pi \in S_n$ and every involution $w \in I_n$,

$$(-1)^{\mathrm{inv}_w(\sigma\pi)} = (-1)^{\mathrm{inv}_w(\pi)} \cdot (-1)^{\mathrm{inv}_{\pi w \pi^{-1}}(\sigma)},$$

and thus

$$\begin{aligned} \rho(\sigma\pi)C_w &= (-1)^{\mathrm{inv}_w(\sigma\pi)} \cdot C_{(\sigma\pi)w(\sigma\pi)^{-1}} \\ &= (-1)^{\mathrm{inv}_w(\pi)} \cdot (-1)^{\mathrm{inv}_{\pi w \pi^{-1}}(\sigma)} C_{\sigma(\pi w \pi^{-1})\sigma^{-1}} \\ &= (-1)^{\mathrm{inv}_w(\pi)} \cdot \rho(\sigma)(C_{\pi w \pi^{-1}}) = \rho(\sigma)\big(\rho(\pi)C_w\big). \end{aligned}$$

This proves that ρ is an S_n-representation, completing the proof of Theorem 1.1.

2.2. *Second proof*

In order to prove that ρ (defined on S) extends to an S_n-representation it suffices to verify the relations:

$$\rho(s)^2 = 1 \quad (\forall s \in S),$$

$$\rho(s)\rho(t) = \rho(t)\rho(s) \quad \text{if } st = ts,$$

$$\rho(s)\rho(t)\rho(s) = \rho(t)\rho(s)\rho(t) \quad \text{if } sts = tst.$$

We will prove the third relation. Verifying the other two relations is easier and will be left to the reader.

Let $s=(i,i+1)$ and $t=(i+1,i+2)$. For every permutation $\pi\in S_n$ let

$$\mathrm{Supp}(\pi):=\left\{i\in[n]\;\middle|\;\pi(i)\neq i\right\}.$$

Denote by $O(w)$ the orbit of an involution w under the conjugation action of $\langle s,t\rangle$, the subgroup of S_n generated by s and t. Since w is an involution $\#O(w)\neq 2$; hence there are three options $\#O(w)=1,3,6$.

Case (a). $\#O(w)=1$. Then $sws=w$ and $twt=w$. Furthermore, in this case $\mathrm{Supp}(w)\cap\{i,i+1,i+2\}=\emptyset$, so that $\mathrm{sign}(s;w)=\mathrm{sign}(t;w)=1$; thus $\rho(s)\rho(t)\rho(s)C_w=\rho(t)\rho(s)\rho(t)C_w=C_w$.

Case (b). $\#O(w)=3$. (This happens, for example, when $w=s$.) With no loss of generality there exists an element v in the orbit such that

$$v,tvt,stvts\quad\text{are distinct elements in the orbit,}$$

while

$$svs=v\quad\text{and}\quad t(stvts)t=stvts. \tag{5}$$

Thus

$$\rho(s)=\begin{pmatrix}x&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

and

$$\rho(t)=\begin{pmatrix}0&1&0\\1&0&0\\0&0&z\end{pmatrix},$$

where $x=\mathrm{sign}(s;v)$ and $z=\mathrm{sign}(t;stvts)$. $\rho(s)\rho(t)\rho(s)=\rho(t)\rho(s)\rho(t)$ holds if and only if $x=z$, which holds if and only if

$$s\in\mathrm{Des}(v)\quad\Longleftrightarrow\quad t\in\mathrm{Des}(stvts). \tag{6}$$

To prove this, observe that for every $w\in S_n$ and $s\in S$ the following holds:

(A) $sws=w$ and $s\notin\mathrm{Des}(w)$ if and only if $\mathrm{Supp}(w)\cap\mathrm{Supp}(s)=\emptyset$.
(B) $sws=w$ and $s\in\mathrm{Des}(w)$ if and only if $w=us$, where $\mathrm{Supp}(u)\cap\mathrm{Supp}(s)=\emptyset$.

Assuming $t\notin\mathrm{Des}(stvts)$ implies, by (5) and (A), that $\mathrm{Supp}(stvts)\cap\mathrm{Supp}(t)=\emptyset$. Hence

$$stvts(i+1)=i+1.$$

On the other hand, assuming $s \in \mathrm{Des}(v)$ implies, by (5) and (B), that there exists $u = vs$ with $i+1 \notin \mathrm{Supp}(u)$. Hence

$$stvts(i+1) = stusts(i+1) = i+2,$$

a contradiction. Similarly, assuming $s \notin \mathrm{Des}(v)$ and $t \in \mathrm{Des}(stvts)$ yields a contradiction (to verify this, replace v by $stvts$ and s by t). This completes the proof of Case (b).

Case (c). $\#O(w) = 6$ (this occurs, for example, when $s = (i, i+1), t = (i+1, i+2)$ and $w = (i, j)(i+1, k)$ where $j, k \neq i+2$). Then, for every element v in the orbit, $svs \neq v$ and $tvt \neq v$. It follows that

$$\rho(s)\rho(t)\rho(s)C_w = C_{stswsts} = C_{tstwtst} = \rho(t)\rho(s)\rho(t)C_w.$$

This completes the proof of the third relation.

3. Characters

3.1. Character formula

The following classical result follows from the work of Frobenius and Schur, see [11, §4] and [21, §7, Example 69].

Theorem 3.1. *Let G be a finite group, for which every complex representation is equivalent to a real representation. Then for every $w \in G$*

$$\sum_{\chi \in G^*} \chi(w) = \#\{u \in G \mid u^2 = w\},$$

where G^* denotes the set of the irreducible characters of G.

It is well known [20] that all complex representations of a Weyl group are equivalent to rational representations. In particular, Theorem 3.1 holds for $G = S_n$. One concludes

Corollary 3.2. *Let $\pi \in S_n$ have cycle structure $1^{d_1}2^{d_2}\cdots n^{d_n}$. Then*

$$\sum_{\chi \in {S_n}^*} \chi(\pi) = \prod_{r=1}^{n} f(r, d_r),$$

where

$$f(r, d_r) := \begin{cases} 0, & \text{if } r \text{ is even and } d_r \text{ is odd;} \\ \binom{d_r}{2,\dots,2} \cdot r^{d_r/2}, & \text{if } r \text{ and } d_r \text{ are even;} \\ \sum_{k=0}^{\lfloor d_r/2 \rfloor} \binom{d_r}{d_r-2k,2,2,\dots,2} \cdot r^k, & \text{if } r \text{ is odd.} \end{cases}$$

In particular, $f(r, 0) = 1$ for all r.

Proof. For every $A \subseteq [n]$ let

$$S_A := \{\pi \in S_n \mid \mathrm{Supp}(\pi) \subseteq A\}$$

be the subgroup of S_n consisting of all the permutations whose support is contained in A. For every $\pi \in S_n$ and $1 \leqslant r \leqslant n$ let $A(\pi, r) \subseteq [n]$ be the set of all letters which appear in cycles of length r in π. In other words,

$$A(\pi, r) := \{i \in [n] \mid \pi^r(i) = i \text{ and } (\forall j < r)\ \pi^j(i) \neq i\}.$$

For example, $A(\pi, 1)$ is the set of fixed points of π.

Denote by $\pi_{|r}$ the restriction of π to $A(\pi, r)$. Then $\pi_{|r}$ may be considered as a permutation in $S_{A(\pi,r)}$.

Observation 3.3. *For every $\pi \in S_n$*

$$\{u \in S_n \mid u^2 = \pi\} = \prod_{r \geqslant 1} \{u_r \in S_{A(\pi,r)} \mid u_r^2 = \pi_{|r}\}.$$

Observation 3.4. *Let $\pi \in S_n$ have cycle type $r^{n/r}$. Then*

$$\#\{u \in S_n \mid u^2 = \pi\} = \begin{cases} 0, & \text{if } r \text{ is even and } n/r \text{ is odd;} \\ \binom{n/r}{2,\ldots,2} \cdot r^{n/2r}, & \text{if } r \text{ and } n/r \text{ are even;} \\ \sum_{k=0}^{\lfloor n/2r \rfloor} \binom{n/r}{n/r-2k,2,2,\ldots,2} \cdot r^k, & \text{if } r \text{ is odd.} \end{cases}$$

Combining these observations with Theorem 3.1 implies Corollary 3.2. □

3.2. Proof of Theorem 1.2

We shall compute the character of the representation ρ and compare it with Corollary 3.2. By (4),

$$\mathrm{Tr}(\rho(\pi)) = \sum_{w \in I_n \cap \mathrm{St}_n(\pi)} (-1)^{\mathrm{inv}_w(\pi)},$$

where $\mathrm{St}_n(\pi)$ is the stabilizer of π under the conjugation action of S_n (i.e., the centralizer of π in S_n).

Observation 3.5. *Let $\pi \in S_n$, $w \in I_n \cap \mathrm{St}_n(\pi)$ and $a_1 \in [n]$ any letter. Then one of the following holds:*

(1) *$(a_1, a_2, \ldots, a_r)$ is a cycle in π $(r \geqslant 1)$; $a_1, a_2, \ldots, a_r$ are fixed points of w.*
(2) *$(a_1, a_2, \ldots, a_r)$ and $(a_{r+1}, \ldots, a_{2r})$ are cycles in π $(r \geqslant 1)$; (a_1, a_{r+1}), (a_2, a_{r+2}), $\ldots$, (a_r, a_{2r}) are cycles in w.*
(3) *$(a_1, a_2, \ldots, a_{2m})$ is a cycle in π $(m \geqslant 1)$; (a_1, a_{m+1}), (a_2, a_{m+2}), $\ldots$, (a_m, a_{2m}) are cycles in w.*

It follows that

Corollary 3.6. *Fix $\pi \in S_n$. Each $w \in I_n \cap \mathrm{St}_n(\pi)$ has a unique decomposition*

$$w = \prod_{r \geqslant 1} w_r,$$

where

$$w_r \in I_{S_{A(\pi,r)}} \cap \mathrm{St}_{S_{A(\pi,r)}}(\pi_{|r}) \quad (\forall r)$$

and $A(\pi, r)$, $\pi_{|r}$ and $S_{A(\pi,r)}$ are defined as in the proof of Corollary 3.2; *and*

$$\mathrm{Inv}_w(\pi) = \bigcup_{r \geqslant 1} \mathrm{Inv}_{w_r}(\pi_{|r}),$$

a disjoint union.

Hence, it suffices to prove that $\mathrm{Tr}(\rho(\pi))$ is equal to the right hand side of the formula in Corollary 3.2, for π of cycle type $r^{n/r}$. Since ρ is a class function, we may assume that

$$\pi = (1, 2, \dots, r)(r+1, \dots, 2r) \cdots (n-r+1, n-r+2, \dots, n). \tag{7}$$

Observation 3.7. *Let r be a positive integer.*

(1) *If i and j are distinct nonnegative integers, π as in* (7) *above, and $w = (ir+1, jr+\sigma(1))(ir+2, jr+\sigma(2)) \cdots (ir+r, jr+\sigma(r))$ (where σ is some power of the cyclic permutation $(1, 2, \dots, r)$), then*

$$(-1)^{\mathrm{inv}_w(\pi)} = 1.$$

(2) *If $r = 2m$ is even, π as in* (7) *above, and $w = (1, m+1)(2, m+2) \cdots (m, 2m)$, then*

$$(-1)^{\mathrm{inv}_w(\pi)} = -1.$$

Lemma 3.8. *For every odd r and a permutation π as in* (7) *above,*

$$\sum_{w \in I_n \cap \mathrm{St}_n(\pi)} (-1)^{\mathrm{inv}_w(\pi)} = \#\big(I_n \cap \mathrm{St}_n(\pi)\big) = \sum_{k=0}^{\lfloor n/2r \rfloor} \binom{n/r}{n/r - 2k, 2, 2, \dots, 2} \cdot r^k.$$

Proof. If r is odd then only cases (1) and (2) in Observation 3.5 are possible. The first equality in the statement of the lemma then follows from Observation 3.7(1). The second equality follows from Observation 3.5(1), (2), counting the involutions $w \in I_n \cap \mathrm{St}_n(\pi)$ with $\#\,\mathrm{Supp}(w) = 2rk$. $\square$

Lemma 3.9. *For every even r and a permutation π as in* (7) *above,*

$$\sum_{w\in I_n\cap \mathrm{St}_n(\pi)} (-1)^{\mathrm{inv}_w(\pi)} = \begin{cases} 0, & \text{if } n/r \text{ is odd;} \\ \binom{n/r}{2,\dots,2}\cdot r^{n/2r}, & \text{if } n/r \text{ is even.} \end{cases}$$

Proof. Let $c_i = (ir+1, ir+2, \dots, ir+r)$ be one of the cycles of π, as in (7). By Observation 3.5, an involution $w \in I_n \cap \mathrm{St}_n(\pi)$ has one of the following three types with respect to c_i:

Type (1): Each element of c_i is a fixed point of w.
Type (2): w maps c_i onto a different cycle of π.
Type (3): $r = 2m$ is even, and c_i is a union of 2-cycles of w:

$$\{ir+t, ir+t+m\} \in \mathrm{Pair}(w) \quad (1 \leqslant t \leqslant m).$$

Denote

$$P_2 := \bigl\{w \in I_n \cap \mathrm{St}_n(\pi) \bigm| w \text{ is of type (2) w.r.t. all cycles of } \pi\bigr\}.$$

For any $w \in (I_n \cap \mathrm{St}_n(\pi)) \setminus P_2$, let

$$i(w) := \min\bigl\{i \bigm| w \text{ is of type (1) or (3) w.r.t. the cycle } c_i\bigr\}.$$

Denote

$$P_1 := \bigl\{w \in \bigl(I_n \cap \mathrm{St}_n(\pi)\bigr) \setminus P_2 \bigm| w \text{ is of type (1) w.r.t. the cycle } c_{i(w)}\bigr\}$$

and

$$P_3 := \bigl\{w \in \bigl(I_n \cap \mathrm{St}_n(\pi)\bigr) \setminus P_2 \bigm| w \text{ is of type (3) w.r.t. the cycle } c_{i(w)}\bigr\}.$$

The map $\varphi : P_1 \to P_3$ which changes the action of w on $c_{i(w)}$ from type (1) to type (3) is clearly a well-defined bijection; and, by Observation 3.7(2), it reverses the sign of $(-1)^{\mathrm{inv}_w(\pi)}$. The contributions of P_1 and P_3 to the sum therefore cancel each other. Each element of the remaining set P_2 contributes 1, by Observation 3.7(1). Lemma 3.9 follows. $\square$

Lemmas 3.8 and 3.9 complete the proof of Theorem 1.2.

4. The Hecke algebra

4.1. A combinatorial lemma

Recall Definition 1.3. In order to prove Theorem 1.4 we need the following combinatorial interpretation of the involutive length $\hat{\ell}$.

Lemma 4.1. *Let $w \in S_n$ be an involution of cycle type $2^k 1^{n-2k}$. Then*

$$\hat{\ell}(w) := \Bigl[\sum_{t\in \mathrm{Supp}(w)} t - \binom{2k+1}{2}\Bigr] + \frac{1}{2}\bigl[\mathrm{inv}(w_{|\mathrm{Supp}(w)}) - k\bigr]. \tag{8}$$

Proof. Denote the right hand side of (8) by $f(w)$. It is easy to verify that $f(w)=0$ when $\hat{\ell}(w)=0$, i.e., when $w=s_1s_3\cdots s_{2k-1}$. Let u and $v=s_ius_i$ be involutions in S_n with $\hat{\ell}(v)=\hat{\ell}(u)+1$. Then $|\{i,i+1\}\cap \mathrm{Supp}(u)|>0$. If $|\{i,i+1\}\cap \mathrm{Supp}(u)|=1$ then

$$\sum_{t\in \mathrm{Supp}(v)} t-\sum_{t\in \mathrm{Supp}(u)} t=\pm 1$$

and $\mathrm{inv}(v_{|\,\mathrm{Supp}(v)})=\mathrm{inv}(u_{|\,\mathrm{Supp}(u)})$. If $|\{i,i+1\}\cap \mathrm{Supp}(u)|=2$ then

$$\sum_{t\in \mathrm{Supp}(v)} t=\sum_{t\in \mathrm{Supp}(u)} t$$

and $\mathrm{inv}(v_{|\,\mathrm{Supp}(v)})-\mathrm{inv}(u_{|\,\mathrm{Supp}(u)})\in\{2,0,-2\}$. Thus in both cases $|f(v)-f(u)|\leqslant 1$. This proves, by induction on $\hat{\ell}$, that $f(w)\leqslant\hat{\ell}(w)$ for every involution w.

On the other hand, if w is an involution with $f(w)>0$ then either $\sum_{t\in\mathrm{Supp}(w)} t>\binom{2k+1}{2}$, or $\sum_{t\in\mathrm{Supp}(w)} t=\binom{2k+1}{2}$ and $\mathrm{inv}(w_{|\,\mathrm{Supp}(w)})>k$. In the first case there exists $i+1\in\mathrm{Supp}(w)$ such that $i\notin\mathrm{Supp}(w)$. Then $f(s_iws_i)=f(w)-1$. In the second case $\mathrm{Supp}(w)=\{1,\dots,2k\}$. Since $\mathrm{inv}(w_{|\,\mathrm{Supp}(w)})>k$, $w\neq s_1s_3\cdots s_{2k-1}$. Thus there must be a minimal i such that $w(i)>i+1$. Let $j:=w(i)-1$; then $w(j)>w(j+1)=i$, so $f(s_jws_j)=f(w)-1$. We conclude that $\hat{\ell}(w)\leqslant f(w)$ for every involution w. $\square$

4.2. Proof of Theorem 1.4

The proof consists of two parts. In the first part we prove that ρ_q is an $H_n(q)$-representation by verifying the defining relations along the lines of the second proof of Theorem 1.1. In the second part we apply Lusztig's version of Tits' deformation theorem to prove that ρ_q is a Gelfand model.

Part 1: Proof of Theorem 1.4(1). First, consider the braid relation $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$. To verify this relation observe that there are six possible types of orbits of an involution w under conjugation by $\langle s_i,s_{i+1}\rangle$, the subgroup of S_n generated by s_i and s_{i+1}. These orbits differ by the action of w on the letters $i,i+1,i+2$:

1. $i,i+1,i+2\notin\mathrm{Supp}(w)$.
2. Exactly one of the letters $i,i+1,i+2$ is in $\mathrm{Supp}(w)$.
3. Exactly two of the letters $i,i+1,i+2$ are in $\mathrm{Supp}(w)$, and these two letters form a 2-cycle in w.
4. Exactly two of the letters $i,i+1,i+2$ are in $\mathrm{Supp}(w)$, and these two letters do not form a 2-cycle in w.
5. $i,i+1,i+2\in\mathrm{Supp}(w)$, and two of these letters form a 2-cycle in w.
6. $i,i+1,i+2\in\mathrm{Supp}(w)$, and no two of these letters form a 2-cycle in w.

Note that an orbit of the first type is of order one; orbits of the second, third and fifth type are of order three; and orbits of the fourth and sixth type are of order six. Moreover, by Lemma 4.1, orbits of the same order form isomorphic intervals in the weak involutive order (see Definition 1.3).

In particular, all orbits of order six have a representative w of minimal involutive length, such that the orbit has the form:

$$\begin{array}{ccccc} & & s_i s_{i+1} s_i w s_i s_{i+1} s_i & & \\ & \swarrow & & \searrow & \\ s_i s_{i+1} w s_{i+1} s_i & & & & s_{i+1} s_i w s_i s_{i+1} \\ \downarrow & & & & \downarrow \\ s_{i+1} w s_{i+1} & & & & s_i w s_i \\ & \searrow & & \swarrow & \\ & & w & & \end{array}. \tag{9}$$

All orbits of order three are linear posets:

$$w <_I s_i w s_i <_I s_{i+1} s_i w s_i s_{i+1} \tag{10}$$

or

$$w <_I s_{i+1} w s_{i+1} <_I s_i s_{i+1} w s_{i+1} s_i. \tag{11}$$

Thus the analysis is reduced into three cases.

Case (a). An orbit of order six. By (3) and (9), the representation matrices of the generators with respect to the ordered basis C_w, $C_{s_i w s_i}$, $C_{s_{i+1} s_i w s_i s_{i+1}}$, $C_{s_i s_{i+1} s_i w s_i s_{i+1} s_i}$, $C_{s_{i+1} w s_{i+1}}$, $C_{s_i s_{i+1} w s_{i+1} s_i}$ are:

$$\rho_q(T_i) = \begin{pmatrix} 1-q & 1 & 0 & 0 & 0 & 0 \\ q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q & 1 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-q & 1 \\ 0 & 0 & 0 & 0 & q & 0 \end{pmatrix}$$

and

$$\rho_q(T_{i+1}) = \begin{pmatrix} 1-q & 0 & 0 & 0 & 1 & 0 \\ 0 & 1-q & 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q \\ q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1-q \end{pmatrix}.$$

It is easy to verify that indeed

$$\rho_q(T_i)\rho_q(T_{i+1})\rho_q(T_i) = \rho_q(T_{i+1})\rho_q(T_i)\rho_q(T_{i+1}).$$

Case (b). An orbit of order three. Without loss of generality, the orbit is of type (10); the analysis of type (11) is analogous. Then $s_{i+1} w s_{i+1} = w$ and $s_i (s_{i+1} s_i w s_i s_{i+1}) s_i = s_{i+1} s_i w s_i s_{i+1}$. By (6), $s_{i+1} \in \mathrm{Des}(w)$ if and only if $s_i \in \mathrm{Des}(s_{i+1} s_i w s_i s_{i+1})$, see second proof of Theorem 1.1.

Given the above, by (3), the representation matrices of the generators with respect to the ordered basis $w <_I s_i w s_i <_I s_{i+1} s_i w s_i s_{i+1}$ are

$$\rho_q(T_i) = \begin{pmatrix} 1-q & 1 & 0 \\ q & 0 & 0 \\ 0 & 0 & x \end{pmatrix}$$

and

$$\rho_q(T_{i+1}) = \begin{pmatrix} x & 0 & 0 \\ 0 & 1-q & 1 \\ 0 & q & 0 \end{pmatrix},$$

where $x \in \{1, -q\}$. These matrices satisfy the required braid relation.

Case (c). An orbit of order one. Then $s_i w s_i = w$, $s_{i+1} w s_{i+1} = w$ and $s_i, s_{i+1} \notin \mathrm{Des}(w)$. By (3), $\rho_q(T_i)\rho_q(T_{i+1})\rho_q(T_i)C_w = \rho_q(T_{i+1})\rho_q(T_i)\rho_q(T_{i+1})C_w = C_w$, completing the proof of the third relation.

The proof of the other two relations is easier and will be left to the reader.

Part 2: Proof of Theorem 1.4(2). Consider the Hecke algebra $H_n(q)$ as the algebra over $\mathbb{Q}(q^{1/2})$ spanned by $\{T_v \mid v \in S_n\}$ with the multiplication rules

$$T_v T_u = T_{vu} \quad \text{if } \ell(vu) = \ell(v) + \ell(u)$$

and

$$(T_s + q)(T_s - 1) = 0 \quad (\forall s \in S).$$

By Lusztig's version of Tits' deformation theorem [15, Theorem 3.1], the group algebra of S_n over $\mathbb{Q}(q^{1/2})$ may be embedded in $H_n(q)$. In particular, every element $w \in S_n$ may be expressed as a linear combination

$$w = \sum_{v \in S_n} m_{v,w}\big(q^{1/2}\big) T_v,$$

where $m_{v,w}$ is a rational function of $q^{1/2}$.

It follows that ρ_q may be considered as an S_n-representation, via

$$\rho_q(w) := \sum_{v \in S_n} m_{v,w}\big(q^{1/2}\big)\rho_q(T_v) \quad (\forall w \in S_n).$$

The resulting character values $\rho_q(w)$ are rational functions of $q^{1/2}$. By discreteness of the S_n character values, each such function is locally constant wherever it is defined, and is thus constant globally.

By Theorem 1.2, $\rho_q|_{q=1} = \rho$ is a model for the group algebra of S_n. This completes the proof.

4.3. Proof of Proposition 1.5

Let SYT_n be the set of all standard Young tableaux of order n, and let $\mathrm{SYT}(\lambda) \subseteq \mathrm{SYT}_n$ be the subset of standard Young tableaux of shape λ. For each partition λ of n, fix a standard Young tableau $P_\lambda \in \mathrm{SYT}(\lambda)$. By [18, Theorem 4], the value of the irreducible $H_n(q)$-character χ_q^λ at T_μ is

$$\chi_q^\lambda(T_\mu) = \sum_{\{w \mapsto (P_\lambda, Q) \mid w \text{ is } \mu\text{-unimodal and } Q \in \mathrm{SYT}(\lambda)\}} (-q)^{\mathrm{des}(w)},$$

where the sum runs over all permutations $w \in S_n$ which are mapped under the Robinson–Schensted (RS) correspondence to (P_λ, Q) for some $Q \in \mathrm{SYT}(\lambda)$. By [21, Lemma 7.23.1], the descent set of $w \in S_n$, which is mapped under RS to (P_λ, Q), is determined by Q. Hence

$$\begin{aligned}
\operatorname{Tr} \rho_q(T_\mu) &= \sum_\lambda \chi_q^\lambda(T_\mu) = \sum_\lambda \sum_{\{w \mapsto (P_\lambda, Q) \mid w \text{ is } \mu\text{-unimodal and } Q \in \mathrm{SYT}(\lambda)\}} (-q)^{\mathrm{des}(w)} \\
&= \sum_\lambda \sum_{\{w \mapsto (Q, Q) \mid w \text{ is } \mu\text{-unimodal and } Q \in \mathrm{SYT}(\lambda)\}} (-q)^{\mathrm{des}(w)} \\
&= \sum_{\{w \mapsto (Q, Q) \mid Q \in \mathrm{SYT}_n \text{ and } w \text{ is } \mu\text{-unimodal}\}} (-q)^{\mathrm{des}(w)} = \sum_{\{w \in I_n \mid w \text{ is } \mu\text{-unimodal}\}} (-q)^{\mathrm{des}(w)}.
\end{aligned}$$

The last equality follows from the well-known property of the RS correspondence: $w \mapsto (P, Q)$ if and only if $w^{-1} \mapsto (Q, P)$ [21, Theorem 7.13.1]. Thus w is an involution if and only if $w \mapsto (Q, Q)$ for some $Q \in \mathrm{SYT}_n$.

5. Remarks and questions

5.1. Classical Weyl groups

Let B_n be the Weyl group of type B, S^B its set of simple reflections, I_n^B its set of involutions, and $V_n^B := \mathrm{span}_{\mathbb{Q}}\{C_w \mid w \in I_n^B\}$ a vector space over $\mathbb{Q}$ formally spanned by the involutions. Recall that $B_n = \mathbb{Z}_2 \wr S_n$, so that each element $w \in B_n$ is identified with a pair (v, σ), where $v \in \mathbb{Z}_2^n$ and $\sigma \in S_n$. Denote $|w| := \sigma$.

Define a map $\rho^B : S^B \to GL(V_n)$ by

$$\rho^B(s) C_w := \mathrm{sign}(s; w) \cdot C_{sws} \quad \left(\forall s \in S^B,\ w \in I_n^B\right)$$

where, for $s = s_0 = ((1, 0, \dots, 0), id)$, the exceptional Coxeter generator, the sign is

$$\mathrm{sign}(s_0; w) := \begin{cases} -1, & \text{if } sws = w \text{ and } s_0 \in \mathrm{Des}(w); \\ 1, & \text{otherwise,} \end{cases}$$

and for a generator $s \neq s_0$ the sign is

$$\mathrm{sign}(s; w) := \begin{cases} -1, & \text{if } sws = w \text{ and } s \in \mathrm{Des}(|w|); \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 5.1. *ρ^B is a Gelfand model for B_n.*

A proof is given in [1].

Models for classical Weyl groups of type D_n for odd n were constructed in [4,5]. These constructions fail for even n. A natural question is whether there exists a signed conjugation (or a representation of type $\rho_s C_w = a_{s,w} C_w + b_{s,w} C_{sws}$) which gives a model for D_{2n}. It is also desired to find representation matrices for the models of the Hecke algebras of types B and D which specialize at $q = 1$ to models of the corresponding group algebra.

We conclude with the following questions regarding an arbitrary Coxeter group W.

Question 5.2. *Find a signed conjugation which gives a Gelfand model for W; Find a representation of the form $\rho_s C_w = a_{s,w} C_w + b_{s,w} C_{sws}$, which gives a Gelfand model for the Hecke algebra of W.*

Question 5.3. *Find a character formula for the Gelfand model of the Hecke algebra of W.*

Appendix A

This appendix was added in proof.

First, it should be acknowledged that an equivalent reformulation of Theorem 1.2, with a different proof, was given by Kodiyalam and Verma [14].

A third proof of Theorem 1.2, along the lines of [10], was suggested by an anonymous referee. Here is a brief outline.

Let $\chi^{\emptyset,(n)}$ denote the one dimensional character of B_n given by the parity of the number of negative signs, and consider the natural embedding of $B_n = \mathbb{Z}_2 \wr S_n$ into S_{2n}. Then

$$\chi^{\emptyset,(n)} \uparrow_{B_n}^{S_{2n}} = \sum_{\lambda \vdash n} \chi^{(2\cdot\lambda)'},$$

where the sum on the right hand side runs through all partitions of $2n$ with even columns only. See, for example, [16, Chapter I, §8, Example 6, and Chapter VII (2.4)]. Combining this with the Littlewood–Richardson rule implies that

$$\left(\left(\chi^{\emptyset,(k)} \uparrow_{B_k}^{S_{2k}}\right) \otimes 1_{S_{n-2k}}\right) \uparrow_{S_{2k} \times S_{n-2k}}^{S_n}$$

is a multiplicity free sum of all irreducible Specht modules indexed by partitions with exactly $n - 2k$ odd columns.

A natural basis for this representation is given by involutions with $n - 2k$ fixed points. Finally, it is straightforward to show that the action of a Coxeter generator s_i on this basis is identical with the signed conjugation defined in (2).

Corollary A.1. *The signed conjugation ρ, when restricted to the conjugacy class of involutions with $n - 2k$ fixed points, is a multiplicity free sum of all irreducible Specht modules indexed by partitions with exactly $n - 2k$ odd columns.*

This gives an algebraic proof to the following combinatorial result: The number of involutions with $n - 2k$ fixed points is equal to the number of standard Young tableaux of shapes with exactly $n - 2k$ odd columns.

Another proof for this enumerative fact may be obtained using the Robinson–Schensted (RS) correspondence. One concludes that the restriction of the signed conjugation ρ to conjugacy classes of involutions is compatible with the RS correspondence; namely, S^λ is a factor of the restriction of ρ to the conjugacy class of cycle type $2^k 1^{n-2k}$ if and only if λ is the shape of some pair of (equal) standard Young tableaux corresponding to an involution of this cycle type.

Acknowledgments

The authors thank Arkady Berenstein, Steve Shnider and Richard Stanley for stimulating discussions and references, and Anna Melnikov for inspiration [17]. We also thank an anonymous referee for his contributions and suggestions.

References

[1] R.M. Adin, A. Postnikov, Y. Roichman, A Gelfand model for wreath products, preprint, 2008.
[2] J.L. Aguado, J.O. Araujo, A Gelfand model for the symmetric group, Comm. Algebra 29 (2001) 1841–1851.
[3] J.O. Araujo, A Gelfand model for a Weyl group of type B_n, Beiträge Algebra Geom. 44 (2003) 359–373.
[4] J.O. Araujo, J.J. Bigeón, A Gelfand model for a Weyl group of type D_n and the branching rules $D_n \hookrightarrow B_n$, J. Algebra 294 (2005) 97–116.
[5] R.W. Baddeley, Models and involution models for wreath products and certain Weyl groups, J. London Math. Soc. (2) 44 (1991) 55–74.
[6] C.T. Benson, C.W. Curtis, On the degrees and rationality of certain characters of finite Chevalley groups, Trans. Amer. Math. Soc. 165 (1972) 251–273, Trans. Amer. Math. Soc. 202 (1975) 405.
[7] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Models of representations of compact Lie groups, Funktsional. Anal. i Prilozhen. 9 (1975) 61–62 (in Russian).
[8] N. Bourbaki, Lie Groups and Lie Algebras, English translation by Andrew Pressley, Springer, 2002.
[9] C.W. Curtis, I. Reiner, Methods of Representation Theory, vol. II. With Applications to Finite Groups and Orders, Pure Appl. Math. (N. Y.), A Wiley–Interscience Publication/John Wiley & Sons, Inc., New York, 1987.
[10] N.F.J. Inglis, R.W. Richardson, J. Saxl, An explicit model for the complex representations of S_n, Arch. Math. (Basel) 54 (3) (1990) 258–259.
[11] I.M. Isaacs, Character Theory of Finite Groups, Dover, New York, 1994.
[12] A.A. Klyachko, Models for complex representations of the groups $\mathrm{GL}(n,q)$ and Weyl groups, Dokl. Akad. Nauk SSSR 261 (1981) 275–278 (in Russian).
[13] A.A. Klyachko, Models for complex representations of groups $\mathrm{GL}(n,q)$, Mat. Sb. (N.S.) 120 (162) (1983) 371–386 (in Russian).
[14] V. Kodiyalam, D.-N. Verma, A natural representation model for symmetric groups, preprint, 2004.
[15] G. Lusztig, On a theorem of Benson and Curtis, J. Algebra 71 (1981) 490–498.
[16] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford Math. Monogr., Oxford Univ. Press, Oxford, 1995.
[17] A. Melnikov, B-orbits of nilpotent order 2 and link patterns, preprint, 2007.
[18] Y. Roichman, A recursive rule for Kazhdan–Lusztig characters, Adv. Math. 129 (1997) 24–45.
[19] P.D. Ryan, Representations of Weyl groups of type B induced from centralisers of involutions, Bull. Austral. Math. Soc. 44 (1991) 337–344.
[20] T.A. Springer, A construction of representations of Weyl groups, Invent. Math. 44 (1978) 279–293.
[21] R.P. Stanley, Enumerative Combinatorics, vol. II, Cambridge Univ. Press, Cambridge, 1999.