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Note

Arithmetic properties of q-Fibonacci numbers and q-pell numbers

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Abstract

We investigate some arithmetic properties of the q-Fibonacci numbers and the q-Pell numbers.

Keywords: q-Fibonacci number; q-Pell number; Jacobi symbol

1. Introduction

The Fibonacci numbers F_n are given by

$$F_0 = 0,\quad F_1 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geqslant 2.$$

For any odd prime p, it is well-known (cf. [11, pp. 44–50]) that

$$F_p \equiv \left(\frac{5}{p}\right) \pmod p, \tag{1.1}$$

$$F_{p+1} \equiv \frac{1}{2}\left(1 + \left(\frac{5}{p}\right)\right) \pmod p \tag{1.2}$$

and

$$F_{p-1} \equiv \frac{1}{2}\left(1 - \left(\frac{5}{p}\right)\right) \pmod p, \tag{1.3}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. Indeed, we have

$$F_p = \frac{(1+\sqrt{5})^p - (1-\sqrt{5})^p}{2^p\sqrt{5}} \equiv \frac{(1+(\sqrt{5})^p) - (1-(\sqrt{5})^p)}{2\sqrt{5}} = 5^{(p-1)/2} \pmod p.$$

For more results on the congruences involving the Fibonacci numbers, the readers may refer to [14–16].

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On the other hand, a sequence of polynomials $\mathscr{F}_n(q)$ was firstly introduced by Schur [13, pp. 117–136]

$$\mathscr{F}_n(q)=\begin{cases}0 & \text{if } n=0,\\ 1 & \text{if } n=1,\\ \mathscr{F}_{n-1}(q)+q^{n-2}\mathscr{F}_{n-2}(q) & \text{if } n\geqslant 2.\end{cases}$$

Also Schur considered another sequence $\widehat{\mathscr{F}}_n(q)$, which is given by

$$\widehat{\mathscr{F}}_n(q)=\begin{cases}0 & \text{if } n=0,\\ 1 & \text{if } n=1,\\ \widehat{\mathscr{F}}_{n-1}(q)+q^{n-1}\widehat{\mathscr{F}}_{n-2}(q) & \text{if } n\geqslant 2.\end{cases}$$

Obviously, both $\mathscr{F}_n(q)$ and $\widehat{\mathscr{F}}_n(q)$ are the q-analogues of the Fibonacci numbers. The sequences $\mathscr{F}_n(q)$ and $\widehat{\mathscr{F}}_n(q)$ have been investigated in several papers (e.g., [2,5–8]). However, seemingly there are no simple expressions for $\mathscr{F}_n(q)$ and $\widehat{\mathscr{F}}_n(q)$.

As usual we set

$$[n]_q=\frac{1-q^n}{1-q}=1+q+\cdots+q^{n-1}$$

for any non-negative integer n. In the proof of a theorem on the number of partitions [1, pp. 302–303], Andrews showed that for any odd prime p with $p\equiv\pm 2 \pmod 5$

$$\mathscr{F}_{p+1}(q)\equiv 0 \pmod{[p]_q},$$

which is a partially q-analogue of (1.2). Here, Andrews' congruence is considered over the ring of the polynomials in q with integral coefficients, rather than the ring of quantum integers [10].

In this paper we shall give the q-analogues of (1.1)–(1.3) for $\mathscr{F}_n(q)$ and $\widehat{\mathscr{F}}_n(q)$. Suppose that n is an odd integer with $5\nmid n$. Let α_n be the integer such that $1\leqslant\alpha_n\leqslant 4$ and $\alpha_n n\equiv 1 \bmod 5$. And let $\Phi_n(q)$ denote the nth cyclotomic polynomial in q, i.e.,

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ (k,n)=1}}(q-e^{2\pi ik/n}).$$

Clearly $\Phi_n(q)$ is a polynomial with integral coefficients, and $\Phi_p(q)=[p]_q$ when p is prime.

Theorem 1.1. *Let $n\geqslant 3$ be an odd integer with $5\nmid n$. Then*

$$\mathscr{F}_{n+1}(q)\equiv\frac{1}{2}\left(1+\left(\frac{5}{n}\right)\right) \pmod{\Phi_n(q)} \tag{1.4}$$

and

$$\mathscr{F}_n(q)\equiv\left(\frac{5}{n}\right)q^{((5-\alpha_n)n+1)/5} \pmod{\Phi_n(q)}, \tag{1.5}$$

where $\left(\frac{\cdot}{n}\right)$ denotes the Jacobi symbol (cf. [9, pp. 56–58]).

Theorem 1.2. *Let $n\geqslant 3$ be an odd integer with $5\nmid n$. Then*

$$\widehat{\mathscr{F}}_{n-1}(q)\equiv\frac{1}{2}\left(1-\left(\frac{5}{n}\right)\right) \pmod{\Phi_n(q)} \tag{1.6}$$

and

$$\widehat{\mathscr{F}}_n(q) \equiv \left(\frac{5}{n}\right) q^{(\alpha_n n-1)/5} \pmod{\Phi_n(q)}. \tag{1.7}$$

The Pell numbers P_n are given by

$$P_0 = 0, \quad P_1 = 1 \quad \text{and} \quad P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geqslant 2.$$

It is easy to check that

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$$

Hence for odd prime p, we have

$$P_p = \frac{(1+\sqrt{2})^p - (1-\sqrt{2})^p}{2\sqrt{2}} \equiv \frac{2(\sqrt{2})^p}{2\sqrt{2}} = 2^{(p-1)/2} \equiv \left(\frac{2}{p}\right) \pmod{p}. \tag{1.8}$$

In [12], Santos and Sills introduced two q-analogues of the Pell numbers. Define the sequences of polynomials $\mathscr{P}_n(q)$ and $\widehat{\mathscr{P}}_n(q)$ by

$$\mathscr{P}_n(q) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ (1+q^{n-1})\mathscr{P}_{n-1}(q) + q^{n-2}\mathscr{P}_{n-2}(q) & \text{if } n \geqslant 2 \end{cases}$$

and

$$\widehat{\mathscr{P}}_n(q) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ (1+q^{n-1})\widehat{\mathscr{P}}_{n-1}(q) + q^{n-1}\widehat{\mathscr{P}}_{n-2}(q) & \text{if } n \geqslant 2. \end{cases}$$

Clearly $\mathscr{P}_n(1) = \widehat{\mathscr{P}}_n(1) = P_n$. Now, we have the q-analogues of (1.8) for $\mathscr{P}_n(q)$ and $\widehat{\mathscr{P}}_n(q)$.

Theorem 1.3. *Let $n \geqslant 3$ be an odd integer. Then*

$$q^{(n^2-1)/8}\mathscr{P}_n(q) \equiv \left(\frac{2}{n}\right) \pmod{\Phi_n(q)} \tag{1.9}$$

and

$$\widehat{\mathscr{P}}_n(q) \equiv \left(\frac{2}{n}\right) q^{(n^2-1)/8} \pmod{\Phi_n(q)}. \tag{1.10}$$

Furthermore, we have

$$\mathscr{P}_{n+1}(q) - \mathscr{P}_n(q) \equiv \widehat{\mathscr{P}}_{n+1}(q) - \widehat{\mathscr{P}}_n(q) \equiv 1 \pmod{\Phi_n(q)}. \tag{1.11}$$

Since $\left(\frac{2}{n}\right) = (-1)^{(n^2-1)/8}$, (1.9) and (1.10) can be, respectively, rewritten as

$$(-q)^{(n^2-1)/8}\mathscr{P}_n(q) \equiv 1 \pmod{\Phi_n(q)}$$

and

$$\widehat{\mathscr{P}}_n(q) \equiv (-q)^{(n^2-1)/8} \pmod{\Phi_n(q)}.$$

The proofs of Theorems 1.1–1.3 will be given in Sections 2 and 3.

2. Proofs of Theorems 1.1 and 1.2

For any $n, m \in \mathbb{Z}$, the q-binomial coefficient $\left[{n \atop m}\right]_q$ is given by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q^m)(1-q^{m-1})\cdots(1-q)}$$

when $m \geqslant 0$, and let $\left[{n \atop m}\right]_q = 0$ if $m < 0$. Obviously $\left[{n \atop m}\right]_q$ is a polynomial in q with integral coefficients since q-binomial coefficients satisfy the recurrence relation

$$\begin{bmatrix} n+1 \\ m \end{bmatrix}_q = q^m \begin{bmatrix} n \\ m \end{bmatrix}_q + \begin{bmatrix} n \\ m-1 \end{bmatrix}_q = \begin{bmatrix} n \\ m \end{bmatrix}_q + q^{n-m+1} \begin{bmatrix} n \\ m-1 \end{bmatrix}_q .$$

Let $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Then for any non-negative integer n, we have

$$\begin{aligned} \mathscr{F}_{n+1}(q) &= \sum_{0 \leqslant 2j \leqslant n} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \\ &= \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} n \\ \lfloor (n-5j)/2 \rfloor \end{bmatrix}_q \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \widehat{\mathscr{F}}_{n+1}(q) &= \sum_{0 \leqslant 2j \leqslant n} q^{j^2+j} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \\ &= \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j-3)/2} \begin{bmatrix} n+1 \\ \lfloor (n+1-5j)/2 \rfloor + 1 \end{bmatrix}_q . \end{aligned} \tag{2.2}$$

Eqs. (2.1) and (2.2) can be considered as the finite forms of the first and the second of the Rogers–Ramanujan identities, respectively (the full proofs of (2.1) and (2.2) can be found in [1]).

Lemma 2.1. *Suppose that n is an arbitrary integer. Let*

$$L(j) = \frac{j(5j+1)}{2} - \binom{\lfloor (n-1-5j)/2 \rfloor + 1}{2},$$

and let

$$\widehat{L}(j) = \frac{j(5j-3)}{2} - \binom{\lfloor (n-1-5j)/2 \rfloor + 2}{2}.$$

Then if n is odd, we have

$$L(2j) - L(2j-1) = \widehat{L}(2j) - \widehat{L}(2j-1) = n.$$

And when n is even,

$$L(2j+1) - L(2j) = \widehat{L}(2j+1) - \widehat{L}(2j) = n.$$

Lemma 2.2. *Suppose that n is an integer prime to* 5. *Let*

$$S_n = \{ j \in \mathbb{Z} : 0 \leqslant \lfloor (n-1-5j)/2 \rfloor \leqslant n-1 \}$$

and

$$\widehat{S}_n = \{j \in \mathbb{Z}:\ 0 \leqslant \lfloor (n-1-5j)/2 \rfloor + 1 \leqslant n-1\}.$$

We have

$$S_n = \{j \in \mathbb{Z}:\ -\lfloor n/5 \rfloor \leqslant j \leqslant \lfloor n/5 \rfloor\}$$

and

$$\widehat{S}_n = \begin{cases} \{j \in \mathbb{Z} : -\lfloor n/5 \rfloor + 1 \leqslant j \leqslant \lfloor n/5 \rfloor\} & \textit{if } n \equiv 1 \bmod 5, \\ \{j \in \mathbb{Z} : -\lfloor n/5 \rfloor \leqslant j \leqslant \lfloor n/5 \rfloor\} & \textit{if } n \equiv 2, 3 \bmod 5, \\ \{j \in \mathbb{Z} : -\lfloor n/5 \rfloor \leqslant j \leqslant \lfloor n/5 \rfloor + 1\} & \textit{if } n \equiv 4 \bmod 5. \end{cases}$$

These two lemmas above can be verified directly, so we omit the details here.

Proof of Theorem 1.1. By the quadratic reciprocity law, we know that

$$\left(\frac{5}{n}\right) = \left(\frac{n}{5}\right) = \begin{cases} 1 & \text{if } n \equiv 1, 4 \bmod 5, \\ -1 & \text{if } n \equiv 2, 3 \bmod 5. \end{cases}$$

Since $\Phi_n(q)$ divides $[n]_q$ and $(\Phi_n(q), [j]_q) = 1$ for each $1 \leqslant j \leqslant n-1$, we have

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q[n-1]_q \ldots [n-m+1]_q}{[m]_q[m-1]_q \ldots [1]_q} \equiv \begin{cases} 1 \pmod{\Phi_n(q)} & \text{if } m = 0 \text{ or } n, \\ 0 \pmod{\Phi_n(q)} & \text{if } 1 \leqslant m \leqslant n-1. \end{cases}$$

Then from (2.1), it follows that

$$\begin{aligned} \mathscr{F}_{n+1}(q) &= \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} n \\ \lfloor (n-5j)/2 \rfloor \end{bmatrix}_q \\ &\equiv \sum_{\lfloor (n-5j)/2 \rfloor = 0 \text{ or } n} (-1)^j q^{j(5j+1)/2} \pmod{\Phi_n(q)}. \end{aligned}$$

It is easy to check that

$$\{j : \lfloor (n-5j)/2 \rfloor = 0 \text{ or } n\} = \begin{cases} \{(n-1)/5\} & \text{if } n \equiv 1 \bmod 5, \\ \{-(n+1)/5\} & \text{if } n \equiv 4 \bmod 5, \\ \emptyset & \text{if } n \equiv 2, 3 \bmod 5. \end{cases}$$

Thus

$$\begin{aligned} \mathscr{F}_{n+1}(q) &\equiv \sum_{\lfloor (n-5j)/2 \rfloor = 0 \text{ or } n} (-1)^j q^{j(5j+1)/2} \\ &= \begin{cases} (-1)^{(n-1)/5} q^{n(n-1)/5} \equiv 1 \pmod{\Phi_n(q)} & \text{if } n \equiv 1 \bmod 5, \\ (-1)^{-(n+1)/5} q^{n(n+1)/5} \equiv 1 \pmod{\Phi_n(q)} & \text{if } n \equiv 4 \bmod 5, \\ 0 \pmod{\Phi_n(q)} & \text{if } n \equiv 2, 3 \bmod 5. \end{cases} \end{aligned}$$

This concludes the proof of (1.4).

Also, applying (2.1) and Lemma 2.2, we deduce that

$$\begin{aligned}\mathscr{F}_n(q) &= \sum_{0\leqslant\lfloor(n-1-5j)/2\rfloor\leqslant n-1}(-1)^j q^{j(5j+1)/2}\begin{bmatrix} n-1 \\ \lfloor(n-1-5j)/2\rfloor\end{bmatrix}_q \\ &= \sum_{j=-\lfloor n/5\rfloor}^{\lfloor n/5\rfloor}(-1)^j q^{j(5j+1)/2}\prod_{k=1}^{\lfloor(n-1-5j)/2\rfloor}\frac{[n-k]_q}{[k]_q} \\ &= \sum_{j=-\lfloor n/5\rfloor}^{\lfloor n/5\rfloor}(-1)^j q^{j(5j+1)/2}\prod_{k=1}^{\lfloor(n-1-5j)/2\rfloor}\frac{[n]_q-[k]_q}{q^k[k]_q} \\ &\equiv \sum_{j=-\lfloor n/5\rfloor}^{\lfloor n/5\rfloor}(-1)^{j+\lfloor(n-1-5j)/2\rfloor}q^{L(j)} \pmod{\Phi_n(q)}.\end{aligned}$$

Assume that $n\equiv 1 \bmod 5$. Noting that $(n-1)/5$ is even and

$$(2j-1)+\lfloor(n-1-5(2j-1))/2\rfloor = 2j+\lfloor(n-1-5\cdot 2j)/2\rfloor+1=(n-1)/2-3j+1,$$

we have

$$\begin{aligned}\mathscr{F}_n(q) &\equiv (-1)^{4(n-1)/5}q^{L(-(n-1)/5)}+\sum_{j=-(n-6)/5}^{(n-1)/5}(-1)^{j+\lfloor(n-1-5j)/2\rfloor}q^{L(j)} \\ &= q^{(n-1)(n-2)/10-n(n-1)/2}+\sum_{j=-(n-11)/10}^{(n-1)/10}(-1)^{(n-1)/2-3j}(q^{L(2j)}-q^{L(2j-1)}) \\ &\equiv q^{(4n+1)/5} \pmod{\Phi_n(q)},\end{aligned}$$

where Lemma 2.1 is applied in the last step. Similarly, we obtain that

$$\mathscr{F}_n(q)\equiv\begin{cases}(-1)^{(n-2)/5}q^{L((n-2)/5)}\equiv -q^{(2n+1)/5} \pmod{\Phi_n(q)} & \text{if } n\equiv 2 \bmod 5,\\ (-1)^{1+4(n-3)/5}q^{L(-(n-3)/5)}\equiv -q^{(3n+1)/5} \pmod{\Phi_n(q)} & \text{if } n\equiv 3 \bmod 5,\\ (-1)^{1+(n-4)/5}q^{L((n-4)/5)}\equiv q^{(n+1)/5} \pmod{\Phi_n(q)} & \text{if } n\equiv 4 \bmod 5.\end{cases}\qquad\square$$

Proof of Theorem 1.2. In view of (2.2),

$$\widehat{\mathscr{F}}_n(q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(5j-3)/2}\begin{bmatrix} n \\ \lfloor(n-5j)/2\rfloor+1\end{bmatrix}_q.$$

And we have

$$\{j:\lfloor(n-5j)/2\rfloor+1=0 \text{ or } n\}=\begin{cases}\{-(n-1)/5\} & \text{if } n\equiv 1 \bmod 5,\\ \{-(n-2)/5\} & \text{if } n\equiv 2 \bmod 5,\\ \{(n+2)/5\} & \text{if } n\equiv 3 \bmod 5,\\ \{(n+1)/5\} & \text{if } n\equiv 4 \bmod 5.\end{cases}$$

Therefore

$$\widehat{\mathscr{F}}_n(q)\equiv\begin{cases}(-1)^{-(n-1)/5}q^{(n+2)(n-1)/10}\equiv q^{(n-1)/5} \pmod{\Phi_n(q)} & \text{if } n\equiv 1 \bmod 5,\\ (-1)^{-(n-2)/5}q^{(n+1)(n-2)/10}\equiv -q^{(3n-1)/5} \pmod{\Phi_n(q)} & \text{if } n\equiv 2 \bmod 5,\\ (-1)^{(n+2)/5}q^{(n+2)(n-1)/10}\equiv -q^{(2n-1)/5} \pmod{\Phi_n(q)} & \text{if } n\equiv 3 \bmod 5,\\ (-1)^{(n+1)/5}q^{(n+1)(n-2)/10}\equiv q^{(4n-1)/5} \pmod{\Phi_n(q)} & \text{if } n\equiv 4 \bmod 5.\end{cases}$$

Now, let us turn to the proof of (1.7). From (2.2), it follows that

$$\widehat{\mathscr{F}}_{n-1}(q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(5j-3)/2}\begin{bmatrix} n-1 \\ \lfloor (n-1-5j)/2\rfloor+1\end{bmatrix}_q$$
$$=\sum_{0\leqslant \lfloor (n-1-5j)/2\rfloor+1\leqslant n-1}^{\infty}(-1)^j q^{j(5j-3)/2}\begin{bmatrix} n-1 \\ \lfloor (n-1-5j)/2\rfloor+1\end{bmatrix}_q.$$

If $n\equiv 1 \bmod 5$, then by Lemmas 2.1 and 2.2, we have

$$\widehat{\mathscr{F}}_{n-1}(q)=\sum_{j=-(n-1)/5+1}^{(n-1)/5}(-1)^j q^{j(5j-3)/2}\begin{bmatrix} n-1 \\ \lfloor (n-1-5j)/2\rfloor+1\end{bmatrix}_q$$
$$\equiv \sum_{j=-(n-6)/5}^{(n-1)/5}(-1)^{j+\lfloor (n-1-5j)/2\rfloor+1}q^{\widehat{L}(j)}$$
$$=\sum_{j=-(n-11)/10}^{(n-1)/10}(-1)^{(n-1)/2-3j+1}(q^{\widehat{L}(2j)}-q^{\widehat{L}(2j-1)})$$
$$\equiv 0 \pmod{\Phi_n(q)}.$$

Similarly when $n\equiv 4 \bmod 5$,

$$\widehat{\mathscr{F}}_{n-1}(q)\equiv \sum_{j=-(n-4)/5}^{(n+1)/5}(-1)^{j+\lfloor (n-1-5j)/2\rfloor+1}q^{\widehat{L}(j)}\equiv 0 \pmod{\Phi_n(q)}.$$

Finally, suppose that $n\equiv 2$ or $3 \bmod 5$. Then

$$\widehat{\mathscr{F}}_{n-1}(q)$$
$$\equiv \sum_{j=-\lfloor n/5\rfloor}^{\lfloor n/5\rfloor}(-1)^{j+\lfloor (n-1-5j)/2\rfloor+1}q^{\widehat{L}(j)}$$
$$\equiv \begin{cases} (-1)^{(n-2)/5+1}q^{\widehat{L}((n-2)/5)}=q^{n(n-7)/10}\equiv 1 \pmod{\Phi_n(q)} & \text{if } n\equiv 2 \bmod 5,\\ (-1)^{-(n-3)/5+(n-1)}q^{\widehat{L}(-(n-3)/5)}=q^{n(1-2n)/5}\equiv 1 \pmod{\Phi_n(q)} & \text{if } n\equiv 3 \bmod 5.\end{cases}$$

All are done. □

Remark. With the similar discussion, it can be showed that for any positive even integer n with $5\nmid n$,

$$\mathscr{F}_n(q)\equiv\left(\frac{n}{5}\right)q^{((5-\alpha_n)n+1)/5} \pmod{\Phi_n(q)},\quad \widehat{\mathscr{F}}_n(q)\equiv\left(\frac{n}{5}\right)q^{(\alpha_n n-1)/5} \pmod{\Phi_n(q)}$$

and

$$\mathscr{F}_{n+1}(q)\equiv\frac{1}{2}\left(1+\left(\frac{n}{5}\right)\right) \pmod{\Phi_n(q)},\quad \widehat{\mathscr{F}}_{n-1}(q)\equiv\frac{1}{2}\left(1-\left(\frac{n}{5}\right)\right) \pmod{\Phi_n(q)}.$$

Also, when $5\mid n$, we have

$$\mathscr{F}_n(q)\equiv\widehat{\mathscr{F}}_n(q)\equiv 0 \pmod{\Phi_n(q)}.$$

3. q-Pell number

To prove Theorem 1.3, we need similar identities as (2.1) and (2.2) for $\mathscr{P}_n(q)$ and $\widehat{\mathscr{P}}_n(q)$, respectively. Fortunately, such identities have been established. Let

$$T_1(n,m,q)=\sum_{j=0}^{n}(-q)^j\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}\begin{bmatrix} 2n-2j \\ n-m-j \end{bmatrix}_q.$$

Lemma 3.1. *Let n be a non-negative integer. Then*

$$\mathscr{P}_{n+1}(q)=\sum_{j=0}^{n}\sum_{k=0}^{j}q^{(j^2+j+k^2-k)/2}\begin{bmatrix} j \\ k \end{bmatrix}_q\begin{bmatrix} n-k \\ j \end{bmatrix}_q = \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}T_1(n+1,4j+1,\sqrt{q}) \quad (3.1)$$

and

$$\widehat{\mathscr{P}}_{n+1}(q)=\sum_{j=0}^{n}\sum_{k=0}^{j}q^{(j^2+j+k^2+k)/2}\begin{bmatrix} j \\ k \end{bmatrix}_q\begin{bmatrix} n-k \\ j \end{bmatrix}_q = \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+j}T_1(n+1,4j+1,\sqrt{q}). \quad (3.2)$$

Eqs. (3.1) and (3.2) are the special cases of an identity due to Berkovich et al. [4, (2.34)]. And another proof of Lemma 3.1 is given in [12].

Proof of Theorem 1.3. Combining identities (2.33) and (2.34) in [3], it is not difficult to deduce that we have another representation of $T_1(n,m,q)$:

$$T_1(n,m,\sqrt{q})=\sum_{\substack{-\infty\leqslant k\leqslant\infty \\ k\equiv n-m \bmod 2}} q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q\begin{bmatrix} n-k \\ (n-m-k)/2 \end{bmatrix}_q. \quad (3.3)$$

Now by (3.1) and (3.3), we have

$$\begin{aligned}\mathscr{P}_n(q)&=\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}T_1(n,4j+1,\sqrt{q})\\ &=\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}\sum_{\substack{-\infty\leqslant k\leqslant-\infty \\ k\equiv n-1 \bmod 2}} q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q\begin{bmatrix} n-k \\ (n-4j-k-1)/2 \end{bmatrix}_q\\ &\equiv\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}\begin{bmatrix} n \\ (n-1)/2-2j \end{bmatrix}_q\\ &\equiv\begin{cases}(-1)^{(n-1)/4}q^{(n-1)^2/8} & \text{if } n\equiv 1 \bmod 4,\\ (-1)^{-(n+1)/4}q^{(n+1)^2/8} & \text{if } n\equiv 3 \bmod 4\end{cases}\\ &\equiv\left(\frac{2}{n}\right)q^{-(n^2-1)/8}\pmod{\Phi_n(q)}.\end{aligned}$$

The proof of (1.10) is very similar. From (3.2), it follows that

$$\begin{aligned}
\widehat{\mathscr{P}}_n(q) &= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \sum_{\substack{-\infty\leqslant k\leqslant\infty \\ k\equiv n-1 \bmod 2}} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ (n-4j-k-1)/2 \end{bmatrix}_q \\
&\equiv \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \begin{bmatrix} n \\ (n-1)/2-2j \end{bmatrix}_q \\
&\equiv \begin{cases} (-1)^{(n-1)/4} q^{(n-1)^2/8+(n-1)/4} & \text{if } n\equiv 1 \bmod 4, \\ (-1)^{-(n+1)/4} q^{(n+1)^2/8-(n+1)/4} & \text{if } n\equiv 3 \bmod 4 \end{cases} \\
&= \left(\frac{2}{n}\right) q^{(n^2-1)/8} \pmod{\Phi_n(q)}.
\end{aligned}$$

Thus, the proofs of (1.9) and (1.10) are completed.

By (3.3), for any integer m we have

$$\begin{aligned}
&T_1(n+1,4m+1,\sqrt{q}) - T_1(n,4m+1,\sqrt{q}) \\
&\quad\equiv T_1(n+1,4m+1,\sqrt{q}) - q^n T_1(n,4m+1,\sqrt{q}) \\
&\quad= \sum_{\substack{-\infty\leqslant k\leqslant\infty \\ k\equiv n \bmod 2}} q^{\binom{k}{2}} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n+1-k \\ (n-4m-k)/2 \end{bmatrix}_q \\
&\qquad - \sum_{\substack{-\infty\leqslant k\leqslant\infty \\ k\equiv n-1 \bmod 2}} q^{n+\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ (n-4m-k-1)/2 \end{bmatrix}_q \\
&\quad= \sum_{\substack{-\infty\leqslant k\leqslant\infty \\ k\equiv n-1 \bmod 2}} q^{\binom{k+1}{2}} \left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q - q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q \right) \begin{bmatrix} n-k \\ (n-4m-k-1)/2 \end{bmatrix}_q \\
&\quad= \sum_{\substack{-\infty\leqslant k\leqslant\infty \\ k\equiv n-1 \bmod 2}} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n-k \\ (n-k-1)/2-2m \end{bmatrix}_q \\
&\quad\equiv q^{\binom{n}{2}} \begin{bmatrix} 1 \\ -2m \end{bmatrix}_q \equiv \delta_m \pmod{\Phi_n(q)},
\end{aligned}$$

where $\delta_m = 1$ if $m = 0$, and 0 otherwise. Thus

$$\begin{aligned}
\mathscr{P}_{n+1}(q) - \mathscr{P}_n(q) &= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2} (T_1(n+1,4j+1,\sqrt{q}) - T_1(n,4j+1,\sqrt{q})) \\
&\equiv \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2} \delta_j \\
&\equiv 1 \pmod{\Phi_n(q)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\widehat{\mathscr{P}}_{n+1}(q)-\widehat{\mathscr{P}}_n(q) &= \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+j}(T_1(n+1,4j+1,\sqrt{q})-T_1(n,4j+1,\sqrt{q}))\\ &\equiv \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+j}\delta_j\\ &\equiv 1 \pmod{\Phi_n(q)}. \qquad \square\end{aligned}$$

Remark. When $n\geqslant 2$ is even, using the same method, we can prove that

$$\mathscr{P}_n(q)\equiv\widehat{\mathscr{P}}_n(q)\equiv 0 \pmod{\Phi_n(q)}$$

and

$$\mathscr{P}_{n+1}(q)-\mathscr{P}_n(q)\equiv\widehat{\mathscr{P}}_n(q)-\widehat{\mathscr{P}}_{n+1}(q)\equiv(-1)^{n/2} \pmod{\Phi_n(q)}.$$

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