The Weak Convergence for Functions of Negatively Associated Random Variables

Li-Xin Zhang

Zhejiang University, Hangzhou, People’s Republic of China
E-mail: lxzhang@mail.hz.zj.cn

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Let \( \{X_n, n \geq 1\} \) be a sequence of stationary negatively associated random variables, \( S_j(l) = \sum_{i=1}^j X_{i+l} \), \( S_n = \sum_{i=1}^n X_i \). Suppose that \( f(x) \) is a real function. Under some suitable conditions, the central limit theorem and the weak convergence for sums

\[
\sum_{j=1}^n f\left( \frac{S_j(l)}{\sqrt{n}} \right), \quad n \geq 1,
\]

are investigated. Applications to limiting distributions of estimators of \( \text{Var} S_n \) are also discussed.

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1. INTRODUCTION AND MAIN RESULTS

Let \( \{X_n; n \geq 1\} \) be a sequence of random variables. \( \{X_n; n \geq 1\} \) is said to be positively associated (PA) if, for any finite subset \( A, B \) of \( \{1, 2, \ldots\} \) and any coordinatewise monotonically increasing \( f, g \), we have

\[
\text{Cov}\{f(X_i; i \in A), g(X_j; j \in B)\} \geq 0,
\]

while \( \{X_n; n \geq 1\} \) is said to be negatively associated (NA) if, for any disjoint finite subset \( A, B \) of \( \{1, 2, \ldots\} \) and any coordinatewise monotonically increasing \( f, g \), we have

\[
\text{Cov}\{f(X_i; i \in A), g(X_j; j \in B)\} \leq 0.
\]

Let \( S_n = \sum_{i=1}^n X_i \) be the partial sum of \( \{X_n; n \geq 1\} \). Throughout this paper the sequence \( \{X_n; n \geq 1\} \) is always assumed to be (strongly) stationary with \( \text{E}X_1 = \mu, \text{E}X_1^2 < \infty \) unless it is specially mentioned. Newman and Wright

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[5] obtained the following theorem on the central limit theorem and weak convergence for the partial sums of a sequence of PA random variables.

**Theorem A.** Let \( \{X_n; n \geq 1\} \) be a sequence of stationary PA random variables with \( \mathbb{E}X_1 = \mu \) and \( \mathbb{E}X_1^2 < \infty \). Let

\[
\sigma^2 := \text{Var} X_1 + 2 \sum_{j=2}^{\infty} \text{Cov}(X_1, X_j).
\]

Suppose \( \sum_{j=2}^{\infty} \text{Cov}(X_1, X_j) < \infty \). Then \( (\text{Var} S_n)/n \to \sigma^2 \) and

\[
\frac{S_n - n\mu}{\sqrt{n}} \overset{D}{\to} N(0, \sigma^2),
\]

\[
W_n \Rightarrow W(\cdot),
\]

where \( W_n(t) = (1/\sqrt{n}) \sum_{j=1}^{[nt]} (X_j - \mathbb{E}X_j), 0 \leq t \leq 1; \{W(t); t \geq 0\} \) is a standard Wiener process, \( N(0, \sigma^2) \) is a normal random variable with mean zero and variance \( \sigma^2 \), and \( \Rightarrow \) denotes the weak convergence in the space \( D[0, 1] \) with Skorohod topology.

Su et al. (1997) obtained a similar theorem on the sequence of NA random variables.

**Theorem B.** Let \( \{X_n; n \geq 1\} \) be a sequence of stationary NA random variables with \( \mathbb{E}X_1 = \mu \) and \( \mathbb{E}X_1^2 < \infty \). Suppose that \( \sigma^2 \) is defined as in (1.1). Then \( (\text{Var} S_n)/n \to \sigma^2 \) and (1.2) holds.

**Remark.** If \( \sigma^2 > 0 \), then Theorem B just is Theorems 3 and 4 of Su et al. (1997). If \( \sigma^2 = 0 \), then Theorem B is obvious. By our Lemmas 1.1 and 1.2 below, if \( \{X_n; n \geq 1\} \) is a sequence of stationary NA random variables with \( \mathbb{E}X_1^2 < \infty \), then one has \( 0 \leq \sigma^2 \leq \text{Var} X_1 \).

Let \( f(x) \) be a real function. If \( f(x) \) is not a monotone function, then usually the sequence \( \{f(X_n); n \geq 1\} \) is not a NA or PA sequence. Studying its central limit theorems and weak convergence becomes difficult. But many statistics, for example, the kernel estimates, can be written as a form of the partial sums of \( \{f(X_n); n \geq 1\} \). So, studying the limit theorems of this kind of sequence is of interest. The purpose of this paper is to investigate the central limit theorem and weak convergence of the sums

\[
\sum_{j=1}^{n} f(X_j), \quad n \geq 1.
\]
Generally, we study the sums
\[ \sum_{j=0}^{n} f \left( \frac{S_j(l) - bt}{\sqrt{l}} \right), \quad n \geq 1, \tag{1.4} \]
where \( S_j(l) = \sum_{i=1}^{j} X_{j+i} \) and \( l = l_n \) (1 \( \leq l \leq n \) is a sequence of positive integers satisfying \( \frac{l}{n} \to 0, \quad n \to \infty. \tag{1.5} \)

It should be noted that the random variables \( \{S_j(l); j = 1, \ldots, n\} \) in (1.4) are not NA. But, \( S_j(l) \) and \( S_l(l) \) are NA if \( |i - j| \geq l \).

The sums of type (1.4) can be found in Peligrad and Shao (1995). Some bootstrap estimates of dependent random variables also can be written as a form of (1.4) (cf. Hünsch, 1989; Shao and Yu, 1997). In the last section of this paper, one can see that the limit theorems of the sums (1.4) play an important role in the investigation of the limiting distribution of the estimates of \( \sigma \) or \( \text{Var} S_n \).

In this paper, the function \( f(x) \) is always assumed to be of bounded variation on any finite interval. The total variation function of \( f(x) \) is denoted by \( V_f(x) \), i.e., \( V_f(x) = V_f^n(f) \) for \( x \geq 0 \), \( V_f(x) = -V_f^0(f) \) for \( x \leq 0 \), where \( V_f^m(f) \) denotes the total variation of \( f(x) \) on \([a, b]\). Let \( f_+(x) \) and \( f_-(x) \) be the positive and negative variation functions of \( f(x) \), respectively; i.e., \( f_+(x) = (V_f(x) + f(x) - f(0))/2 \), \( f_-(x) = (V_f(x) - f(x) + f(0))/2 \). Obviously, \( V_f(x) \) and \( f_\pm(x) \) are all monotonically increasing functions, and \( |f_\pm(x)| \leq 2 |V_f(x)| \). \( V_f(x) = f_+(x) + f_-(x) \), \( f(x) = f_+(x) - f_-(x) + f(0) \).

Without loss of generality, we assume \( f(0) = 0 \). Otherwise, we can replace \( f(x) \) by \( f(x) - f(0) \). If \( f(x) \) is absolutely continuous, i.e., \( f(x) = \int_0^x f'(t) \, dt + f(0) \), then \( V_f(x) = \int_0^x |f'(t)| \, dt \), \( f_+(x) = \int_0^x f'^+ (t) \, dt \), \( f_-(x) = \int_0^x f'^- (t) \, dt \), where \( f'(x) \) denotes the derivative of \( f(x) \), \( x^+ = x \vee 0, \quad x^- = x \wedge 0 \). When \( f(x) \) is an absolutely continuous function, without loss of generality we may assume that \( f(x) \) is differentiable everywhere.

**Remark.** One may think that the limit theorems for the sums of type (1.3) or (1.4) are obvious when \( f(x) \) is a function with bounded variation on any finite interval, since \( \{f_+(X_n); n \geq 1\} \) and \( \{f_-(X_n); n \geq 1\} \) are both sequences of NA random variables and \( f(x) = f_+(x) - f_-(x) \). Actually, some of the limit theorems are easy to get. For example, from the strong law of large numbers for NA random variables one has
\[
\frac{1}{n} \sum_{j=1}^{n} \{f_+(X_n) - EF_+(X_n)\} \to 0, \quad \text{a.s.,}
\]
\[
\frac{1}{n} \sum_{j=1}^{n} \{f_-(X_n) - EF_-(X_n)\} \to 0, \quad \text{a.s.,}
\]
under the condition $E |V_f(X_1)| < \infty$. It follows obviously that
\[
\frac{1}{n} \sum_{j=1}^{n} \{ f(X_n) - Ef(X_n) \} \rightarrow 0, \text{ a.s.,}
\]
since $f(X_n) = f_+(X_n) - f_-(X_n)$. So, nothing is new for the strong law of large numbers. But, if we have
\[
\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \{ f_+(X_n) - Ef_+(X_n) \} \rightarrow N_1, \text{ in distribution,}
\]
\[
\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \{ f_-(X_n) - Ef_-(X_n) \} \rightarrow N_2, \text{ in distribution,}
\]
we may not get
\[
\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \{ f(X_n) - Ef(X_n) \} \rightarrow N_1 + N_2, \text{ in distribution.}
\]
So, the central limit theorem for \( \{ f(X_n); n \geq 1 \} \) is not so obvious.

Let
\[
T_n = \sum_{j=1}^{n} \left\{ f\left(\frac{S_j(l) - \mu}{\sqrt{l}}\right) - Ef\left(\frac{S_j(l) - \mu}{\sqrt{l}}\right)\right\}, \quad (1.6)
\]
\[
U_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \left\{ f\left(\frac{S_j(l) - \mu}{\sqrt{l}}\right) - Ef\left(\frac{S_j(l) - \mu}{\sqrt{l}}\right)\right\}, \quad 0 \leq t \leq 1. \quad (1.7)
\]
We write, for short,
\[
Y_{l,j} = f\left(\frac{S_j(l) - \mu}{\sqrt{l}}\right).
\]
Our main results read as follows.

**THEOREM 1.** Let \( \{ X_n; n \geq 1 \} \) be a sequence of stationary NA random variables with \( E X_1 = \mu, f(x) \) be a function with bounded variation on any finite interval, and \( l \) be a fixed positive integer. Suppose
\[
E V_f^2\left(\frac{S_j(l) - \mu}{\sqrt{l}}\right) < \infty. \quad (1.8)
\]
Denote
\[ \hat{\sigma}_f^2 = \text{Var} Y_{l,0} + 2 \sum_{j=1}^{\infty} \text{Cov}(Y_{l,0}, Y_{l,j}). \]  

Then
\[ U_n \Rightarrow W\left( \frac{1}{\sqrt{l}} \hat{\sigma}_f \right). \]  

**Theorem 2.** Let \( \{X_n; n \geq 1\} \) be a sequence of stationary NA random variables with \( E X_1 = \mu \) and \( E X_1^2 < \infty \), \( f(x) \) be an absolute continuous function, \( l = l_n \) satisfy \( l/n \to 0 \), and \( l \to \infty \). Suppose that
\[ V^2_f \left( \frac{S(lj) - lj^2}{\sqrt{j}} \right), \quad l \geq 1, \quad \text{are uniformly integrable.} \]  

Then
\[ U_n \Rightarrow W(\sigma_f), \]  

where \( \sigma_f^2 = 2 \int_0^1 \text{Cov}\{f(\sigma W(1)), f(\sigma(1 + t) - W(t))\} \, dt \) and \( \{W(t); t \geq 0\} \) is standard Wiener process.

The following theorem is a corresponding result for PA sequences.

**Theorem 1′.** Let \( \{X_n; n \geq 1\} \) be a sequence of stationary PA random variables with \( E X_1 = \mu \), \( f(x) \) be a function with bounded variation on any finite interval, and \( l \) be a fixed positive integer. Suppose that condition (1.8) is satisfied and
\[ \sum_{j=1}^{\infty} \text{Cov} \left\{ V_f \left( \frac{S(lj) - lj^2}{\sqrt{j}} \right), V_f \left( \frac{S(lj) - lj^2}{\sqrt{j}} \right) \right\} \]  

is convergent.  

Then
\[ U_n \Rightarrow W\left( \frac{1}{\sqrt{l}} \hat{\sigma}_f \right). \]  

In particular, if \( l = 1 \), taking \( f(x) \) to be \( f(x + \sqrt{j} \mu) \) in Theorems 1 and 1′ yields the following results.

**Corollary 1.** Suppose that \( \{X_n; n \geq 1\} \) is a sequence of stationary NA random variables with \( E X_1 = \mu \) and \( E V_f^2(X_1) < \infty \). Denote
\[ \sigma_f^2 = \text{Var} f(X_1) + 2 \sum_{j=2}^{\infty} \text{Cov}\{f(X_1), f(X_j)\}. \]  

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Then
\[ U_n \Rightarrow W(\sigma_f^2). \]

**Corollary 1.** Let \( \{ X_n; n \geq 1 \} \) be a sequence of stationary PA random variables with \( EX_1 = \mu \) and \( EV_f^2(X_1) < \infty \). Suppose
\[
\sum_{j=2}^{\infty} \text{Cov}(V_f(X_1), V_f(X_j)) < \infty. \tag{1.15}
\]
Then
\[ U_n \Rightarrow W(\sigma_f^2). \]

To illustrate that \( \hat{\sigma}_f^2 \) in (1.9) and \( \tilde{\sigma}_f^2 \) in (1.14) are well defined, we need the following lemmas.

**Lemma 1.** Let \( \{ \xi_n; n \geq 1 \} \) be a sequence of stationary random variables. If \( \sum_{j=2}^{\infty} \text{Cov}(\xi_1, \xi_j) \) is convergent, then
\[
A^2 = \text{Var} \xi_1 + 2 \sum_{j=2}^{\infty} \text{Cov}(\xi_1, \xi_j) \geq 0,
\]
and
\[
\frac{\text{Var}(\sum_{j=1}^{N} \xi_j)}{N} \to A^2.
\]

**Proof.** Obvious.

**Lemma 2.** Let \( \{ \xi_n; n \geq 1 \} \) be a sequence of stationary NA random variables, and \( f(x) \) be a function with bounded variation on any finite interval. If \( EV_f^2(\xi_1) < \infty \), then \( \sum_{j=2}^{\infty} \text{Cov}(f(\xi_1), f(\xi_j)) \) is convergent.

**Proof.** First, we suppose \( f(x) = x \). Note that \( \text{Cov}(\xi_1, \xi_j) \leq 0 \), \( j \geq 2 \). For any \( \forall N > 1 \), we have, for \( n \) large enough,
\[
0 \leq \frac{1}{n} \text{Var} \left( \sum_{j=1}^{n} \xi_j \right) = \text{Var} \xi_1 + 2 \frac{1}{n} \sum_{i=1}^{n} \sum_{j=2}^{n+1} \text{Cov}(\xi_1, \xi_j)
\leq \text{Var} \xi_1 + 2 \frac{1}{n} \sum_{i=N}^{n} \sum_{j=2}^{N+1} \text{Cov}(\xi_1, \xi_j)
\leq \text{Var} \xi_1 + \left( 1 - \frac{N}{n} \right) \sum_{j=2}^{N+1} \text{Cov}(\xi_1, \xi_j).\]
Letting \( n \to \infty \) yields
\[
-2 \sum_{j=2}^{\infty} \text{Cov}(\xi_1, \xi_j) \leq \text{Var} \xi_1.
\]
Then
\[
-\sum_{j=2}^{\infty} \text{Cov}(\xi_1, \xi_j) \leq -\sum_{j=2}^{\infty} \text{Cov}(\xi_1, \xi_j) \leq \frac{1}{2} \text{Var} \xi_1.
\]
Hence \( \sum_{j=2}^{\infty} \text{Cov}(\xi_1, \xi_j) \) is convergent. Now suppose \( \mathbb{E} V_j^2(X_i) < \infty \). Since \( V_j(x) \) is a monotonic function, \( \{ V_j(\xi_j); j \geq 1 \} \) is a sequence of stationary NA random variables. Thus \( \sum_{j=2}^{\infty} \text{Cov} \{ V_j(\xi_1), V_j(\xi_j) \} \) is convergent. By Corollary 2.1 of our Lemma 2.1 below, it follows that
\[
\sum_{j=2}^{\infty} \left| \text{Cov} \{ f(X_1), f(X_j) \} \right| \leq \sum_{j=2}^{\infty} \text{Cov} \{ V_j(\xi_1), V_j(\xi_j) \} < \infty.
\]
Hence, we have proved the lemma.

By Lemmas 1.1 and 1.2, we see that \( \sigma_j^2 \) in (1.9) and \( \sigma_j^2 \) in (1.14) are well defined, and
\[
\sigma_j^2 \geq 0, \sigma_j^2 \geq 0.
\]

**Remark.** If \( f(x) \) is a convex or concave function, then \( |V_j(x)| \approx |f(x)| + C \).

### 2. The Proof of the Theorem

We first need some lemmas.

**Lemma 2.1.** Let \( \{ X_i; i \in A \cup B \} \) be NA random variables and \( A \) and \( B \) be two disjoint sets of positive integers. Suppose that \( h_1(s_1, \ldots, s_m) \) and \( h_2(t_1, \ldots, t_n) \) are two real functions with bounded partial derivatives, \( s_p(s), t_q(t), \) \( p = 1, \ldots, m; t_q(t), q = 1, \ldots, n, \) are one-dimensional real functions with bounded variation on any finite interval; and \( f_p(x); \mathbb{R}^A \to \mathbb{R}, g_q(x); \mathbb{R}^B \to \mathbb{R} \) are coordinatewise monotonically increasing functions. Denote \( S = (s_1, \ldots, s_m), T = (t_1, \ldots, t_n), F = (f_1, \ldots, f_p), G = (g_1, \ldots, g_q). \) Then
\[
|\text{Cov} \{ h_1 \circ S \circ F(X_i; i \in A), h_2 \circ T \circ G(X_j; j \in B) \}| \leq - \sum_{p=1}^{n} \sum_{q=1}^{m} \left| \frac{\partial h_1}{\partial s_p} \right| \left| \frac{\partial h_2}{\partial t_q} \right| \text{Cov} \{ V_p \circ f_p(X_i; i \in A), V_q \circ g_q(X_j, j \in B) \}.
\]
If \( \{X_i; i \in A \cup B\} \) are PA random variables, then, similarly,

\[
|\text{Cov}\{h_1 \circ S \cdot f(X_i; i \in A), h_2 \circ T \cdot g(X_j; j \in B)\}| \\
\leq \sum_{p=1}^{m} \sum_{q=1}^{n} \left| \frac{\partial h_1}{\partial s_p} \right|_{\infty} \left| \frac{\partial h_2}{\partial t_q} \right|_{\infty} \text{Cov}\{V_{s_p} \cdot f_p(X_i; i \in A), V_{t_q} \cdot g_q(X_j; j \in B)\},
\]

and \( A \cap B \) may not be empty. If \( h_1, h_2 \) are complex functions, the above inequalities are also true if the terms on the right-hand side of “\( \leq \)” are replaced by the four times of them.

To prove this lemma, we need another lemma.

**Lemma 2.1.** Let \( g(x) \) be a real function with \( \|g'\|_{\infty} < \infty \) and \( f(x) \) be a function with bounded variation on any finite interval. Then \( \|g'\|_{\infty} V_f(x) \pm g(f(x)) \) are monotonically increasing functions.

**Proof.** Let \( x_1 < x_2 \). Then

\[
|g(f(x_1)) - g(f(x_2))| \leq \|g'\|_{\infty} |f(x_1) - f(x_2)| \\
\leq \|g'\|_{\infty} V_f(x_2) - V_f(x_1).
\]

It follows that

\[
\|g'\|_{\infty} V_f(x_1) \pm g(f(x_1)) \leq \|g'\|_{\infty} V_f(x_2) \pm g(f(x_2)).
\]

The proof is completed.

The proof of Lemma 2.1. Let

\[
\tilde{h}_1 = \sum_{p=1}^{m} \left| \frac{\partial h_1}{\partial s_p} \right|_{\infty} V_{s_p}, \\
\tilde{h}_2 = \sum_{q=1}^{n} \left| \frac{\partial h_2}{\partial t_q} \right|_{\infty} V_{t_q}.
\]

It follows from Lemma 2.2 that \( \tilde{h}_1 \pm h_1 \circ S \) and \( \tilde{h}_2 \pm h_2 \circ T \) are coordinate-wise monotonically increasing functions. Thus

\[
\tilde{h}_1 \circ F \pm h_1 \circ S \circ F, \quad \text{and} \quad \tilde{h}_2 \circ G \pm h_2 \circ T \circ G
\]
are coordinatewise monotonically increasing functions. By the property of NA, it follows that
\[
\text{Cov}\{\tilde{h}_1 \circ F(X_i; i \in A), \tilde{h}_2 \circ G(X_j; j \in B)\} \leq 0.
\]

It follows easily that
\[
|\text{Cov}\{\tilde{h}_1 \circ F(X_i; i \in A), \tilde{h}_2 \circ G(X_j; j \in B)\}|
\leq -\text{Cov}\{\tilde{h}_1 \circ F(X_i; i \in A), \tilde{h}_2 \circ G(X_j; j \in B)\}.
\]

The following corollary follows immediately from Lemma 2.1.

**Corollary 2.1.** Let \(X\) and \(Y\) be NA random variables and \(f(x)\) be a function with bounded variation on any finite interval. Then
\[
|\text{Cov}\{f(X), g(Y)\}| \leq -\text{Cov}\{V_f(X), V_f(Y)\}.
\]

If \(X\) and \(Y\) are PA random variables, then
\[
|\text{Cov}\{f(X), g(Y)\}| \leq \text{Cov}\{V_f(X), V_f(Y)\}.
\]

**Lemma 2.3.** Let \(X\) and \(Y\) be NA random variables and \(f(x)\) and \(g(y)\) be real functions with bounded derivations. Then
\[
|\text{Cov}\{f(X), g(Y)\}| \leq -\|f'\|_\infty \|g'\|_\infty \text{Cov}(X, Y).
\]

If \(X\) and \(Y\) are PA random variables, then
\[
|\text{Cov}\{f(X), g(Y)\}| \leq \|f'\|_\infty \|g'\|_\infty \text{Cov}(X, Y).
\]

**Proof.** Note that
\[
P(X \leq x, Y \leq y) - P(X \leq x) P(Y \leq y) \leq 0, \quad \forall x, y,
\]
by the property of NA. By Theorem 2.3 of Yu (1993) we have
\[
|\text{Cov}\{f(X), g(Y)\}|
= \left| \int_{\mathbb{R}^2} f'(x) g'(y) P(X \leq x, Y \leq y) - P(X \leq x) P(Y \leq y) \, dx \, dy \right|
\leq \|f'\|_\infty \|g'\|_\infty \int_{\mathbb{R}^2} |P(X \leq x, Y \leq y) - P(X \leq x) P(Y \leq y)| \, dx \, dy
\]
\[ \leq \|f\|_\infty \|g\|_\infty \int_{\mathbb{R}^2} \mathbf{P}(X \leq x, Y \leq y) - \mathbf{P}(X \leq x) \mathbf{P}(Y \leq y) \, dx \, dy \]
\[ \leq - \|f\|_\infty \|g\|_\infty \text{Cov}(X, Y). \]

**Lemma 2.4.** Let \( \{X_1, \ldots, X_n\} \) be NA random variables and \( p \geq 2 \) be a real number. Then

\[
\mathbf{E} \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} (X_j - \mathbf{E}X_j) \right|^p \leq C_p \left\{ \left( \sum_{j=1}^{n} \text{Var} X_j \right)^{p/2} + \sum_{j=1}^{n} \mathbf{E} \left| X_j - \mathbf{E}X_j \right|^p \right\}.
\]

**Proof.** See Su et al. (1997) or Shao (2000).

**Lemma 2.5.** Let \( f(x) \) be of bounded variation on any finite interval, \( \{X_1, \ldots, X_n\} \) be NA random variables, and \( p \geq 2 \) be a real number. Then

\[
\mathbf{E} \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} \left\{ f(X_j) - \mathbf{E}f(X_j) \right\} \right|^p \leq C_p \left\{ \left( \sum_{j=1}^{n} \mathbf{E} f^2(X_j) \right)^{p/2} + \sum_{j=1}^{n} \mathbf{E} \left| f(X_j) \right|^p \right\}.
\]

**Proof.** By Lemma 2.4 and noting that \( \{ f_X(X_j); j = 1, \ldots, n \} \) are NA random variables, we have

\[
\mathbf{E} \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} \left\{ f_X(X_j) - \mathbf{E}f_X(X_j) \right\} \right|^p \leq C_p \left\{ \left( \sum_{j=1}^{n} \mathbf{E} f^2_X(X_j) \right)^{p/2} + \sum_{j=1}^{n} \mathbf{E} \left| f_X(X_j) \right|^p \right\}.
\]

This completes the proof.

The following lemma follows easily from Lemma 2.5.

**Lemma 2.6.** Let \( \{X_1, \ldots, X_n\} \) be NA random variables, \( p \geq 2 \) be a real number, and \( l \) be a positive integer. Denote \( Z_j = f(S_j(l)/\sqrt{l}) \). Then

\[
\mathbf{E} \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} (Z_j - \mathbf{E}Z_j) \right|^p \leq C_{l,nf} \max_{1 \leq j \leq n} \left| V_f \left( \frac{S_j(l)}{\sqrt{l}} \right) \right|^p.
\]
Lemma 2.7. Let \( l \) be a fixed positive integer, \( \{ X_n; n \geq 1 \} \) be a sequence of stationary NA random variables, and \( f(x) \) be of bounded variation on any finite interval. Suppose \( \mathbb{E} V^2_f(S_0(l)/\sqrt{l}) < \infty \). Then
\[
\frac{1}{nl} \text{Var} \left( \sum_{i=0}^{n} f \left( \frac{S_i(l)}{\sqrt{l}} \right) \right) \to \frac{\sigma_f^2}{l}, \quad n \to \infty.
\]

Proof. From Lemma 1.2, it follows that \( \sum_{i=0}^{\infty} \text{Cov} \{ f(S_m(l)/\sqrt{l}), f(S_{m+1}(l)/\sqrt{l}) \} \) is convergent for each \( m = 0, \ldots, l - 1 \). Thus \( \sum_{i=0}^{\infty} \text{Cov} \{ f(S_0(l)/\sqrt{l}), f(S_i(l)/\sqrt{l}) \} \) is convergent. By Lemma 1.1, we complete the proof.

Lemma 2.8. Let \( f(x) \) be a real function and \( \{ X_n; n \geq 1 \} \) be a sequence of stationary NA random variables with \( \mathbb{E} X_1 = \mu = 0 \) and \( \mathbb{E} X_1^2 < \infty \). Suppose that \( \{ f^2(S_0(l)/\sqrt{l}); l \geq 1 \} \) is uniformly integrable. Then
\[
\lim_{l \to \infty} \frac{1}{l} \sum_{j=0}^{l-1} \text{Cov} \left\{ f \left( \frac{S_0(l)}{\sqrt{l}} \right), f \left( \frac{S_j(l)}{\sqrt{l}} \right) \right\}
= \int_{0}^{\infty} \text{Cov} \{ f(\alpha W(1)), f(\alpha (W(1+t) - W(t))) \} \, dt.
\]

Proof. The proof is similar to that in Peligrad and Shao (1995). See also Lemma 2.2 of Peligrad and Suresh (1995).

Lemma 2.9. Let \( f(x) \) be an absolutely continuous function and \( \{ X_n; n \geq 1 \} \) be a sequence of stationary NA random variables with \( \mathbb{E} X_1 = \mu = 0 \) and \( \mathbb{E} X_1^2 < \infty \). Suppose that (1.11) is satisfied and \( h \) satisfies \( l \to 0 \) and \( l \to \infty \). Then
\[
\frac{1}{nl} \text{Var} \left( \sum_{j=0}^{n} f \left( \frac{S_j(l)}{\sqrt{l}} \right) \right) \to \sigma_f^2, \quad n \to \infty.
\]

Proof. Without loss of generality, we can assume \( \|f\|_\infty \leq A < \infty \) and \( \|f\|_\infty \leq A < \infty \) (cf. (2.3) and (2.4) below). Write
\[
I = \left\{ \frac{n+1}{nl} \text{Var} Y_{i,0} + \sum_{i=0}^{n} \sum_{j=i+1}^{n} \text{Cov}(Y_{i,j}, Y_{i,j}) \right\} =: I_1 + I_2.
\]
Obviously, \( I_1 \to 0 \). Now, we consider \( I_2 \). Let \( \{ u_n \} \) be positive integers with \( u_n \to \infty \) and \( u_n = o(l_0) \). Then
\[ I_2 = \frac{2}{nl} \left[ \sum_{i=0}^{n-l} \sum_{j=i+1}^{i+u} \text{Cov}(Y_{L,i}, Y_{L,j}) + \sum_{i=0}^{n-l} \sum_{j=i+1}^{i+u} \text{Cov}(Y_{L,i}, Y_{L,j}) \right] \]

\[ + \frac{2}{nl} \left( \sum_{i=0}^{n-l} \sum_{j=i+1}^{i+u+1} \text{Cov}(Y_{L,i}, Y_{L,j}) + \sum_{i=n-l+1}^{n-1} \sum_{j=i+1}^{i+u+1} \text{Cov}(Y_{L,i}, Y_{L,j}) \right) \]

\[ = \frac{2}{nl} [J_1 + J_2 + J_3 + J_4]. \]

From Lemma 2.8, it follows that

\[ \lim_{n \to \infty} \frac{2J_1}{nl} = 2 \lim_{n \to \infty} \frac{\sum_{i=0}^{n-l} \sum_{j=0}^{i+u} \text{Cov}(Y_{L,0}, Y_{L,j})}{l} = \sigma_f^2. \]

From Lemma 2.6, it follows that

\[ |J_2| \leq \sum_{i=0}^{n-l} \left| \text{Cov} \left( Y_{L,0}, \sum_{j=i+1}^{i+u} Y_{L,j} \right) \right| \leq n \| Y_{L,0} \|_2 \left\| \sum_{j=i+1}^{i+u} Y_{L,j} \right\|_2 \]

\[ \leq Cn(uu^2)^{1/2} \left\| V_f \left( \frac{S_0(l)}{\sqrt{l}} \right) \right\|_2 \leq Cn(uu^2). \]

Hence

\[ \frac{|J_2|}{nl} \leq C \left( \frac{u}{n} \right)^{1/2} \to 0 \quad (n \to \infty). \]

Similarly,

\[ \frac{J_4}{nl} \leq C \left( \frac{l}{n} \right)^{1/2} = C \left( \frac{l}{n} \right)^{1/2} \to 0 \quad (n \to \infty). \]

Finally, from Lemmas 2.1 and 2.3 it follows that

\[ \frac{|J_3|}{nl} \leq \frac{1}{nl} \sum_{i=0}^{n-l} \sum_{j=i+1}^{i+u+1} \left\{ -\text{Cov} \left( V_f \left( \frac{S_0(l)}{\sqrt{l}} \right), V_f \left( \frac{S_0(l)}{\sqrt{l}} \right) \right) \right\} \]

\[ \leq \frac{1}{nl} \sum_{i=0}^{n-l} \sum_{j=i+1}^{i+u+1} \| f' \|_\infty^2 \left\{ -\text{Cov} \left( \frac{S_0(l)}{\sqrt{l}}, \frac{S_0(l)}{\sqrt{l}} \right) \right\} \]

\[ \leq A^2 \sum_{j=u}^{\infty} \left\{ -\text{Cov}(X_1, X_j) \right\} \to 0 \quad (n \to \infty). \]

The proof is completed.
The Proof of Theorem 1. Without loss of generality, we can assume \( \mu = 0 \). Let
\[
\tilde{\sigma}_f^2 = \frac{\sigma_f^2}{I} = \frac{1}{I} \text{Var} \left( Y_{l, 0} + 2 \sum_{j=1}^{\infty} \text{Cov}(Y_{l, 0}, Y_{l, j}) \right).
\]

By Lemma 2.7, we have
\[
\text{Var} \left( \sum_{j=0}^{n} \left( \frac{S_j(l)}{\sqrt{I}} \right) \right) \rightarrow \tilde{\sigma}_f^2, \quad n \rightarrow \infty. \tag{2.1}
\]

For \( R > 0 \), let \( f_R(x) = f(-R \wedge (x \wedge R)), \tilde{f}_R(x) = f(x) - f_R(x) \), and
\[
U_{R, n}(t) = \frac{1}{\sqrt{nl}} \sum_{j=0}^{\lfloor nt \rfloor} \left( \tilde{f}_R \left( \frac{S_j(l)}{\sqrt{I}} \right) - \mathbb{E} \tilde{f}_R \left( \frac{S_j(l)}{\sqrt{I}} \right) \right),
\]
\[
\bar{U}_{R, n}(t) = \frac{1}{\sqrt{nl}} \sum_{j=0}^{\lfloor nt \rfloor} \left( \tilde{f}_R \left( \frac{S_j(l)}{\sqrt{I}} \right) - \mathbb{E} \tilde{f}_R \left( \frac{S_j(l)}{\sqrt{I}} \right) \right), \quad 0 < t < 1. \tag{2.2}
\]

By Lemma 2.6 we have
\[
\sup_n \mathbb{E} \max_{0 \leq t < 1} \bar{U}_{R, n}(t)
\leq \sup_n \mathbb{E} \max_{1 \leq k \leq n} \left| \sum_{j=0}^{k} \left( \tilde{f}_R \left( \frac{S_j(l)}{\sqrt{I}} \right) - \mathbb{E} \tilde{f}_R \left( \frac{S_j(l)}{\sqrt{I}} \right) \right) \right|^2
\leq C \sup_l \mathbb{E} \nu^2_{f_R} \left( \frac{S_d(l)}{\sqrt{I}} \right)
\leq C \sup_l \mathbb{E} \nu^2_{f} \left( \frac{S_d(l)}{\sqrt{I}} \right) \mathbb{P} \left( \left| \frac{S_d(l)}{\sqrt{I}} \right| \geq R \right) \rightarrow 0, \quad R \rightarrow \infty. \tag{2.3}
\]

It follows easily from (2.3) that
\[
\tilde{\sigma}_f^2 \rightarrow \sigma_f^2, \quad R \rightarrow \infty. \tag{2.4}
\]

Hence it is sufficient to show that for any fixed \( R \) we have
\[
U_{R, n} \Rightarrow W(\tilde{\sigma}_f^2).
\]

So, without loss of generality we can assume \( \| f \|_{\infty} \leq A < \infty \) and \( \| V_f \|_{\infty} \leq A < \infty \).
First we prove the central limit theorem, i.e.,
\[ I_n := \frac{1}{\sqrt{n}} \sum_{j=0}^{n} \left\{ f \left( \frac{S(j)}{\sqrt{j}} \right) - \mathbb{E} f \left( \frac{S(j)}{\sqrt{j}} \right) \right\} \xrightarrow{d} N(0, \sigma^2). \]  

Let \{r = r_n\} and \{l' = l'_n\} be two sequences of positive integers with \( r = o(n), \ l = o(l'), \ l' = o(r), \ l' \to \infty, \) and \( r \to \infty. \) Let
\[ \xi_{m,n} = \sum_{j=0}^{m2^l+r-1} \left\{ f \left( \frac{S(j)}{\sqrt{j}} \right) - \mathbb{E} f \left( \frac{S(j)}{\sqrt{j}} \right) \right\}, \]
\[ \eta_{m,n} = \sum_{j=0}^{(m+1)2^l+r-1} \left\{ f \left( \frac{S(j)}{\sqrt{j}} \right) - \mathbb{E} f \left( \frac{S(j)}{\sqrt{j}} \right) \right\}, \]  

\[ m = 0, 1, ..., k = k_n := \left\lceil \frac{n+1}{2l'+r} \right\rceil - 1. \]

Then
\[ I_n = \frac{1}{\sqrt{n}} \sum_{m=0}^{k_n} \xi_{m,n} + \frac{1}{\sqrt{n}} \sum_{m=0}^{k_n} \eta_{m,n} \]
\[ + \frac{1}{\sqrt{n}} \sum_{j=0}^{n} \left\{ f \left( \frac{S(j)}{\sqrt{j}} \right) - \mathbb{E} f \left( \frac{S(j)}{\sqrt{j}} \right) \right\} \]
\[ =: I_{1,n} + I_{2,n} + I_{3,n}. \]  

From Lemma 2.6 it follows that
\[ \text{Var} I_{3,n} \leq C \frac{1}{n} \sqrt{l(2l'+r)} \mathbb{E} V_2 \left( \frac{S(l)}{\sqrt{l}} \right) \leq C \frac{2l'+r}{n} \to 0, \quad n \to \infty. \]

With \( f_\pm \) taking the place of \( f, \) we define \( \xi_{m,n}^\pm \) and \( \eta_{m,n}^\pm \) similarly. Then \( \{ \xi_{m,n}^\pm \} \) and \( \{ \eta_{m,n}^\pm \} \) are both sequences of NA random variables. By noting \( f = f_+ - f_- \) and Lemma 2.6, we have
\[ \text{Var} I_{2,n} \leq 2 \text{Var} \left( \frac{1}{\sqrt{n}} \sum_{m=0}^{k_n} \eta_{m,n}^+ \right) + 2 \text{Var} \left( \frac{1}{\sqrt{n}} \sum_{m=0}^{k_n} \eta_{m,n}^- \right) \]
\[ \leq \frac{2(k_n+1)}{nl} (\text{Var} \eta_{0,n}^+ + \text{Var} \eta_{0,n}^-) \]
\[ \leq C \frac{n+1}{n} \cdot \frac{1}{2l'+r} \left\{ \mathbb{E} \left( f_+ \left( \frac{S(l)}{\sqrt{l}} \right) \right)^2 + \mathbb{E} \left( f_- \left( \frac{S(l)}{\sqrt{l}} \right) \right)^2 \right\} \]
\[ \leq C \frac{l'}{r} \to 0 \quad (n \to \infty). \]
So, to prove the central limit theorem, it is sufficient to show

$$I_{1,n} \xrightarrow{d} N(0, \sigma_f^2). \quad (2.8)$$

With $V_f(x)$ taking the place of $f(x)$ in (2.6), we define $\tilde{\xi}_{m,n}$ and $\tilde{\eta}_{m,n}$, i.e.,

$$\tilde{\xi}_{m,n} = \sum_{j=m(2^r + r) + r-1}^{m(2^r + r) + r-1} V_f \left( \frac{S_j(l)}{\sqrt{T}} \right),$$

$$\tilde{\eta}_{m,n} = \sum_{j=m(2^r + r) + r}^{(m+1)(2^r + r) - 1} V_f \left( \frac{S_j(l)}{\sqrt{T}} \right). \quad (2.9)$$

From Lemma 2.1 it follows that

$$|E \sum_{m=0}^{k} \tilde{\xi}_{m,s} - \sum_{m=0}^{k} E \sum_{m=0}^{k} \tilde{\xi}_{m,s}| \leq 4T^2 \sum_{m=0}^{k} \sum_{j=m+1}^{k} \text{Cov} \left\{ V_f \left( \frac{S_j(l)}{\sqrt{T}} \right), V_f \left( \frac{S_j(l)}{\sqrt{T}} \right) \right\} \quad (2.10)$$

By the induction method, one can show that

$$|E \sum_{m=0}^{k} \tilde{\xi}_{m,s} - \sum_{m=0}^{k} E \tilde{\xi}_{m,s}| \leq 4T^2 \sum_{1 \leq j \neq m \leq k} \left\{ - \text{Cov}(\tilde{\xi}_{j,n}, \tilde{\xi}_{m,n}) \right\}. \quad (2.11)$$

It follows that

$$|E \sum_{m=0}^{k} \tilde{\xi}_{m,s} - \sum_{m=0}^{k} \sum_{1 \leq j \neq m \leq k} \text{Cov} \left\{ V_f \left( \frac{S_j(l)}{l} \right), V_f \left( \frac{S_j(l)}{l} \right) \right\} | \to 0 \quad (n \to \infty), \quad (2.12)$$

since $\sum_{j=1}^{\infty} \text{Cov} \{ V_f(S_j(l)/\sqrt{I}), V_f(S_j(l)/\sqrt{I}) \}$ is convergent and $l' \to \infty$.

Let $\{\tau_{m,n}, 0 \leq m \leq k_n\}$ be an array of independent random variables such that for each $0 \leq m \leq k_n$, $\tau_{m,n}$ and $\tilde{\xi}_{m,n}$ are identically distributed. By (2.12), it is enough to show that

$$\frac{1}{\sqrt{nl}} \sum_{m=0}^{k} \tau_{m,n} \xrightarrow{d} N(0, \sigma_f^2).$$
Note that $\xi_{m,n}$ and $\tau_{m,n}$ are identically distributed. By (2.1), we have
\[
\frac{1}{m} \text{Var} \left( \sum_{m=0}^{k_n} \tau_{m,n} \right) = \frac{(k_n + 1) \text{Var} \xi_{0,m}}{m}
\]
\[
= \frac{(k_n + 1) m}{n} \frac{1}{rl} \text{Var} \left( \sum_{j=0}^{r-1} f \left( \frac{S_j(l)}{\sqrt{l}} \right) \right) \to \delta_f^2,
\]
i.e.,
\[
\text{Var} \left( \sum_{m=0}^{k_n} \tau_{m,n} \right) \sim n \delta_f^2,
\quad n \to \infty.
\]
Hence, it is enough to prove that when $\delta_f^2 > 0$, $\{ \tau_{m,n}; m = 0, \ldots, k_n \}$ satisfies Linderberg’s condition. Thus, it is sufficient to prove that $\{(1/\text{Var} \xi_{0,n}) \xi_{0,n}; n \geq 1 \}$ are uniformly integrable. Note that $\text{Var} \xi_{0,n} \sim r \delta_f^2$. It is sufficient to show that
\[
\lambda_n := \frac{1}{rl} \left( \sum_{j=0}^{r-1} f \left( \frac{S_j(l)}{\sqrt{l}} \right) - \mathbb{E} f \left( \frac{S_j(l)}{\sqrt{l}} \right) \right)^2,
\]
$n \geq 1$ are uniformly integrable.

By Lemma 2.6, we have
\[
\mathbb{E} |\lambda_n|^2 \leq C \left( \frac{1}{rl} \right)^2 (rl)^{4A} \mathbb{E} |V_f(X_1)|^4 \leq CA^4 < \infty,
\]
hence $\{\lambda_n; n \geq 1 \}$ are uniformly integrable. The central limit theorem (2.5) is proved. Next, we show the tightness. Since
\[
\mathbb{P}(\sup_{0 \leq s \leq \theta} |U_n(s)| \geq \varepsilon) \leq \frac{1}{\varepsilon^2 (nl)^2} \mathbb{E} \max_{0 \leq m < n \leq \theta} \left| \sum_{j=0}^{m} f \left( \frac{S_j(l)}{\sqrt{l}} \right) - \mathbb{E} f \left( \frac{S_j(l)}{\sqrt{l}} \right) \right|^4
\]
\[
\leq C(\varepsilon) \frac{1}{(nl)^2} (n \delta l)^{4A} \mathbb{E} \left| f \left( \frac{S_0(l)}{\sqrt{l}} \right) \right|^4 \leq C(\varepsilon) \delta^2 A^4,
\]
it follows that
\[
\sup_n \mathbb{P}(\sup_{|t-x| < \theta} |U_n(t) - U_n(x)| \geq \varepsilon)
\]
\[
\leq 2 \delta^{-1} \mathbb{P}(\sup_{0 \leq s \leq \theta} |U_n(s)| \geq \varepsilon) \leq C(\varepsilon) A^4 \delta \to 0 \quad (\delta \to 0).
\]
So, \( \{ U_n(\cdot) \} \) is tight and, if \( U_n \Rightarrow X \), then \( P(X \in C) = 1 \). Note that for any \( 0 < s_1 < s_2 < s_3 \leq 1 \),

\[
U_n(s_1) \xrightarrow{d} N(0, \sigma_n^2 s_1), \quad U_n(s_3) - U_n(s_2) \xrightarrow{d} N(0, \sigma_n^2(s_3 - s_2)),
\]

by the central limit theorem. We only need to show that \( X(s_3) \) and \( X(s_2) \) are independent. From Lemma 2.1, it follows that

\[
|E e^{it U_n(s_1) + it U_n(s_3) - U_n(s_2)} - E e^{it U_n(s_1)} E e^{it U_n(s_3) - U_n(s_2)}| \leq 4 |t_1| |t_2| \sum_{j=0}^n \sum_{l=0}^{n_2 - n_1 - 1} \text{Cov} \left\{ V_f \left( \frac{S_j(l)}{\sqrt{t}} \right), V_f \left( \frac{S_d(l)}{\sqrt{t}} \right) \right\} \to 0,
\]

\[ (n \to \infty) , \]

(2.14)

which implies the independence of \( X(s_3) \) and \( X(s_2) \). The proof of Theorem 1 is completed.

The Proof of Theorem 2. We can also assume \( \mu = 0 \), \( \| V_f \|_{\infty} < A < \infty \), and \( \| f' \|_{\infty} < A < \infty \). With Lemma 2.9 taking the place of (2.1), Theorem 2 can be proved along the lines of the proof of Theorem 1, with only the proof of (2.12) and (2.14) being modified. Now, by (2.12) and Lemma 2.3, we have

\[
\left| E e^{it(1/\sqrt{n}) \sum_{n=0}^k \xi_{n,n} - \prod_{m=0}^{k} E e^{it(1/\sqrt{n}) \xi_{n,n}}} \right| \leq 4 t^2 \left( \sum_{j=0}^{n_2 - n_1} \text{Cov} \left\{ V_f \left( \frac{S_j(l)}{\sqrt{t}} \right), V_f \left( \frac{S_d(l)}{\sqrt{t}} \right) \right\} \right) \to 0 \quad (n \to \infty).
\]

Equation (2.14) can be showed in the same way.
The Proof of Theorem 1'. Assume $\mu = 0$. Now, we cannot use the truncation method. We shall directly prove (2.5), (2.14), and that, for any $\varepsilon > 0$,

$$\lim_{\lambda \to \infty} \lim_{n \to \infty} \lambda^2 \mathbf{P} \left( \max_{0 \leq m \leq n} \left| \sum_{j=0}^{m} f \left( \frac{S_j(l)}{\sqrt{l}} \right) - \mathbf{E} f \left( \frac{S_0(l)}{\sqrt{l}} \right) \right| \geq \lambda \sqrt{nl} \right) = 0.$$  

(2.15)

Using the second part of Lemma 2.1 instead of the first one, the proof of (2.14) is similar. Now, let $\{ \xi_{m,n} \}$, $\{ \eta_{m,n} \}$ be defined as in (2.6), and $I_{k,n}$, $i = 1, 2, 3$, be as in (2.7). Then

$$\begin{align*}
\Var I_{k,n} &= \frac{n - (k_n + 1)(2l' + r) + 1}{n} \\
&\quad \times \frac{1}{(n - (k_n + 1)(2l' + r) + 1)} \Var \left\{ \sum_{j=0}^{n - (k_n + 1)(2l' + r)} f \left( \frac{S_j(l)}{l} \right) \right\} \\
&= o(1) \cdot \sigma_f^2 = o(1), \quad n \to \infty.
\end{align*}$$

On the other hand, from Lemma 2.1 it follows that

$$\Var I_{2,n} = \frac{k_n + 1}{nl} \Var \eta_{0,n} + 2 \frac{1}{nl} \sum_{m=0}^{k_n - 1} \sum_{j=1}^{k_n - m - 1} \Cov \{ \eta_{0,n}, \eta_{p,n} \}$$

$$\leq \frac{2l'(k_n + 1)}{n} \frac{1}{2l'!} \Var \left\{ \sum_{j=0}^{2l'-1} f \left( \frac{S_j(l)}{\sqrt{l}} \right) \right\}$$

$$+ \frac{k_n + 1}{nl} \sum_{j \geq r, r \neq l} \Cov \left\{ V_{j} \left( \frac{S_0(l)}{\sqrt{l}} \right), V_{j} \left( \frac{S_{j}(l)}{l} \right) \right\}$$

$$= o(1) \cdot \sigma_f^2 + o(1) = o(1), \quad n \to \infty.$$  

So, to prove (2.5) it is enough to prove (2.8). Along the same lines as those in the proof of Theorem 1, it is sufficient to show that

$$\begin{align*}
\hat{\lambda} &= \frac{1}{nl} \left\{ \sum_{j=0}^{r-1} \left[ f \left( \frac{S_j(l)}{\sqrt{l}} \right) - \mathbf{E} f \left( \frac{S_0(l)}{\sqrt{l}} \right) \right] \right\}^2 \\
r \geq l \text{ are uniformly integrable}. \quad (2.16)
\end{align*}$$
Note that
\[
\frac{1}{\sqrt{rl}} \sum_{j=0}^{r-l} \left[ f \left( \frac{S_j(l)}{l} \right) - \mathbb{E} \left( \frac{S_j(l)}{l} \right) \right] = \frac{1}{\sqrt{l}} \sum_{i=0}^{l-1} \frac{1}{\sqrt{l}} \sum_{j=0}^{l} \left[ f \left( \frac{S_{j+1}(l)}{l} \sqrt{l} \right) - \mathbb{E} \left( \frac{S_{j+1}(l)}{l} \sqrt{l} \right) \right] \\
+ \frac{1}{\sqrt{rl}} \sum_{j=\lfloor (r-l)/l \rfloor + 1}^{r-1} \left[ f \left( \frac{S_j(l)}{l} \sqrt{l} \right) - \mathbb{E} \left( \frac{S_j(l)}{l} \sqrt{l} \right) \right].
\]

It is easy to see that
\[
\mathbb{E} \left( \frac{1}{\sqrt{rl}} \sum_{j=\lfloor (r-l)/l \rfloor + 1}^{r-1} \left[ f \left( \frac{S_j(l)}{l} \sqrt{l} \right) - \mathbb{E} \left( \frac{S_j(l)}{l} \sqrt{l} \right) \right] \right)^2 \\
= \frac{1}{l} \text{Var} \left( \sum_{j=0}^{\lfloor (r-l)/l \rfloor \to 1} f \left( \frac{S_j(l)}{l} \sqrt{l} \right) \right) = o(1), \quad r \to \infty.
\]

So, it is sufficient to show that, for each \( i \),
\[
\frac{1}{\sqrt{rl}} \sum_{j=\lfloor (r-l)/l \rfloor + 1}^{r-1} \left[ f \left( \frac{S_j(l)}{l} \sqrt{l} \right) - \mathbb{E} \left( \frac{S_j(l)}{l} \sqrt{l} \right) \right] ;
\]
\( r \geq l \) are uniformly square integrable.

By the property of stationarity, it is sufficient to prove that
\[
\{ \eta_r^2; r \geq 1 \} \text{ are uniformly integrable, (2.17)}
\]

where
\[
\eta_r = \frac{1}{\sqrt{r}} \sum_{j=0}^{\lfloor r/l \rfloor} \left[ f_+ \left( \frac{S_j(l)}{l} \sqrt{l} \right) - \mathbb{E} \left( \frac{S_j(l)}{l} \sqrt{l} \right) \right] .
\]

Note that
\[
\eta_r = \frac{1}{\sqrt{r}} \sum_{j=0}^{\lfloor r/l \rfloor} \left[ f_+ \left( \frac{S_j(l)}{l} \sqrt{l} \right) - \mathbb{E} \left( \frac{S_j(l)}{l} \sqrt{l} \right) \right] \\
- \frac{1}{\sqrt{r}} \sum_{j=0}^{\lfloor r/l \rfloor} \left[ f_+ \left( \frac{S_j(l)}{l} \sqrt{l} \right) - \mathbb{E} \left( \frac{S_j(l)}{l} \sqrt{l} \right) \right] =: \eta_{r+} - \eta_{r-} .
\]
and that \( \{ f_+ (S_{jl}(l)/\sqrt{l}) \colon j \geq 1 \} \) is a sequence of stationary NA random variables satisfying the conditions in Theorem A. Hence

\[
\eta_{r+} \xrightarrow{\mathcal{D}} N(0, \Sigma_+^2), \quad r \to \infty, \\
E \eta_{r+}^2 \to \Sigma_+^2, \quad r \to \infty;
\]

where

\[
\Sigma_+^2 = \operatorname{Var} \left\{ f_+ \left( \frac{S_0(l)}{\sqrt{l}} \right) \right\} + 2 \sum_{j=1}^{\infty} \operatorname{Cov} \left\{ f_+ \left( \frac{S_0(l)}{\sqrt{l}} \right), f_+ \left( \frac{S_{jl}(l)}{\sqrt{l}} \right) \right\}.
\]

It follows easily that

\[
\{ \eta_{r+}^2 \colon r \geq 1 \}
\]

are uniformly integrable.

Similarly, we have that

\[
\{ \eta_{r-}^2 \colon r \geq 1 \}
\]

are uniformly integrable.

Hence (2.17) is proved. And then (2.16) is proved. Finally, we prove (2.15). We have

\[
\frac{1}{\sqrt{ml}} \max_{0 \leq m \leq n} \left\{ \sum_{j=0}^{m} \left| f_+ \left( \frac{S_j(l)}{\sqrt{l}} \right) - Ef_+ \left( \frac{S_j(l)}{\sqrt{l}} \right) \right| \right\} \\
\leq \frac{1}{l} \sum_{i=0}^{l-1} \frac{1}{\sqrt{ml}} \max_{0 \leq m \leq [n/l]} \left\{ \sum_{j=0}^{m} \left| f_+ \left( \frac{S_{j+i}(l)}{\sqrt{l}} \right) - Ef_+ \left( \frac{S_{j+i}(l)}{\sqrt{l}} \right) \right| \right\} \\
+ \frac{1}{\sqrt{ml}} \max_{[n/l] < m \leq n} \left| \sum_{j=0}^{m} \right| \left| f_+ \left( \frac{S_{j+i}(l)}{\sqrt{l}} \right) - Ef_+ \left( \frac{S_{j+i}(l)}{\sqrt{l}} \right) \right| \\
\leq \frac{1}{l} \sum_{i=0}^{l-1} \frac{1}{\sqrt{ml}} \max_{0 \leq m \leq [n/l]} \left| \sum_{j=0}^{m} \right| \left( f_+ \left( \frac{S_{j+i}(l)}{\sqrt{l}} \right) - Ef_+ \left( \frac{S_{j+i}(l)}{\sqrt{l}} \right) \right) \\
+ \frac{1}{\sqrt{ml}} \max_{[n/l] < m \leq n} \left| \sum_{j=0}^{m} \right| \left( f_+ \left( \frac{S_{j+i}(l)}{\sqrt{l}} \right) - Ef_+ \left( \frac{S_{j+i}(l)}{\sqrt{l}} \right) \right) + \frac{1}{\sqrt{ml}} \max_{0 \leq m \leq n} \left| \sum_{j=0}^{m} \right| \left( f_+ \left( \frac{S_{j+i}(l)}{\sqrt{l}} \right) - Ef_+ \left( \frac{S_{j+i}(l)}{\sqrt{l}} \right) \right) \right.$
It is easy to see that the last term above tends to zero in probability. By the property of stationarity, similar to the proof of (2.16), it is enough to show

\[
\lim_{\lambda \to 0} \limsup_{n \to \infty} \lambda^2 \mathbb{P}\left( \max_{0 \leq m \leq n} \left| \sum_{j=0}^{m} f_+ \left( \frac{S_{\rho(l)}(l)}{\sqrt{l}} \right) - \mathbf{E} f_+ \left( \frac{S_{\rho(l)}(l)}{\sqrt{l}} \right) \right| \geq \lambda \sqrt{n} \right) = 0.
\]  

(2.18)

Note that \( \{ f_+ (S_{\rho(l)}(l)/\sqrt{l}); j \geq 1 \} \) are PA random variables. By (11) of Newman and Wright (1981) and

\[
s_n^2 := \text{Var} \left\{ \sum_{j=0}^{n} f_+ \left( \frac{S_{\rho(l)}(l)}{\sqrt{l}} \right) - \mathbf{E} f_+ \left( \frac{S_{\rho(l)}(l)}{\sqrt{l}} \right) \right\} \leq n \Sigma^2,
\]

it follows that, for \( \lambda^2 > 2 \Sigma^2 \),

\[
\mathbb{P}\left( \max_{0 \leq m \leq n} \left| \sum_{j=0}^{m} f_+ \left( \frac{S_{\rho(l)}(l)}{\sqrt{l}} \right) - \mathbf{E} f_+ \left( \frac{S_{\rho(l)}(l)}{\sqrt{l}} \right) \right| \geq \lambda \sqrt{n} \right) 
\leq 2 \left( 1 - \frac{s_n^2}{2 \lambda^2} \right)^{-1} \mathbb{P}\left( \left| \sum_{j=0}^{n} f_+ \left( \frac{S_{\rho(l)}(l)}{\sqrt{l}} \right) - \mathbf{E} f_+ \left( \frac{S_{\rho(l)}(l)}{\sqrt{l}} \right) \right| \geq \lambda \sqrt{n} \right) 
\leq 4 \mathbb{P}(\eta_{n+} | \geq \lambda) \leq \frac{4}{\lambda^2} \mathbf{E} \eta_{n+}^2 I\{\eta_{n+} | \geq \lambda\}.
\]

Since \( \{ \eta_{n+}^2; \tau \geq 1 \} \) are uniformly integrable, (2.18) is proved. Hence (2.15) is proved. The proof of Theorem 1 is now completed.

3. SOME APPLICATIONS

Let \( \{ X_n; n \geq 1 \} \) be a sequence of stationary NA random variables. From Theorem B, it follows that if \( \mathbf{E} X_1^2 < \infty, \mathbf{E} X_1 = \mu, \) and \( \sigma^2 > 0, \) then

\[
\frac{S_n - n \mu}{\sqrt{n} \sigma} \Rightarrow N(0, 1), \quad \frac{\text{Var} S_n}{n} \to \sigma^2.
\]  

(3.1)

Usually, the value of \( \sigma \) is unknown. In the case that \( \{ X_n; n \geq 1 \} \) are i.i.d. random variables, \( \sigma^2 = \text{Var} X_1 \) and there are many statistical methods to estimate \( \sigma \). But, if \( \{ X_n; n \geq 1 \} \) are dependent random variables, the method to estimate \( \sigma \) is not as simple as in the i.i.d. case. When \( \{ X_n; n \geq 1 \} \) are
stationary $\rho$-mixing random variables, Peligrad and Shao (1995) gave two estimators $B_{n, p}$ and $\hat{B}_{n, p}$ of $\sigma$ defined as

$$B_{n, p} = \frac{c_p}{n-1} \sum_{j=0}^{n-l} \left( \frac{|S_j(l) - lX|}{\sqrt{l}} \right)^p$$

and

$$\hat{B}_{n, p} = \frac{c_p}{n-1} \sum_{j=0}^{n-l} \left( \frac{|S_j(l) - l\mu|}{\sqrt{l}} \right)^p,$$  \hspace{1cm} (3.2)

where $\hat{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$, $c_p = 1 \mathbb{E} |N|^p$, $N = N(0, 1)$ is a standard normal variable, and $p \geq 1$. Obviously, $B_{n, p}$ and $\hat{B}_{n, p}$ are estimators of $\sigma$ corresponding to the case in which the mean $\mu$ are known and unknown, respectively.

In the case of PA, Peligrad and Suresh (1995) used particular $B_{n,1}$ and $\hat{B}_{n,1}$ to estimate $\sigma$ and studied their limiting properties. Zhang and Shi (1998) investigated the limiting properties of the general estimators $B_{n, p}$ and $\hat{B}_{n, p}$ of $\sigma$ in the case of NA. But neither Peligrad and Suresh nor Zhang and Shi obtained the exact limiting distributions of $B_{n, p}$ or $\hat{B}_{n, p}$. By using our theorems in Section 1, we can now obtain their limiting distributions.

**Theorem 3.** Let $\{X_n; n \geq 1\}$ be a sequence of stationary NA random variables with $\mathbb{E} |X|^p < \infty$ and $\mathbb{E} X_1 = \mu$ and $\sigma^2 > 0$, where $p \geq 1$. Let $l = l_n$ satisfy $l_n \to \infty$, $l_n/n \to 0$. Then

$$\sqrt{n} \left( \hat{B}_{n, p} - (\mathbb{E} \hat{B}_{n, p})^{1/p} \right) \overset{D}{\to} N(0, \sigma^2 A_p),$$  \hspace{1cm} (3.3)

where $A_p = 2(c_p/p)^2 \int_0^1 \text{Cov} \left( |W(1)|^p, |W(1 + t) - W(t)|^p \right) dt$. If $p > 1$, then

$$\sqrt{n} \left( B_{n, p} - (\mathbb{E} B_{n, p})^{1/p} \right) \overset{D}{\to} N(0, \sigma^2 A_p).$$  \hspace{1cm} (3.4)

Also, $\mathbb{E} \hat{B}_{n, p}$ in (3.3) and (3.4) can be replaced by $\mathbb{E} B_{n, p}^\mu \cdot (\mathbb{E} B_{n, p})^\mu$, $(\mathbb{E} B_{n, p})^\mu$, or $c_p \mathbb{E} \left( (S_j(l) - l\mu)^{1/2} \right)^p$.

**Proof.** Without loss of generality, we may assume $\mu = 0$. Let $f(x) = |x|^p$, then $|V_p(x)| = |x|^p = f(x)$. By the condition $\mathbb{E} |X|^p < \infty$ and using Lemma 2.2 and the truncation method, one can show that

$$\left\{ \frac{|S_j(l)|^{2p}}{\sqrt{l}} ; l \geq 1 \right\}$$

are uniformly integrable.  \hspace{1cm} (3.5)
Hence, from Theorem 1 it follows that
\[
\sqrt{n} \left( \hat{B}_{n, p} - \mathbf{E} B_{n, p} \right) \xrightarrow{D} N(0, \Sigma_p^2), \tag{3.6}
\]
where \( \Sigma_p^2 = 2c_p^2 \sigma^2 \int_0^1 \text{Cov}\{ |W(1)|^p, |W(1 + t) - W(t)|^p \} \, dt \). By (3.5) and Theorem B, we have
\[
\mathbf{E} B_{n, p} = c_p \left( \mathbf{E} B_{n, p} \right)^{1/p}. \tag{3.7}
\]
From (3.6) and (3.7), it follows that \( B_{n, p} \sim \mathbf{E} B_{n, p} \), i.e., \( B_{n, p} \sim \sigma \). Then
\[
\frac{1}{\rho^2} \left( \frac{\hat{B}_{n, p} - (\mathbf{E} B_{n, p})}{B_{n, p} - \mathbf{E} B_{n, p}} \right)^{1/p} \xrightarrow{p} \frac{1}{\rho^2} \tag{3.8}
\]
Hence
\[
\sqrt{n} \left( \hat{B}_{n, p} - (\mathbf{E} B_{n, p}) \right)^{1/p} \xrightarrow{D} N(0, \sigma^2 A_n). \tag{3.9}
\]
Equation (3.3) is thus proved. Now, let
\[
A_n = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left\{ f \left( \frac{S_j(l) - b}{\sqrt{l}} \right) - f \left( \frac{S_j(l) - b}{\sqrt{l}} \right) \right\} + \sqrt{l}(\bar{X}_n - \mu) \sigma^{-1} E f'(N), \tag{3.10}
\]
where \( f(x) = |x|^p, (x^+)^p, \) or \((x^-)^p\). Using Taylor’s formula, Zhang and Shi (1998) proved that if \( p > 1 \) and \( \mathbf{E} |X_1|^{2p} < \infty \), then
\[
A_n \to 0 \quad \text{in} \ L_1, \tag{3.11}
\]
Note that if \( f(x) = |x|^p \) here, we have \( f'(x) = |x|^{p-1} \text{sgn } x \), and then \( \mathbf{E} f'(N) = 0 \). It follows that
\[
\sqrt{n} \left( B_{n, p}^* - \hat{B}_{n, p} \right) \to 0 \quad \text{in} \ L_1. \tag{3.12}
\]
Hence $EB_{n,p}^+ \to \sigma^p$, $B_{n,p} \xrightarrow{p} \sigma$, and
\[
\sqrt{\frac{m}{T}} (B_{n,p}^p - \mathbf{E}B_{n,p}^p) \xrightarrow{p} N(0, \Sigma_p^2).
\]
(3.13)

Similar to (3.9), we have
\[
\sqrt{\frac{m}{T}} (B_{n,p} - (\mathbf{E}B_{n,p})^{1/p})
= \frac{B_{n,p}(\mathbf{E}B_{n,p})^{1/p} - 1}{(\mathbf{E}B_{n,p})^{(p-1)/p}}
\to N(0, \sigma^2 P).
\]
(3.14)

So, (3.4) is proved.

Finally, from (3.12) it follows that $\mathbf{E}B_{n,p} = c_p \mathbf{E} \left( \frac{(S(l) - \mu)^2}{\sqrt{l}} \right)^p$ can be replaced by $\mathbf{E}B_{n,p}^p$. To show that $(\mathbf{E}B_{n,p})^{1/p}$ can be replaced by $\mathbf{E}B_{n,p}$ and $\mathbf{E}B_{n,p}$, it is enough to show that the random variables on the left-hand side of "=" in (3.9) and (3.14) are uniformly integrable. By noting (3.8), (3.9), (3.14), and that $(x - 1)/(x^p - 1)$ is a bounded function, it is sufficient to show that $\sqrt{m/l} (B_{n,p}^p - \mathbf{E}B_{n,p})$ and $\sqrt{m/l} (\mathbf{E}B_{n,p} - \mathbf{E}B_{n,p})$ are uniformly integrable. By (3.12) again, it is sufficient to show that $\sqrt{m/l} (\mathbf{E}B_{n,p} - \mathbf{E}B_{n,p})$ are uniformly integrable, which can be implied by
\[
\mathbf{E} \left\{ \sqrt{\frac{m}{T}} (\mathbf{E}B_{n,p} - \mathbf{E}B_{n,p}) \right\}^2 \leq CE \left\{ \left( \frac{S(l) - \mu}{\sqrt{l}} \right)^p \right\}^2 \leq CE \left| X_1 \right|^{2p} < \infty.
\]

The proof is completed.

Zhang and Shi (1998) also gave the following estimators of $\sigma$:
\[
B_{n,p}^+ = \frac{c_p}{n-l+1} \sum_{j=0}^{n-l} \left( \frac{(S(l) - \mu)^2}{\sqrt{l}} \right)^p,
\]
\[
\hat{B}_{n,p} = \frac{c_p}{n-l+1} \sum_{j=0}^{n-l} \left( \frac{(S(l) - \mu)^2}{\sqrt{l}} \right)^p.
\]
(3.15)

The following gives the limiting distributions of $B_{n,p}^+$. 

FUNCTIONS OF NA RANDOM VARIABLES
Theorem 4. Suppose that the conditions in Theorem 3 are satisfied. Then
\[
\sqrt{n}\left(\hat{\beta}_{n,p} - \left(\mathbf{E}\hat{\beta}_{n,p}\right)^{1/p}\right) \overset{d}{\to} N(0, \sigma^2 \tilde{A}_p),
\] (3.16)
where \( \tilde{A}_p = 8(c_p/p)^2 \sum_i \text{Cov}(\{ [(W(1))^\pm]_i \}, [(W(1) - W(t))^\pm]_i) \text{ dt} \). If \( p > 1 \), then
\[
\sqrt{n}\left(\hat{\beta}_{n,p} - \left(\mathbf{E}\hat{\beta}_{n,p}\right)^{1/p}\right) \overset{d}{\to} N(0, \sigma^2(\tilde{A}_p - (c_p/p)^2)).
\] (3.17)

Proof. By choosing \( f(x) = (x^+)^p \), the proof of (3.16) is similar to that of (3.3). From (3.11) and (3.10), it follows that
\[
1 - \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left\{ f\left(\frac{S_j(l) - lX_a}{\sqrt{l}}\right) - f\left(\frac{S_j(l) - \mu}{\sqrt{l}}\right) - \frac{S_j(l) - \mu}{\sqrt{l}} \sigma^{-1} \mathbf{E} f'(N) \right\} \overset{d}{\to} 0
\]
i.e.,
\[
1 - \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left\{ \left(\frac{(S_j(l) - lX_a)^+}{\sqrt{l}}\right)^p - \left(\frac{(S_j(l) - \mu)^+}{\sqrt{l}}\right)^p - \frac{S_j(l) - \mu}{\sqrt{l}} 2 \sigma^{-1} c_{p-1} \right\} \overset{d}{\to} 0.
\] (3.18)

By letting \( f(x) = (x^+)^p - \frac{p}{\sigma} \sigma^{-1} c_{p-1} x \), from (3.18) and Theorem 2 it follows that
\[
\sqrt{n}\left(\hat{\beta}_{n,p} - \left(\mathbf{E}\hat{\beta}_{n,p}\right)^{1/p}\right) \overset{d}{\to} N(0, \Sigma_p^2),
\]
where \( \Sigma_p^2 = 8c_p \sum_i \text{Cov}(\{ f(\sigma W(1)), f(\sigma(W(1) - W(t))\}) \text{ dt} \). Let \( N_1 = W(1), N_2 = W(1 + t) - W(t) \), then \( \text{Cov}(N_1, N_2) = 1 - t \) and
\[
\text{Cov}(N_1^+, N_2) = \text{Cov}(N_1, N_2) \frac{1}{2} c_{p-1}^{-1} = \text{Cov}(N_1, N_2) \frac{1}{2} c_{p-1}^{-1}.
\]
Then
\[
\text{Cov}\left\{ N_1^+, N_2 \right\} = \frac{p}{2} c_{p-1}^{-1} N_1, N_2^+ = \frac{p}{2} c_{p-1}^{-1} N_2 \}
\]
\[
= \text{Cov}(N_1^+, N_2^+) = \frac{p^2}{4} c_{p-1}^{-2}(1 - t).
\]
It follows that \( \sum_{i=1}^{n} p_i = p^2 \gamma^2 \left( \bar{A}_p - (c_p/c_{p-1})^2 \right) \). The remainder of the proof is similar to that of Theorem 3.

Finally, we consider the case of PA. The following result improved Theorem 1.3 and Corollary 1.4 of Peligrad and Suresh (1995).

**Theorem 5.** Let \( \{X_n; n \geq 1\} \) be a sequence of stationary PA random variables with \( \mathbb{E} X_1 = \mu, \mathbb{E} X_2 < \infty \), and \( \sigma^2 := \text{Var}(X_1) + 2 \sum_{j=2}^{\infty} \text{Cov}(X_1, X_j) < \infty \).

Then
\[
\frac{\sqrt{n}}{\sqrt{\pi}} \left( \bar{B}_{n,1} - \frac{\sum_{j=1}^{k_n} \mathbb{E} S_j(l) - l \mu}{\sqrt{l}} \right) \xrightarrow{d} N \left( 0, \frac{3\pi - 8}{4} \sigma^2 \right). \quad (3.19)
\]

**Proof.** Without loss of generality, we assume \( \mu = 0 \). Let \( f(x) = |x| \) and
\[
V_i = \sum_{j=0}^{l-1} \{ |S_{(i-1)l+j}(l)| - \mathbb{E} |S_{(i-1)l+j}| \},
\]
\[
\bar{V}_i = \sum_{j=0}^{l-1} \{ S_{(i-1)l+j}(l) - \mathbb{E} S_{(i-1)l+j}(l) \}, \quad i = 1, \ldots, k_n = \left\lfloor \frac{n-l}{T} \right\rfloor - 1.
\]

Note that \( \|f'\|_{\infty} \leq 1 \). Similarly to (2.12') we have
\[
\left| \mathbb{E} e^{i(u \sqrt{\pi} \sum_{j=1}^{k_n} V_j - \sqrt{\frac{n}{\pi}} f'(V_j))} - \sum_{1 \leq i < j < k_n} \text{Cov}(\bar{V}_i, \bar{V}_j) \right| \leq 4 \frac{\pi^2}{l^2} \|f'\|_{\infty} \sum_{1 \leq i < j < k_n} \text{Cov}(\bar{V}_i, \bar{V}_j).
\]

The remainder of the proof is similar to that of Theorem 1.3 in Peligrad and Suresh (1995).

**Remark.** Note that
\[
\frac{\sqrt{n}}{\sqrt{\pi}} |B_{n,1} - \bar{B}_{n,1}| \leq \frac{|S_n|}{\sqrt{n}} \frac{\sqrt{\pi}}{\sqrt{\pi}}.
\]

By Theorem A, from (3.19) it follows that
\[
\limsup_{n \to \infty} \mathbb{P} \left\{ \frac{\sqrt{n}}{\sqrt{\pi}} |B_{n,1} - \bar{B}_{n,1}| \leq \frac{|S_n|}{\sqrt{n}} \frac{\sqrt{\pi}}{\sqrt{\pi}} \right\} \leq 2 \mathbb{P}(|N| \geq x), \quad (3.20)
\]
where \( A = \sqrt{\frac{\pi}{2}} \left( \sqrt{\frac{3\pi - 8}{4}} + 1 \right) \sigma \).
REFERENCES