On the congruence of square real matrices

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Abstract

We show that if A and B are real n by n matrices which are ∗-congruent (i.e., P∗AP = B for some invertible complex matrix P), then A and B are congruent over the real numbers (i.e., QT AQ = B for some invertible real matrix Q). This statement remains true if P is assumed to be an invertible quaternionic matrix.

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1. Introduction

If two real n by n matrices are similar over complex numbers, C, then they are also similar over the real numbers, R. More generally, if two matrices over a field F are similar over an extension field E, then they are also similar over F. Of course, these facts are well known.

Let us mention another result of similar nature: If two real n by n matrices are unitarily similar, then they are also similar by a real orthogonal matrix. The proof of this fact, based on Specht’s theorem, can be found, e.g., in Kaplansky’s book...
[3, Theorem 65, p. 75] and is much harder than the proof of the results mentioned above. A direct proof of this fact was given recently by Merino [4] (see also [2] or [8, Theorem 5.13, p. 153]).

In the same spirit we can ask the following natural question (which is the main topic of this article): If \(A\) and \(B\) are real \(n\) by \(n\) matrices such that \(P^*AP = B\) for some invertible complex matrix \(P\), is it true that there is an invertible real matrix \(Q\) such that \(Q^T AQ = B\)? (For a complex matrix \(X\) we denote its transpose by \(X^T\) and its transpose conjugate matrix by \(X^*\).)

The answer is again affirmative but the proof, that we give later, is not elementary.

**Theorem 1.1.** If \(P^*AP = B\) where \(A\) and \(B\) are \(n\) by \(n\) real matrices and \(P\) is an invertible complex matrix, then there exists an invertible real matrix \(Q\) such that \(Q^T AQ = B\).

We point out the trivial fact that the assertion of the theorem becomes false if \(P^*\) is replaced by \(P^T\). It may be also of interest to mention that there is a nice expository paper of Thompson [7] on the canonical forms for pairs of complex or real matrices that are symmetric or skew-symmetric.

Our proof is based on the paper of Riehm and Shrader-Frechette [6] and its predecessor, the paper of Riehm [5] and an addendum to it by Gabriel [1]. This paper solves the equivalence problem for sesquilinear forms on finitely generated modules over semisimple (Artinian) rings. We need only to apply this general theory in two simple settings: The bilinear forms over \(\mathbb{R}\) and the sesquilinear forms over \(\mathbb{C}\) (with respect to the ordinary complex conjugation). Let us recast the above theorem in the language of bilinear and sesquilinear forms.

**Definition 1.2.** A sesquilinear form on a complex vector space \(V\) is a mapping \(f: V \times V \to \mathbb{C}\) which is complex linear in the second variable and anti-linear in the first:

\[
\begin{align*}
    f(x, ay + bz) &= af(x, y) + bf(x, z), \\
    f(ax + by, z) &= \bar{a}f(x, z) + \bar{b}f(y, z)
\end{align*}
\]

for all \(x, y, z \in V\) and \(a, b \in \mathbb{C}\). A **sesquilinear space** is a pair \((V, f)\) where \(V\) is a finite-dimensional complex vector space and \(f\) a sesquilinear form on \(V\).

The definition of equivalence of two sesquilinear forms is the usual one.

**Definition 1.3.** Two sesquilinear forms \(f: V \times V \to \mathbb{C}\) and \(g: W \times W \to \mathbb{C}\) are **equivalent** if there exists a vector space isomorphism \(\varphi: V \to W\) such that \(g(\varphi(x), \varphi(y)) = f(x, y)\).
\( \psi(y) = f(x, y) \ \forall x, y \in V \). In that case, assuming that \( V \) and \( W \) are finite-dimensional, we also say that the sesquilinear spaces \((V, f)\) and \((W, g)\) are isometric and that \( \psi \) is an isometry.

Let \( f \) and \( g \) be as in the above definition and assume that \( \dim(V) = \dim(W) = n < \infty \). We fix a basis \( \{v_1, v_2, \ldots, v_n\} \) of \( V \). Then the \( n \) by \( n \) matrix \( A = (a_{ij}) \) where \( a_{ij} = f(v_i, v_j) \) is the matrix of \( f \) with respect to this basis. Similarly, let \( B = (b_{ij}) \) be the matrix of \( g \) with respect to a fixed basis \( \{w_1, w_2, \ldots, w_n\} \) of \( W \). To say that \( f \) and \( g \) are equivalent is the same as to say that \( P^*AP = B \) for some invertible complex matrix \( P \).

Assume now that the matrices \( A \) and \( B \) are real and let \( V' \) (resp. \( W' \)) be the \( \mathbb{R} \)-span of the above basis of \( V \) (resp. \( W \)). Then the restrictions \( f' : V' \times V' \to \mathbb{R} \) and \( g' : W' \times W' \to \mathbb{R} \) are real bilinear forms.

By using these notations, our theorem can be restated as follows.

**Theorem 1.4.** \( \text{If } f \text{ and } g \text{ are equivalent, then } f' \text{ and } g' \text{ are equivalent.} \)

The proof of this theorem will be given in the next five sections. In the last section we state a theorem analogous to Theorem 1.1 where \( \mathbb{C} \) is replaced by the algebra \( \mathbb{H} \) of real quaternions.

**2. Real and complex Kronecker modules**

In this section we introduce two special classes of Kronecker modules, real and complex, by specializing the general Kronecker modules discussed in \([1,6]\), and describe their relationship to real bilinear and sesquilinear spaces, respectively. The reader should consult these references for more details.

We define a real Kronecker module as a four-tuple \((X, u, v, Y)\) where \( X \) and \( Y \) are finite-dimensional real vector spaces and \( u, v : X \to Y \) are linear maps. If \( Z \) is a finite-dimensional real vector space and \( h : Z \times Z \to \mathbb{R} \) a real bilinear form then we shall say that \((Z, h)\) is a real bilinear space. To such a space we assign the real Kronecker module \( K(Z) = (Z, h_l, h_r, Z^*) \), where \( Z^* \) is the dual space of \( Z \) and \( h_l, h_r : Z \to Z^* \) are defined by

\[
\begin{align*}
    h_l(x)(y) &= h(x, y), \\
    h_r(x)(y) &= h(y, x)
\end{align*}
\]

\( \forall x, y \in Z \).

Every real Kronecker module is a direct sum of indecomposable ones which are unique up to ordering and isomorphism. There are five types of indecomposable real Kronecker modules \((X, u, v, Y)\):

I Both \( u \) and \( v \) are isomorphisms and \( u^{-1}v \) is indecomposable (i.e., it has only one elementary divisor).

II The spaces \( X \) and \( Y \) have the same dimension and the pencil \( \lambda u + \mu v \), with respect to suitable bases of \( X \) and \( Y \), has the matrix
II* Similar to II with the matrix
\[
\begin{bmatrix}
\mu & \lambda \\
\mu & \lambda \\
\lambda & \mu \\
\end{bmatrix}
\]

III In this case \(\dim(Y) = \dim(X) + 1\) and the pencil \(\lambda u + \mu v\) has the matrix
\[
\begin{bmatrix}
\lambda & \mu \\
\mu & \lambda \\
\end{bmatrix}
\]

III* In this case \(\dim(X) = \dim(Y) + 1\) and the matrix is
\[
\begin{bmatrix}
\mu & \lambda \\
\mu & \lambda \\
\end{bmatrix}
\]

If \((Z, h)\) is a real bilinear space and \(Z = Z_1 + Z_2\) is a direct decomposition of \(Z\) such that \(h(Z_1, Z_2) = 0\) and \(h(Z_2, Z_1) = 0\), then we say that this space is an orthogonal direct sum of the bilinear spaces \((Z_1, h_1)\) and \((Z_2, h_2)\), where \(h_1\) and \(h_2\) are the corresponding restrictions of \(h\). The following theorem is a very special case of the general result stated in [6, Section 9].

**Theorem 2.1.** Every real bilinear space \((Z, h)\) can be decomposed into an orthogonal direct sum

\[Z = Z_1 + Z_{II} + Z_{III}\]

such that all indecomposable direct summands of \(K(Z_1)\) are of type I, those of \(K(Z_{II})\) are of type II or II*, and those of \(K(Z_{III})\) are of type III or III*. If \(h_1\) is the restriction of \(h\) to \(Z_1 \times Z_1\), etc., the bilinear spaces \((Z_1, h_1), (Z_{II}, h_{II}), (Z_{III}, h_{III})\) are uniquely determined by \((Z, h)\) up to isometry.

Moreover, if \(Z = Z_{II}\) or \(Z = Z_{III}\), then \(K(Z)\) determines \((Z, h)\) up to isometry.

A complex Kronecker module is a four-tuple \((X, u, v, Y)\) where \(X\) and \(Y\) are finite-dimensional complex vector spaces and \(u, v : X \rightarrow Y\) are anti-linear maps. To a
3. Reduction to the nondegenerate case

We now begin the proof of Theorem 1.4. We warn the reader that the notation and the definitions of the invariants of bilinear spaces in [5,6] do not agree. The second paper is more general and we shall exclusively use the definitions given there.

By Theorem 2.1 there is an orthogonal direct decomposition

\[ V' = V'_I + V'_II + V'_III, \]

where the summands \( V'_I, V'_II \) and \( V'_III \) have the properties stated there. We have also a similar decomposition \( W' = W'_I + W'_II + W'_III \). If \( V_I = V'_I + iV'_I \) is the complexification of \( V'_I \), etc. then we obtain orthogonal direct decompositions of the sesquilinear spaces:

\[ V = V_I + V_{II} + V_{III}, \quad W = W_I + W_{II} + W_{III}. \]

By the complex version of Theorem 2.1, the following sesquilinear spaces are isometric:

\[ V'_I \cong W'_I, \quad V'_II \cong W'_II, \quad V'_III \cong W'_III. \] (3.1)

We deduce that \( K_c(V'_I) \cong K_c(W'_I) \) and \( K_c(V'_III) \cong K_c(W'_III) \).

We claim that the bilinear spaces \( V'_II \) and \( W'_II \) are isometric, and so are \( V'_III \) and \( W'_III \). The real Kronecker module \( K(V'_II) \) is a direct sum of indecomposable summands of type II or II*. After complexification, these summands remain indecomposable and their matrices remain the same. Hence \( K_c(V'_II) \cong K_c(W'_II) \) implies that \( K(V'_III) \cong K(W'_III) \). The same argument shows that also \( K(V'_III) \cong K(W'_III) \). Now the last assertion of Theorem 2.1 shows that our claim is true.

It remains to show that the real bilinear spaces \( V'_I \) and \( W'_I \) are isometric. Since the sesquilinear spaces \( V_I \) and \( W_I \) are isometric, the proof of our theorem has been reduced to the nondegenerate case, i.e., the case where \( A \) (and \( B \)) is a nonsingular matrix.

4. Reduction to the primary case

We assume in this section that \( f \) (and \( g \)) is nondegenerate. We shall use a number of results of Riehm and Shrader-Frechette [6] without explicit reference and the reader should consult this paper for the claims made but not proved here.

We recall from [6] that the asymmetry of \( f \) is the linear operator \( \alpha : V \rightarrow V \) such that \( f(x,y) = f(\alpha(y),x) \ \forall x, y \in V \). Its matrix, with respect to our fixed basis of
The matrix of the asymmetry \( \alpha' \) of \( f' \). Similarly, let \( \beta \) (resp. \( \beta' \)) be the asymmetry of \( g \) (resp. \( g' \)) and let \( G \) be their common matrix.

The equivalence of sesquilinear forms \( f \) and \( g \) entails that their asymmetries \( \alpha \) and \( \beta \) are similar operators, i.e., the matrices \( F \) and \( G \) are similar over \( \mathbb{C} \). As \( F \) and \( G \) are real matrices, they are also similar over \( \mathbb{R} \). We conclude that the asymmetries of \( f' \) and \( g' \) are similar operators.

The monic irreducible polynomials \( p \) in \( \mathbb{R}[X] \) are either linear or quadratic. For such \( p \neq X \) we define the monic irreducible polynomial \( p^* \in \mathbb{R}[X] \) by \( p^*(x) = p(0)^{-1}x^d p(x^{-1}) \), where \( d \) is the degree of \( p \). Let us decompose \( V' \) into primary components with respect to \( \alpha' \)

\[
V' = \bigoplus_p V'_p,
\]

where the sum is over the monic irreducible polynomials \( p \in \mathbb{R}[X], \; p \neq X \). Similarly, we have

\[
W' = \bigoplus_p W'_p.
\]

The subspaces \( V'_p \) and \( V'_q \) are orthogonal, i.e., \( f'(V'_p, V'_q) = 0 \) and \( f'(V'_q, V'_p) = 0 \) if \( q \neq p^* \). If \( p \) has its roots on the unit circle in the complex plane, then \( p^* = p \) and otherwise \( p^* \neq p \). If \( p^* \neq p \) then \( V'_p + V'_p^* \), and \( W'_p + W'_p^* \) are isometric bilinear spaces by [6, Theorem 16, Corollary]. It remains to deal with the case \( p^* = p \).

The monic irreducible polynomials \( p \in \mathbb{C}[X] \) are all linear, say \( p(X) = X - c \). If \( c \neq 0 \) then \( p^* \) is defined by \( p^*(x) = X - (\bar{c})^{-1} \). The primary decomposition of \( V \) with respect to the asymmetry \( \alpha \) is \( V = \bigoplus_c V_c \), where \( c \) runs through nonzero complex numbers and \( V_c \) is the primary component corresponding to \( X - c \). The primary components \( V_c \) and \( V_{\bar{d}} \) are orthogonal iff \( d \neq (\bar{c})^{-1} \).

It is easy to describe the complexification of the primary component \( V'_p \). If \( p \) is quadratic with roots \( c \) and \( \bar{c} \) (both on the unit circle), then the complexification of \( V'_p \) is \( V_c + V_{\bar{c}} \). Otherwise \( p = X - c \) with \( c = \pm 1 \), and the complexification of \( V'_p \) is just \( V_c \). In both cases this complexification is isometric, as a sesquilinear space, to the complexification of \( W'_p \) (see loc. cit.).

Hence the proof of Theorem 1.4 has been reduced to the primary case: The minimal polynomial of \( \alpha' \) is a power of \( p \), where \( p = p^* \) is an irreducible polynomial in \( \mathbb{R}[X] \).

5. Reduction to the homogeneous primary case

In this section we consider the primary case as just described above. By [6, Proposition 25] there exists an orthogonal direct decomposition
\[ V' = \bigoplus_{s \geq 1} V'_s \]
such that \( V'_s \subseteq \ker(p(\alpha')^s) \) and the induced map
\[ V'_s / p(\alpha')(V'_s) \to \ker \left( (p(\alpha')^s) \big/ \ker \left( p(\alpha')^{s-1} + p(\alpha') \ker (p(\alpha')^{s+1}) \right) \right) \]
is an isomorphism for each \( s \). Denote by \( V_s' \) the complexification of \( V'_s \). Let \( f'_s : V'_s \times V'_s \to \mathbb{R} \) (resp. \( f_s : V_s \times V_s \to \mathbb{C} \)) be the restriction of \( f' \) (resp. \( f \)). Define the forms \( g'_s \) and \( g_s \) analogously. Since \( f \) and \( g \) are equivalent, [6, Theorem 27] implies that \( f_s \) and \( g_s \) are equivalent for each \( s \). Hence, without any loss of generality, we may assume that \( V' = V'_s \) for some \( s \). In other words, we may assume that \( \alpha' \) has only one elementary divisor with arbitrary multiplicity. We refer to this case as the \textit{homogeneous primary case}.  

6. The homogeneous primary case

The minimal polynomial of \( \alpha' \) is a power of \( p \), say \( p^r \), where \( p \) is as in the previous section, and all elementary divisors of \( \alpha' \) are equal to \( p^r \). Let \( r \) be the number of these elementary divisors, i.e., \( n = rsd \) where \( d \) is the degree of \( p \) (\( d = 1 \) or \( 2 \)). In order to prove that \( f' \) and \( g' \) are equivalent, it suffices to check that they have the same invariant attached to them by [6, Theorem 27]. This will be so if the invariants of the sesquilinear form \( f \) determine uniquely the invariant of \( f' \). That is exactly what we are going to show.

Assume that \( p = X - 1 \) and set \( \pi' = 1 - (\alpha')^{-1} \) and \( \pi = 1 - \alpha^{-1} \). Define \( \tilde{V}' = V' / p'(V') \) and \( \tilde{V} = V / \pi(\tilde{V}) \). If \( s \) is even, then the invariant attached to \( f' \) (see [6, p. 517]) is a nondegenerate skew-symmetric real bilinear form on the \( r \)-dimensional space \( \tilde{V}' \). (Hence if \( s \) is even then \( r \) must be even.) Therefore the invariant is unique up to equivalence. We now assume that \( s \) is odd, in which case the invariant is the nondegenerate symmetric bilinear form \( \tilde{f}' \) on \( \tilde{V}' \) defined by
\[ \tilde{f}'(\tilde{x}, \tilde{y}) = f'(\pi'(x), y) \quad \forall x, y \in V', \]
where \( \tilde{x} \) denotes the canonical image of \( x \) in \( \tilde{V}' \).

The invariant of the sesquilinear form \( f \) is the unique nondegenerate hermitian form \( \tilde{f} \) on \( \tilde{V} \) such that
\[ \tilde{f}(\tilde{x}, \tilde{y}) = f(\pi^{-1}(x), y) \quad \forall x, y \in V, \]
where now \( \tilde{x} \) is the canonical image of \( x \) in \( \tilde{V} \). As \( \tilde{f}' \) is just the restriction of \( \tilde{f} \), the equivalence class of \( \tilde{f}' \) is uniquely determined by that of \( \tilde{f} \).

In the case \( p = X + 1 \) the proof is similar.

It remains to consider the case where \( p \) is quadratic with roots \( e \) and \( \bar{e} \) on the unit circle. We set \( \pi' = p((\alpha')^{-1}) \) and \( \pi = 1 - c\alpha^{-1} \). Let \( V_e \) and \( V_{\bar{e}} \) be the primary
components of $\alpha$ for the irreducible polynomials $X - c$ and $X - \bar{c}$, respectively. Define again $\tilde{V} = V'/\pi'(V')$ and $\tilde{V}_c = V_c/\pi(V_c)$. We can identify the quotient ring $\mathbb{R}[X]/(p)$ with $\mathbb{C}$ via the isomorphism which sends the image of $X$ to $c$. The space $V'$ is naturally a module over $\mathbb{R}[X]/(p)$ with the image of $X$ acting as $\alpha'$, and so, by using the above identification, we can make $\tilde{V}$ into a complex vector space. The invariant of the bilinear form $f'$ is the unique nondegenerate $c^{2t-1}$-hermitian form $\tilde{f}'$ on this complex vector space such that

$$2 \Re \tilde{f}'(\tilde{x}, \tilde{y}) = f'((\pi')^{t-1}(x), y) \quad \forall x, y \in V'.$$

Recall that a sesquilinear form $h$ is called $\mu$-hermitian if $h(y, x) = \mu h(x, y)$ for all vectors $x, y$.

The sesquilinear space $V$ is an orthogonal direct sum of the two primary components $V_c$ and $V_{\bar{c}}$ and has, apart from asymmetry, two invariants attached to it corresponding to $X - c$ and $X - \bar{c}$. We need only the former: The nondegenerate $(-1)^{t-1}c$-hermitian form $f_c$ on $\tilde{V}_c$ such that

$$f_c(\tilde{x}, \tilde{y}) = f((\pi)^{t-1}(x), y) \quad \forall x, y \in V_c.$$

We have to show that the equivalence class of $\tilde{f}'$ is uniquely determined by that of $f_c$. In fact we shall prove that

$$\tilde{f}' = (1 - c^2)^{t-1} f_c \circ (\psi \times \psi),$$

where $\psi : \tilde{V} \to \tilde{V}_c$ is a canonical isomorphism which will be defined below.

Note that $V'$ is a real form of $V$, i.e., $V$ is the direct sum of the real subspaces $V'$ and $iV'$. For a vector $x = y + iz$ with $y, z \in V'$ we define $\tilde{x} = y - iz$ and we refer to the map $x \to \tilde{x}$ as the conjugation with respect to $V'$. Note that $\alpha$ is the complexification of $\alpha'$, i.e., we have $\alpha(x) = \alpha'(y) + i\alpha'(z)$ for $x \in V$ as above (with real part $y$ and imaginary part $z$). It follows that $\alpha$ commutes with the conjugation, i.e. $\alpha(\tilde{x}) = \bar{\alpha(x)}$.

An arbitrary vector $x \in V'$ can be written uniquely as $x = y + z$ with $y \in V_c$ and $z \in V_{\bar{c}}$. Since the conjugation interchanges $V_c$ and $V_{\bar{c}}$, we must have $\bar{z} = y$. The map $\varphi : V' \to V_c$ defined by $\varphi(x) = y$ is an isomorphism of real vector spaces. Thus

$$x = \varphi(x) + \overline{\varphi(x)} \quad \forall x \in V'.$$

It is easy to check that

$$\varphi\alpha' = \alpha \varphi.$$

The isomorphism $\varphi$ induces an isomorphism $\psi : \tilde{V}' \to \tilde{V}_c$. This is an isomorphism of complex vector spaces if we use the complex vector space structure on $V'$ introduced above.
By (6.3) and the fact that \( \pi s^{-1}(V_c) \) is the eigenspace of \( \alpha \) for the eigenvalue \( c \), we obtain the following formula valid for all \( x, y \in V' \):

\[
\phi \circ (\pi')s^{-1}(x) = \phi \circ p((\alpha')s^{-1})(x) = p(\alpha^{-1})s^{-1}\phi(x) = (1 - c\alpha^{-1})s^{-1}\pi s^{-1}\phi(x) = (1 - (\bar{c})^2)s^{-1}\pi s^{-1}\phi(x).
\]

Let us define the sesquilinear form \( h \) on \( \tilde{\mathbb{V}}' \) by

\[
h(\tilde{x}, \tilde{y}) = (1 - c^2)s^{-1}f_c(\psi(\tilde{x}), \psi(\tilde{y})).
\]

(6.4)

The formula proved above implies that

\[
h(\tilde{x}, \tilde{y}) = f(\phi \circ (\pi')s^{-1}(x), \phi(y)) \quad \forall x, y \in V'.
\]

By using (6.2) and the facts that \( f(u, v) = f(\bar{u}, \bar{v}) \) for all \( u, v \in V \), and \( V_c \) and \( V_{\bar{c}} \) are orthogonal with respect to \( f \), we obtain

\[
2\text{Re} h(\tilde{x}, \tilde{y}) = f(\phi \circ (\pi')s^{-1}(x), \phi(y)) + f(\phi \circ (\pi'\bar{s})^{-1}(x), \psi(y))
\]

\[
= f(\phi \circ (\pi')s^{-1}(x) + \phi \circ (\pi'\bar{s})^{-1}(x), \phi(x) + \phi(y))
\]

\[
= f((\pi')s^{-1}(x), y)
\]

\[
= f'(((\pi')s^{-1}(x), y).
\]

Hence indeed \( \tilde{f}' = h \) and the formula (6.1) is valid. As \( f_c \) is \((-1)^{s-1}c\)-hermitian, it follows immediately from the definition (6.4) that \( h \) is \( c^{2s-1}\)-hermitian. Hence the proof is completed.

7. The quaternionic case

Let \( \mathbb{H} \) be the division algebra of real quaternions and let \( 1, i, j, k \) be the standard quaternionic units. Write \( a \in \mathbb{H} \) as a linear combination \( a = a_0 + a_1i + a_2j + a_3k \) with \( a_0, a_1, a_2, a_3 \in \mathbb{R} \). The conjugation of \( \mathbb{H} \) is the involutory anti-automorphism \( a \to \bar{a} \) where \( \bar{a} = a_0 - a_1i - a_2j - a_3k \). This involution of \( \mathbb{H} \) extends to an involution \* on the \( \mathbb{R} \)-algebra \( M_n(\mathbb{H}) \) of \( n \) by \( n \) quaternionic matrices: For \( A = (a_{ij}) \in M_n(\mathbb{H}) \), the matrix \( A^* \) is the conjugate transpose of the matrix \( A \), i.e. the \((i, j)\)th entry of \( A^* \) is \( \bar{a}_{ji} \). We can consider the \( \mathbb{R} \)-algebra \( M_n(\mathbb{R}) \) of \( n \) by \( n \) real matrices as a subalgebra of \( M_n(\mathbb{H}) \). Then the \* operation on \( M_n(\mathbb{H}) \) extends the transposition operation on \( M_n(\mathbb{R}) \).
We can now state the following quaternionic version of Theorem 1.1.

**Theorem 7.1.** If $P^*AP = B$ where $A$ and $B$ are $n$ by $n$ real matrices and $P$ is an invertible quaternionic matrix, then there exists an invertible real matrix $Q$ such that $Q^TAQ = B$.

The proof is similar to that given in the previous sections and will be omitted. Of course, one can reformulate this theorem in the language of sesquilinear forms $f : V \times V \rightarrow \mathbb{H}$ where $V$ is a finite-dimensional right $\mathbb{H}$-vector space.

8. **Acknowledgments**

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**References**