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# Existence of fractional differential equations

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**Abstract**

Consider the fractional differential equation

$$D^{\alpha}x = f(t,x),$$

where $\alpha \in (0,1)$ and $f(t,x)$ is a given function. We obtained a sufficient condition for the existence for the solutions of this equation, improving previously known results.


*Keywords:* Existence; Fractional differential equations

## 1. Introduction

The qualitative properties of the solution of the fractional differential equation

$$D^{\alpha}x = f(t,x), \tag{1}$$

where $\alpha \in (0,1)$ and $f(t,x)$ is a given function, have been the subject of many investigations. This equation has been extensively studied by many authors [1–4,6]; El-Sayed [1] gave a systematic study. Since 1988 when El-Sayed [1] obtained a theorem of existence and uniqueness for Eq. (1), the existence and uniqueness of Eq. (1) has also been discussed

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extensively in the literature. Now it is well known (see [2–4]) that Eq. (1) has a unique solution, provided that Lipschitz condition holds.

Therefore, our first aim in this paper is to establish a theorem which improves the existing results in the literature [1–4].

The paper is organized as follows. In Section 2 we recall the definitions of fractional integral and derivative and related basic properties used in the text. Section 3 contains results for solutions which are continuous at the origin.

## 2. Definitions and lemmas

In this section we first give some definitions used in the text.

First, Let $I = [0, T]$ and $D = I \times C(I)$, where $C(I)$is the class of all continuous functions defined on $I$, with the norm

$$\|x\| = \max\left|x(t)\right|, \quad t \in I,\ x(t) \in C(I).$$

**Definition 2.1** [3]. The fractional primitive of order $\alpha > 0$ of a function $f : R^+ \to R$ is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(\theta)\, d\theta,$$

provided the right side is pointwise defined on $R^+$.

**Definition 2.2** [3]. The fractional derivate of order $0 < \alpha < 1$ of a function $f : R^+ \to R$ is given by

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\theta)^{-\alpha} f(\theta)\, d\theta,$$

provided the right side is pointwise defined on $R^+$.

Now we need the following lemmas:

**Lemma 2.1** [2]. *Let $x(t) \in C(I)$ and $f(t, x(t)) \in C(D)$, if a solution of Eq.* (1) *exists, then it is given by*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f\bigl(\theta, x(\theta)\bigr) d\theta,$$

*where $\Gamma$ is the gamma function.*

**Lemma 2.2** [5] (Schauder fixed-point theorem). *If $U$ is a close bounded convex subset of a Banach space $X$ and $T : U \to U$ is completely continuous, then $T$ has a fixed point in $U$.*

## 3. Theorem of existence

**Theorem 3.1.** *If $f$ is a continuous function on $I \times R$ and suppose the following holds*:

$$\left|f(t,x)-f(t,y)\right| \leqslant \lambda(t)h(r), \tag{2}$$

*where $h(r)$is continuous on $[0,\infty)$ and $h(0)=0$, $r=|x-y|$, $|I^\alpha\lambda(t)|<M$ for $t\in I$, then there exists a continuous solution $x(t)$ of Eq.* (1), *which is defined on $[0,\bar{\mu}]$ for a suitable $\bar{\mu}<T$.*

**Proof.** Let $I_\mu=[0,\mu]$, fix $\nu>0$ and denote norm of $\|f\|=\max|f(t,x)|$, $t\in I$ and $|x|\leqslant\nu$.

Choose $\bar{\mu}$, $\bar{\nu}$ so that $0\leqslant\bar{\mu}\leqslant T$, $0\leqslant\bar{\nu}\leqslant\nu$, $\frac{\|f\|\bar{\mu}^\alpha}{\Gamma(\alpha+1)}\leqslant\bar{\nu}$, and define the set $A=A(\bar{\mu},\bar{\nu})$ of function $\phi$ in $C(I_\mu,R)$ which satisfies $\phi(0)=0$ and $|\phi(t)|\leqslant\bar{\nu}$ for all $t\in I_{\bar{\mu}}$.

The set $A$ is a closed, bounded and convex.

For any $\phi$ in $A$, define the function $G\phi$ by the relation

$$G\phi(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\theta)^{\alpha-1}f\big(\theta,\phi(\theta)\big)\,d\theta,\quad t\in I_{\bar{\mu}}. \tag{3}$$

The fixed points of $G$ in $A$ coincide with the solutions of equation in $A$. We can apply the Schauder theorem to prove the existence of a fixed point of $G$ in $A$.

For any $\phi$ in $A$, the operator $G$ is well defined since $f(\theta,\phi(\theta))$ is bounded for $\phi$ in $A$.

Also, $G\phi(0)=\phi(0)=0$, and $G\phi(t)$ is continuous for $t\in I$, $\bar{\mu}\in I$.

Moreover, using the condition of (4), we have

$$\begin{aligned}\left|G\phi(t)\right| &\leqslant \left|\frac{1}{\Gamma(\alpha)}\int_0^t(t-\theta)^{\alpha-1}f\big(\theta,\phi(\theta)\big)\,d\theta\right|\\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t\left|(t-\theta)^{\alpha-1}f\big(\theta,\phi(\theta)\big)\right|d\theta\leqslant\frac{\|f\|\bar{\mu}^\alpha}{\Gamma(\alpha+1)}\leqslant\bar{\nu}\end{aligned}$$

for all $t\in I_{\bar{\mu}}$.

Therefore, $G: A\to A$.

The operator $G$ is continuous on $A$. In fact, if $\phi_n$, $n=0,1,2,3\ldots$, and $\phi\in A$ with $|\phi_n-\phi|\to 0$ as $n\to\infty$.

Then for $t\in I$, $\bar{\mu}\in I$, we have

$$\begin{aligned}&|G\phi_n-G\phi|\\ &\quad=\left|\frac{1}{\Gamma(\alpha)}\int_0^t(t-\theta)^{\alpha-1}f\big(\theta,\phi_n(\theta)\big)\,d\theta-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\theta)^{\alpha-1}f\big(\theta,\phi(\theta)\big)\,d\theta\right|\\ &\quad\leqslant\left|\frac{1}{\Gamma(\alpha)}\int_0^t(t-\theta)^{\alpha-1}\left|f\big(\theta,\phi_n(\theta)\big)-f\big(\theta,\phi(\theta)\big)\right|d\theta\right|\end{aligned}$$

$$\leqslant \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} \lambda(\theta) h\big(|\phi_n(\theta) - \phi(\theta)|\big)\, d\theta \right|$$
$$\leqslant \left| I^\alpha \lambda(t) \right| h(r).$$

Since $h(r)$ is continuous on $[0,\infty)$ and $h(0)=0$, we have $h(r) \to 0$ as $r \to 0$. On the other hand, $r = |\phi_n - \phi| \to 0$ as $n \to \infty$ and $|I^\alpha \lambda(t)| < M$, therefore $G : A \to A$ is continuous.

For any $\phi$ in $A$, let $t, \tau$ in $I$ and $t \leqslant \tau$, $\bar{\mu} \in I$,

$$\begin{aligned}
&\left| G\phi(t) - G\phi(\tau) \right| \\
&\quad = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f\big(\theta, \phi(\theta)\big)\, d\theta - \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-\theta)^{\alpha-1} f\big(\theta, \phi(\theta)\big)\, d\theta \right| \\
&\quad \leqslant \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f\big(\theta, \phi(\theta)\big)\, d\theta - \frac{1}{\Gamma(\alpha)} \int_0^t (\tau-\theta)^{\alpha-1} f\big(\theta, \phi(\theta)\big)\, d\theta \right| \\
&\qquad + \left| \frac{1}{\Gamma(\alpha)} \int_0^t (\tau-\theta)^{\alpha-1} f\big(\theta, \phi(\theta)\big)\, d\theta - \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-\theta)^{\alpha-1} f\big(\theta, \phi(\theta)\big)\, d\theta \right| \\
&\quad \leqslant \frac{\|f\|}{\Gamma(\alpha+1)} \big[ \left| \tau^\alpha - t^\alpha - (\tau - t)^\alpha \right| + (\tau - t)^\alpha \big].
\end{aligned}$$

So we obtain that the set $GA$ is an equicontinuous set of $C(I_{\bar{\mu}}, R)$. It is also uniformly bounded. This proves $GA$ is relatively compact and, thus, $G$ is completely continuous. The Schauder fixed point theorem implies the existence point in $A$ and the theorem is proved.

**Remark 3.2.** If $\lambda(t) = L > 0$ is a constant, then condition (3) reduces to the Osgood condition.

If $\lambda(t) = L > 0$ and $h(|x-y|) = |x-y|$, then condition (3) reduces to the Lipschitz condition.

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