MATHEMATICS

A DECISION METHOD FOR THE INTUITIONISTIC THEORY OF SUCCESSOR

BY

E. G. K. LOPEZ-ESCOBAR

College Park, Maryland [1])

(Communicated by Prof. A. HEYTING at the meeting of September 28, 1968)

By the (elementary) intuitionistic theory of successor, TS, we understand the theory obtained from the formal theory given in IM ([1]) p. 82 by the following modifications: (1) omit from the list of formal symbols (page 69 in IM) the function symbols $+$ and $\cdot$, (2) omit from the list of postulates the following: 8°, 18, 19, 20, 21 and (3) add the following as postulates: $8^I: \neg A \supset (A \supset B)$ and $*100: a=a$.

Atomic formulas of the form: $r''''^{\cdots\prime}=s$ or $\neg r''''^{\cdots\prime}=s$ where r, s are either variables or the individual constant 0 will be called *basic formulas* (e.g. $\neg 0''''''=a$ is basic formula while $0'''=0''$ is not).

Theorem. *To every formula A of TS we can effectively associate a formula A^* of TS such that A^* is a positive Boolean combination of basic formulas, A^* has exactly the same free variables as A and $\vdash_{TS} A \sim A^*$.*

Proof is by induction on the length of A. If A is an atomic formula then the result is immediate. (e.g. $a''=b''''$ is equivalent in TS to $b''=a$ which is a basic formula). Also if A_1 and A_2 are TS-equivalent to positive Boolean combinations of basic formulas, then clearly so are A_1 & A_2 and $A_1 \vee A_2$. For $\neg$ and $\supset$ we must first prove the following:

Lemma: *If B is a quantifier-free formula of TS which is substitution instance of a classical tautology, then $\vdash_{TS} B$.*

Proof of lemma. First, for a quantifier-free formula B, it may be shown by induction on the length of B that $\vdash_{TS} B \vee \neg B$. Using this result it can be shown, again by induction, that if B' is obtained from B by replacing occurrences of $\vee$ by $\neg(\neg \wedge \neg)$ then $\vdash_{TS} B \sim B'$. Finally by results of Gödel it is known that if B is a substitution instance of a tautology, then B' is provable in the intuitionistic propositional calculus (c.f. Theorem 60 in IM, page 495). Hence the lemma follows.

Now suppose we have shown that A is TS-equivalent to a positive Boolean combination of basic formulas. Because the distributive laws are

[1]) This research was partially supported by NSF grant GP 6897.

intuitionistically valid we may assume that A is TS-equivalent to a conjunction of disjunctions of basic formulas, say $\bigwedge_{i<n} \bigvee_{j<m} F_{ij}$. Since the basic formulas are quantifier free we may apply the lemma to obtain that $\vdash_{TS} \neg \bigwedge_{i<n} \bigvee_{j<m} F_{ij} \sim \bigvee_{i<n} \bigwedge_{j<m} \neg F_{ij}$. Furthermore $\neg F_{ij}$ is either a basic formula or TS-equivalent to a basic formula (again by the lemma). Thus $\neg A$ is TS-equivalent to a positive Boolean combination of basic formulas. The situation $A_1 \supset A_2$ is similarly treated.

Since in the intuitionistic predicate calculus & transfers across $\exists$ (c.f. *91: $A \,\&\, \exists x B(x) \sim \exists x(A \,\&\, B(x))$, page 162 in *IM*) the case of the existential quantifier is treated as in the corresponding classical theory, e.g.

$$\vdash_{TS} \exists b(a''=b \;\&\; b'=c \;\&\; c'''=d) \sim a'''=c \;\&\; c'''=d.$$

Although in the intuitionistic predicate calculus $\vee$ does not transfer across $\forall$ (c.f. Theorem 58, page 487 in *IM*) it can still be shown that if x does not occur free in C, $\neg C \supset [\forall x(C \vee B(x)) \supset \forall x B(x)]$, from which follows that $C \vee \neg C \supset [\forall x(C \vee B(x)) \sim C \vee \forall x B(x)]$ is provable in the intuitionistic predicate calculus. Thus if F is a basic formula in which x does not occur free then $\vdash_{TS} \forall x(F \vee B(x)) \sim F \vee \forall x B(x)$. Thus in order to complete the proof of the theorem it suffices to show how to find positive Boolean combinations of basic formulas which are TS-equivalent to formulas of the form $\forall x(F_0(x) \vee F_1(x) \vee \ldots F_n(x))$ where each of the $F_i(x)$ are basic formulas in which the variable x occurs. The latter, although laborious, is routine.

Let TS^c be like TS except that postulate 8^I is replaced by $8°$: $\neg\neg A \supset A$ (i.e. TS^c is the classical theory of successor). Then a consequence of the theorem and its proof is the following

Corollary. *There is a (primitive) recursive function f such that if n is a (Gödel) number of a proof in TS^c of a sentence A then $f(n)$ is a (Gödel) number of a proof in TS of A.*

In other words the classical and intuitionistic (first-order) theory of successor are *really and truly* the same (unlike when $+$ and $\cdot$ are permitted because then there are sentences provable in the classical system which are not provable in the intuitionistic system).

REFERENCE

1. KLEENE, S. C., *Introduction to metamathematics*, Van Nostrand, 1952.