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# Spectral transformations of the Laurent biorthogonal polynomials. I. $q$-Appel polynomials

Luc Vinet[a], Alexei Zhedanov[b,*]

[a] *Centre de recherches mathématiques, Université de Montréal, C.P. 6128, succ. Centre-Ville, Montréal QC H3C 3J7, Canada*

[b] *Donetsk Institute for Physics and Technology, Donetsk 340114, Ukraine*

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**Abstract**

We study the simplest closure conditions of chains of spectral transformations of the Laurent biorthogonal polynomials (LBP). It is shown that the 1-periodic $q$-closure condition leads to the $q$-Appel LBP which are a special case of the LBP introduced by Pastro. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The Laurent biorthogonal polynomials (LBP) $P_n(z)$ appeared in problems connected with the two-points Padé approximations (see, e.g., [6]).

We shall recall their definition and general properties (see, e.g. [6,4,5], where equivalent Laurent *orthogonal functions* are considered).

Let $\mathscr{L}$ be some linear functional defined on all possible monomials $z^n$ by the moments

$$c_n = \mathscr{L}\{z^n\}, \quad n = 0, \pm 1, \pm 2 \ldots . \tag{1.1}$$

In general the moments $c_n$ are arbitrary complex numbers. The functional $\mathscr{L}$ is thus defined on the space of Laurent polynomials $\mathscr{P}(z) = \sum_{n=-N_1}^{N_2} a_n z^n$ where $a_n$ are arbitrary complex numbers and $N_{1,2}$ arbitrary integers:

$$\mathscr{L}\{\mathscr{P}(z)\} = \sum_{n=-N_1}^{N_2} a_n c_n .$$

* Corresponding author.
*E-mail address:* zhedanov@kinetic.ac.donetsk.ua (A. Zhedanov)


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The monic LBP $P_n(z)$ are defined by the determinant [4]

$$P_n(z) = (\Delta_n)^{-1} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_0 & \cdots & c_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1-n} & c_{2-n} & \cdots & c_1 \\ 1 & z & \cdots & z^n \end{vmatrix},$$

where $\Delta_n$ is defined as

$$\Delta_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{-1} & c_0 & \cdots & c_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1-n} & c_{2-n} & \cdots & c_0 \end{vmatrix}.$$

It is obvious from definition (1.2) that the polynomials $P_n(z)$ satisfy the orthogonality property

$$\mathscr{L}\{P_n(z)z^{-k}\} = h_n\delta_{kn}, \quad 0 \leqslant k \leqslant n, \tag{1.2}$$

where the normalization constants $h_n$ are

$$h_0 = c_0, \quad h_n = \Delta_{n+1}/\Delta_n. \tag{1.3}$$

This orthogonality property can be rewritten as the biorthogonal relation [9,4],

$$\mathscr{L}\{P_n(z)Q_m(1/z)\} = h_n\delta_{nm}, \tag{1.4}$$

where the polynomials $Q_n(z)$ are defined by the formula

$$Q_n(z) = (\Delta_n)^{-1} \begin{vmatrix} c_0 & c_{-1} & \cdots & c_{-n} \\ c_1 & c_0 & \cdots & c_{1-n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n-1} & c_{n-2} & \cdots & c_{-1} \\ 1 & z & \cdots & z^n \end{vmatrix}. \tag{1.5}$$

We note that the polynomials $Q_n(z)$ are again LBP with moments $c_n^{\{Q\}} = c_{-n}$.

In what follows we will assume that

$$\Delta_n \neq 0, \quad n = 1, 2, \dots \tag{1.6}$$

and that

$$\Delta_n^{(1)} \neq 0, \quad n = 1, 2, \dots, \tag{1.7}$$

where by $\Delta_n^{(j)}$ we denote the determinants

$$\Delta_0^{(j)} = 1,$$

$$\Delta_n^{(j)} = \begin{vmatrix} c_j & c_{j+1} & \cdots & c_{n+j-1} \\ c_{j-1} & c_j & \cdots & c_{n+j-2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1+j-n} & c_{2+j-n} & \cdots & c_j \end{vmatrix}. \tag{1.8}$$

If conditions (1.6) and (1.7) are fulfilled, the polynomials $P_n(z)$ satisfy the recurrence relation (see, e.g., [3])

$$P_{n+1}(z) + (d_n - z)P_n(z) = zb_nP_{n-1}(z), \quad n \geqslant 1, \tag{1.9}$$

where the recurrence coefficients are

$$d_n = -\frac{P_{n+1}(0)}{P_n(0)} = h_n^{-1}\frac{T_{n+1}}{T_n} \neq 0, \quad n = 0, 1, \ldots, \tag{1.10}$$

$$b_n = d_n\frac{h_n}{h_{n-1}} \neq 0, \quad n = 1, 2, \ldots \tag{1.11}$$

with $T_n = \Delta_n^{(1)}$.

There is a one-to-one correspondence between the moments $c_n$ and the recurrence coefficients $b_n, d_n$ (provided restrictions $b_n\, d_n \neq 0$ are fulfilled).

Recurrence relation (1.9) will be our starting point.

In addition to the polynomials $P_n(z)$ it is useful to introduce the $j$-associated polynomials $P_n^{(j)}(z)$ that satisfy the recurrence relation

$$P_{n+1}^{(j)}(z) + d_{n+j}P_n^{(j)}(z) = z(P_n^{(j)}(z) + b_{n+j}\, P_{n-1}^{(j)}(z)) \tag{1.12}$$

with $P_0^{(j)} = 1$, $P_1^{(j)}(z) = z - d_j$. Note that $P_n(z)$ and $P_{n-j}^{(j)}(z)$ are two independent solutions of the difference equation

$$\phi_{n+1} + d_n\phi_n = z\,(\phi_n + b_n\phi_{n-1}). \tag{1.13}$$

There is an obvious scaling property of the LBP.

Assume that the polynomials $P_n(z)$ are LBP with $c_n$ as moments and $d_n$, $b_n$ as recurrence coefficients. The polynomials $\tilde{P}_n(z) = \kappa^{-n} \quad P_n(\kappa z)$ are also LBP with the scaled recurrence parameters $\tilde{b}_n = b_n/\kappa, \quad \tilde{d}_n = d_n/\kappa$, and moments $\tilde{c}_n = \kappa^{-n}c_n$. The proof of this statement is obvious. This property means that simultaneously multiplying the recurrence coefficients by the same constant merely leads to a rescaling of the argument $z$ of the polynomials $P_n(z)$.

We say that the LBP are regular if $b_nd_n \neq 0$. It can be shown that for regular LBP the following Lemma holds (see, e.g., [15]).

**Lemma 1.** *Let $P_n(z)$ be a set of regular LBP. Assume that these $P_n(z)$ satisfy the relation*

$$S_1(z;n)\,P_n(z) + S_2(z;n)\,P_{n-1}(z) = 0, \quad n = 1, 2, \ldots, \tag{1.14}$$

*where $S_{1,2}(z;n)$ are polynomials in $z$ whose coefficients depend on $n$ but whose degrees are fixed and independent of $n$. Then the polynomials $S_{1,2}$ should identically vanish*:

$$S_1(z;n) = S_2(z;n) \equiv 0. \tag{1.15}$$

In what follows we will use the so-called rescaled LBP

$$\tilde{P}_n(z) = q^nP_n(z/q), \quad n = 0, 1, \ldots \tag{1.16}$$

with some nonzero parameter $q$. It is easily verified that the rescaled polynomials $\tilde{P}_n(z)$ are monic LBP satisfying the recurrence relation

$$\tilde{P}_{n+1}(z) + (\tilde{d}_n - z)\tilde{P}_n(z) = z\tilde{b}_n\tilde{P}_{n-1}(z)$$

with

$$\tilde{d}_n = qb_n, \quad \tilde{b}_n = qb_n. \tag{1.17}$$

Thus rescaled LBP $\tilde{P}_n(z)$ differ from initial LBP $P_n(z)$ only by a trivial rescaling of recurrence parameters.

It is interesting to note that the LBP were identified recently in the physical context of the relativistic Toda chain [7].

The main purpose of this paper is to study chains of spectral transformations of LBP and their closure properties. For the ordinary orthogonal polynomials this was developed in [11,14]. We show that the simplest quasi-periodic closure leads to $q$-Appel LBP.

## 2. Spectral transformations of LBP

In this section we describe the two simplest spectral transformations of LBP, namely the Christoffel (CT) and the Geronimus (GT) transformations which are the exact analogs of the spectral transformations of ordinary orthogonal polynomials bearing the same name (see, e.g., [11,14]). In what follows we will assume the standard normalization condition $c_0 = 1$.

By CT we mean the following transformation:

$$P_n^{(C)}(z) = \frac{P_{n+1}(z) - U_n P_n(z)}{z - \mu}, \tag{2.1}$$

where $\mu$ is an arbitrary parameter and

$$U_n = \frac{P_{n+1}(\mu)}{P_n(\mu)}.$$

It is easily verified that the polynomials $P_n^{(C)}(z)$ constructed according to (2.1) are again monic LBP with moments

$$c_n^{(C)} = (c_1 - \mu)^{-1}(c_{n+1} - \mu c_n), \quad n = 0, \pm 1, \pm 2, \dots. \tag{2.2}$$

In terms of the moment functional $\mathscr{L}$, relation (2.2) can be rewritten in the form

$$\mathscr{L}^{(C)} = (c_1 - \mu)^{-1}(z - \mu)\mathscr{L}. \tag{2.3}$$

Formulas (2.1) and (2.3) coincide with those defining the Christoffel transformation of the ordinary orthogonal polynomials [1,12].

If the polynomials $P_n(z)$ satisfy recurrence relation (1.9), the polynomials $P_n^{(C)}(z)$ satisfy the recurrence relation

$$P_{n+1}^{(C)}(z) + d_n^{(C)} P_n^{(C)}(z) = z(P_n^{(C)}(z) + b_n^{(C)} P_{n-1}^{(C)}(z)), \tag{2.4}$$

where

$$b_n^{(C)} = b_n \frac{b_{n+1} + U_n}{b_n + U_{n-1}}, \tag{2.5}$$

$$d_n^{(C)} = d_n \frac{d_{n+1} + U_{n+1}}{d_n + U_n}. \tag{2.6}$$

The reciprocal of the CT of LBP is the Geronimus transformation (GT) of LBP defined as

$$P_n^{(G)}(z) = V_n P_n(z) + z(1 - V_n)P_{n-1}(z) \tag{2.7}$$

with some coefficients. Compatibility of relation (2.7) with three-term recurrence relations (1.9) of the polynomials $P_n(z)$ and $P_n^{(G)}(z)$ leads to the following restriction:

$$\frac{b_n V_n V_{n+1}}{1 - V_n} + d_n V_{n+1} = \mu, \tag{2.8}$$

where $\mu$ is a constant. From the initial condition $P_0(z) = 1$, we also have

$$V_0 = 1. \tag{2.9}$$

Consider first the case $\mu \neq 0$. Then we can linearize relation (2.8) introducing a new sequence $\phi_n$:

$$V_n = \frac{\mu}{\mu - \phi_n/\phi_{n-1}}, \quad n = 1, 2, \dots. \tag{2.10}$$

From (2.8) it then follows that $\phi_n$ can be chosen as an arbitrary solution of the second-order difference equation

$$\phi_{n+1} + d_n\phi_n = \mu(\phi_n + b_n\phi_{n-1}). \tag{2.11}$$

The polynomials $P_n^{(G)}(z)$ then satisfy recurrence relation (2.4) with recurrence coefficients

$$b_n^{(G)} = b_{n-1}\frac{1 - V_n}{1 - V_{n-1}}, \quad n = 2, 3, \dots, \tag{2.12}$$

$$d_n^{(G)} = d_n V_{n+1}/V_n, \quad n = 0, 1, \dots. \tag{2.13}$$

One can easily show that the CT and the GT are reciprocal.

Indeed, we have the following:

**Proposition 2.** *Assume that the LBP $R_n(z)$ have the recurrence coefficients $d_n$, $b_n$ and are related to the LBP $P_n(z)$ by the formula*

$$R_n(z) = V_n P_n(z) + z(1 - V_n)P_{n-1}(z), \tag{2.14}$$

*where $V_n$ are some coefficients. Then*

$$P_n(z) = \frac{R_{n+1}(z) - U_n R_n(z)}{z - \mu}, \tag{2.15}$$

*where*

$$\mu = V_{n+1}\left(d_n + \frac{V_n b_n}{1 - V_n}\right) = \text{const.}, \tag{2.16}$$

$$U_n = \frac{b_n\, V_{n+1}}{1 - V_n} = \frac{R_{n+1}(\mu)}{R_n(\mu)}. \tag{2.17}$$

**Proof.** Consider relation (2.14) with $n \to n+1$:

$$R_{n+1}(z) = V_{n+1}P_{n+1}(z) + z(1 - V_{n+1})P_n(z). \tag{2.18}$$

Excluding $P_{n+1}(z)$ in (2.18) (using recurrence relation (1.9))

$$R_{n+1}(z) = (z(1 - V_{n+1}) + V_{n+1}(z - d_n))P_n(z) + zV_{n+1}b_nP_{n-1}(z) \tag{2.19}$$

and solving system (2.14) and (2.19) we obtain

$$P_n(z) = \frac{R_{n+1}(z) - (b_nV_{n+1}/(1 - V_n))R_n(z)}{z - d_nV_{n+1} - b_nV_nV_{n+1}/(1 - V_n)}. \tag{2.20}$$

Relation (2.16) then follows from (2.8). The second relation in (2.17) follows from the fact that the l.h.s of (2.15) cannot have a pole at $z = \mu$.

We thus proved that the polynomials $P_n(z)$ are expressed in terms of $R_n(z)$ through the CT (2.15).

Note that the GT contains (with respect to the CT) an additional parameter because the function $\phi_n = P_n(\mu) + \chi P^{(1)}_{n-1}(\mu)$ is a linear combination of two independent solutions of the recurrence relation (2.11).

When $\mu \to 0$ then both the CT and the GT become extremely simple. In this case for the CT we have (note that $P_n(0) \neq 0$ due to the 1-regularity)

$$P^{(C)}_n(z) = z^{-1}(P_{n+1} + d_nP_n(z)) = P_n(z) + b_nP_{n-1}(z). \tag{2.21}$$

The moments are then merely shifted,

$$c^{(C)}_n = \frac{c_{n+1}}{c_1}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{2.22}$$

and the recurrence coefficients transformed according to

$$b^{(C)}_n = b_n\frac{b_{n+1} - d_n}{b_n - d_{n-1}}, \quad n = 1, 2, \ldots \tag{2.23}$$

$$d^{(C)}_0 = d_0 - b_1, \qquad d^{(C)}_n = d_{n-1}\frac{b_{n+1} - d_n}{b_n - d_{n-1}}, \quad n = 1, 2, \ldots. \tag{2.24}$$

For the GT, when $\mu = 0$ we have from (2.8) and (2.9) that

$$d_n + \frac{b_nV_n}{1 - V_n} = 0$$

and thus

$$P^{(G)}_n(z) = \frac{P_{n+1}(z) - zP_n(z)}{b_n - d_n} = \frac{zb_nP_{n-1}(z) - d_nP_n(z)}{b_n - d_n}, \tag{2.25}$$

$$c^{(G)}_n = \frac{c_{n-1}}{c_{-1}}, \tag{2.26}$$

$$b^{(G)}_1 = -\frac{b_1d_0}{b_1 - d_1}, \qquad b^{(G)}_n = b_n\frac{b_{n-1} - d_{n-1}}{b_n - d_n}, \quad n = 2, 3, \ldots, \tag{2.27}$$

$$d^{(G)}_0 = -\frac{d_0d_1}{b_1 - d_1}, \qquad d^{(G)}_n = d_{n+1}\frac{b_n - d_n}{b_{n+1} - d_{n+1}}, \quad n = 1, 2, \ldots. \tag{2.28}$$

In what follows we will denote (for brevity) by $\mathscr{C}(\mu)\{P_n(z)\}$ and $\mathscr{G}(\mu,\chi)\{P_n(z)\}$ the result of the Christoffel and Geronimus transformations of the LBP $P_n(z)$ (i.e., (2.1) and (2.7)). For $\mu=0$ we will write $\mathscr{C}(0)\{P_n(z)\}$ and $\mathscr{G}(0)\{P_n(z)\}$ with formulas (2.21) and (2.25) in mind. Note that for $\mu=0$ the Geronimus transformation does not depend on an auxiliary parameter.

## 3. Involutions of the LBP

Apart from the spectral transformations (i.e., Christoffel and Geronimus transformations) there are two fundamental transformations of the LBP. These transformations are *involutions*, i.e., they yield the identity transformation when applied twice.

The first of these involutions is the transformation of the LBP $P_n(z)$ into their biorthogonal partners $Q_n(z)$ introduced in (1.5). In terms of $P_n(z)$ the polynomials $Q_n(z)$ are written as

$$Q_n(z)=\frac{z^nP_{n+1}(1/z)-z^{n-1}P_n(1/z)}{P_{n+1}(0)}, \tag{3.1}$$

where $P_{n+1}(0)=(-1)^{n+1}\ d_0d_1\cdots d_n$. It is verified that $Q_n(z)$ are monic LBT satisfying recurrence relation (1.9) with recurrence parameters

$$\tilde b_n=\frac{b_n\ (d_{n+1}-b_{n+1})}{d_n\,d_{n+1}(d_n-b_n)}, \tag{3.2}$$

$$\tilde d_n=\frac{d_{n+1}-b_{n+1}}{d_{n+1}(d_n-b_n)}. \tag{3.3}$$

The moments $\tilde c_n$ corresponding to the polynomials $Q_n(z)$ are

$$\tilde c_n=c_{-n}. \tag{3.4}$$

Introduce the operator $\mathscr{Q}$ (acting on the space of polynomials) such that $\mathscr{Q}\ \{P_n(z)\}=Q_n(z)$. It is then obvious from (3.4) that $\mathscr{Q}^2=I$, where $I$ is identity operator: $I\{P_n(z)\}=P_n(z)$. Thus the transformation $\mathscr{Q}$: $P_n(z)\to Q_n(z)$ is indeed an involution.

The second involution is the transformation of the polynomials $P_n(z)$ into the polynomials $T_n(z)$ with the argument inverted:

$$T_n(z)=\frac{z^n\,P_n(1/z)}{P_n(0)}. \tag{3.5}$$

It is verified that $T_n(z)$ are monic LBP having the recurrence coefficients

$$\tilde b_n=\frac{b_n}{d_n\,d_{n+1}}, \tag{3.6}$$

$$\tilde d_n=\frac{1}{d_n}. \tag{3.7}$$

The moments $\tilde c_n$ corresponding to the polynomials $T_n(z)$ are

$$\tilde c_n=\frac{c_{1-n}}{c_1}. \tag{3.8}$$

Introduce the operator $\mathscr{T}$ such that $\mathscr{T}\ \ \{P_n(z)\}=T_n(z)$. Again it is clear from (3.8) (or from (3.6) and (3.7)) that $\mathscr{T}^2=I$ and hence the transformation $\mathscr{T}$ is an involution.

Consider composition $\mathscr{T}\,\mathscr{Q}$ of these involutions. (This means that we first perform the transformation $\mathscr{Q}$ and then the transformation $\mathscr{T}$.) For the corresponding moments we have

$$\mathscr{T}\,\mathscr{Q}\{c_n\} = \frac{c_{n+1}}{c_1}. \tag{3.9}$$

This means that $\mathscr{T}\,\mathscr{Q} = \mathscr{C}(0)$. The special Christoffel transformation $\mathscr{C}(0)$ has been defined at the end of Section 2.

Analogously, $\mathscr{Q}\,\mathscr{T} = \mathscr{G}(0)$, where $\mathscr{G}(0)$ is the special Geronimus transformation also defined in Section 2.

The composition of these two involutions leads therefore to special spectral transformations.

It is interesting to note that these two involutions and their compositions are connected with the discrete-time dynamics of the relativistic Toda chain (for details see [7]).

## 4. Algebraic interpretation of the spectral transformations

In this section we consider the algebraic interpretation of the Christoffel transformation of the LBP. (The Geronimus transformation can also be interpreted in the same manner but we do not do this here.)

Let $|n\rangle$, $n = 0, 1, 2, \ldots$ be some basis. Consider four 2-diagonal operators:

$$A|n\rangle = |n-1\rangle + d_n|n\rangle, \qquad n = 1, 2, \ldots, \quad A|0\rangle = d_0|0\rangle, \tag{4.1}$$

$$B|n\rangle = |n\rangle + b_{n+1}|n+1\rangle, \quad n = 0, 1, \ldots, \tag{4.2}$$

$$J|n\rangle = |n-1\rangle + \xi_n|n\rangle, \quad n = 1, 2, \ldots, J|0\rangle = \xi_0|0\rangle, \tag{4.3}$$

$$K|n\rangle = |n-1\rangle + \eta_n|n\rangle, \quad n = 1, 2, \ldots, K|0\rangle = \eta_0|0\rangle, \tag{4.4}$$

where $\xi_n, \eta_n$ are some coefficients to be determined.

Consider the generalized eigenvalue problem (GEVP)

$$A|\psi\rangle = z\,B\,|\psi\,\rangle, \tag{4.5}$$

where $z$ is the (generalized) eigenvalue and $|\psi\rangle$ is the (generalized) eigenstate. Expand the eigenstate $|\psi\rangle$ over the basis $|n\rangle$:

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n(z)|n\rangle, \tag{4.6}$$

where $C_n(z)$ are some coefficients depending on $z$. We can normalize these coefficients by choosing $C_0 = 1$. It is then easily seen that $C_n(z) = P_n(z)$ with $P_n(z)$ the LBP satisfying recurrence relation (1.9). Thus, the LBP $P_n(z)$ can be interpreted as objects arising from the generalized eigenvalue problem (4.5).

Assume moreover that the following operator relations

$$\tilde{A}J = KA, \tag{4.7}$$

$$\tilde{B}J = KB \tag{4.8}$$

take place where the operators $\tilde{A}$ and $\tilde{B}$ have the same form as (4.1) and (4.2) with $d_n, b_n$ replaced by $\tilde{d}_n, \tilde{b}_n$. We then have the following:

**Proposition 3.** *If* $|\psi\rangle$ *is an eigenstate of the GEVP* (4.5) *with eigenvalue* $z$ *then the state*

$$|\tilde{\psi}\rangle = F(z)J|\psi\rangle \tag{4.9}$$

(*where* $F(z)$ *is an arbitrary function of* $z$) *is the eigenstate of another GEVP*

$$\tilde{A}|\tilde{\psi}\rangle = z\tilde{B}|\tilde{\psi}\rangle \tag{4.10}$$

*with the same eigenvalue* $z$.

The proof of this proposition is obvious.

We have correspondingly for the coefficients, the transformation law

$$\tilde{C}_n(z) = F(z)(C_{n+1}(z) - \xi_n C_n(z)). \tag{4.11}$$

If we demand that both $C_n(z) = P_n(z)$ and $\tilde{C}_n(z) = \tilde{P}_n(z)$ are monic LBP, the conditions

$$\xi_n = -\frac{P_{n+1}(\mu)}{P_n(\mu)}, \tag{4.12}$$

$$\eta_n = \xi_{n-1}\frac{\xi_n - b_{n+1}}{\xi_{n-1} - b_n}, \tag{4.13}$$

$$F(z) = 1/(z-\mu), \tag{4.14}$$

where $\mu$ is an arbitrary parameter are seen to hold. Transformation law (4.11) then becomes simply the Christoffel transformation (2.1) with $U_n = \xi_n$. We thus obtain an algebraic (operator) interpretation of the CT in terms of relations (4.7) and (4.8).

If $Q$ is the operator such that $Q|n\rangle = q^n|n\rangle$, it is verified that

$$qQAQ^{-1}|n\rangle = |n-1\rangle + qd_n|n\rangle, \tag{4.15}$$

$$QBQ^{-1}|n\rangle = |n\rangle + qb_{n+1}|n+1\rangle. \tag{4.16}$$

We thus see that the action of the operators

$$A(q) = qQAQ^{-1}, \quad B(q) = QBQ^{-1} \tag{4.17}$$

has the same structure as that of the operators $A, B$ but with the coefficients $b_n, d_n$ replaced with $qb_n, qd_n$.

## 5. Self-similar chains of spectral transformations

From given polynomials $P_n(z)$ we can construct a chain of new polynomials

$$P_n(z;\mu_1,\mu_2,\ldots,\mu_N) = \mathscr{C}(\mu_N)\mathscr{C}(\mu_{N-1})\ldots\mathscr{C}(\mu_1)\{P_n(z)\} \tag{5.1}$$

by applying successively $N$ CT at the points $\mu_1,\mu_2,\ldots,\mu_N$.

In general, LBP $P_n(z;\mu_1,\mu_2,\ldots,\mu_N)$ essentially differ from initial LBP $P_n(z)$. It is therefore natural to ask when the polynomials $P_n(z;\mu_1,\mu_2,\ldots,\mu_N)$ are "almost the same" as $P_n(z)$. By the term "almost the same" we mean that LBP $P_n(z;\mu_1,\mu_2,\dots,\mu_N)$ are *rescaled* $P_n(z)$ as defined by (1.16). We will call this situation "closure condition" for a chain of spectral transformations:

$$P_n(z;\mu_1,\mu_2,\dots,\mu_N)=q^nP_n(z/q) \tag{5.2}$$

with $q$ some nonzero parameter.

Note that the analogous closure conditions for the ordinary orthogonal polynomials were introduced and analyzed in [11]. In this case, the closure condition (5.2) leads to a special class of semi-classical polynomials (on special exponential grids) introduced by Magnus [8].

From (1.17) it follows that closure condition (5.2) is equivalent to the conditions for recurrence coefficients:

$$b_n^{(N)}=qb_n,\qquad d_n^{(N)}=qd_n, \tag{5.3}$$

where $b_n^{(N)}$ stands for the coefficients obtained from $b_n$ by the application of $N$ CT at the points $\mu_1,\mu_2,\dots,\mu_N$. For arbitrary $N$ relations (5.3) become very complicated nonlinear difference equations. However in particular cases, these equations can be solved in terms of elementary functions.

In operator form, closure condition (5.2) can be written as

$$A(q)J(N)=K(N)A,\quad B(q)J(N)=K(N)B, \tag{5.4}$$

where the operators $A(q),B(q)$ are as defined in (4.17) and the operators $K(N)$, $J(N)$ specified by the following procedure. Assume that we start from the polynomials $P_n(z)$. In the first step we get the polynomials $P_n(z;\mu_1)$ by applying a CT at the point $\mu_1$. We then have the operator relations $A_1J_1=K_1A$, $B_1J_1=K_1B$ where the operators $A_1,B_1$ correspond to the eigenproblem for the LBP $P_n(z;\mu_1)$. In the second step, we choose the point $\mu_2$ and get the relation $A_2J_2=K_2A_1$, $B_2J_2=K_2B_1$ where $A_2,B_2$ correspond to the polynomials $P_n(z;\mu_1,\mu_2)$. Clearly we have $A_2J_2J_1=K_2K_1A$, $B_2J_2,J_1=K_2K_2B$. Repeating this procedure we get relation (5.4) where

$$J(N)=J_NJ_{N-1},\dots,J_1,\quad K(N)=K_NK_{N-1},\dots,K_1. \tag{5.5}$$

## 6. The simplest $N=1$ closure

In this section we consider the simplest $q$-closure with $N=1$, i.e., it means that

$$\frac{P_{n+1}(z)-U_nP_n(z)}{z-\mu}=q^nP_n(z/q), \tag{6.1}$$

where $\mu$ is an arbitrary parameter.

From formulas (2.5) and (2.6) we get the relations

$$\frac{b_{n+1}+U_n}{b_n+U_{n-1}}=q, \tag{6.2}$$

$$\frac{d_{n+1}+U_{n+1}}{d_n+U_n}=q, \tag{6.3}$$

where the coefficients $U_n=P_{n+1}(\mu)/P_n(\mu)$ obey the relation

$$U_n+d_n=\mu(1+b_n/U_{n-1}),\quad n=1,2,\dots \tag{6.4}$$

as follows from the recurrence relation (1.9). Moreover, the initial condition yields

$$U_0 = \mu - d_0. \tag{6.5}$$

It is easily found that the most general solution of relations (6.2) and (6.3) (compatible with initial condition (6.5)) is

$$U_n \equiv -\beta, \tag{6.6}$$

$$b_n = \beta(1 - q^n), \tag{6.7}$$

$$d_n = \mu q^n + \beta, \tag{6.8}$$

where $\beta$ is an arbitrary nonzero constant.

The polynomials $P_n(z)$ thus obtained have their recurrence coefficients $b_n, d_n$ expressed in terms of elementary functions (6.7), (6.8) and depending on two parameters $\beta$ and $\mu$. In fact, only one of these parameters (say, $\mu$) is essential because the parameter $\beta$ can be set equal to 1 with the help of a scaling transformation.

In order to find the moments corresponding to these polynomials we exploit the formula (2.2) for the transformation of the moments $c_n$ with respect to the CT. Taking into account that the scaling transformation $P_n(z) \to q^n P_n(z/q)$ leads to the transformation $c_n \to q^n c_n$, we see that closure condition (6.1) is equivalent to the equation

$$c_{n+1} = (\beta q^n + \mu)c_n, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{6.9}$$

whence

$$c_n = \mu^n(-\beta/\mu; q)_n, \tag{6.10}$$

where $(a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1})$ is the $q$-shifted factorial (for negative $n$ the $q$-shifted factorials are defined [2] as $(a;q)_{-n} = 1/(aq^{-n};q)_n$).

## 7. $q$-Appel LBP

The LBP satisfying the property

$$\mathscr{D}_q\{P_{n+1}(z)\} = [n+1]P_n(z) \tag{7.1}$$

will be called $q$-Appel LBP. The symbol $\mathscr{D}_q$ used above denotes the $q$-derivative operator defined by

$$\mathscr{D}_q\psi(z) = \frac{\psi(qz) - \psi(z)}{z(q-1)} \tag{7.2}$$

while $[n] = (q^n - 1)/(q-1)$ is the so-called $q$-number. Let us show that condition (7.1) is equivalent to closure condition (6.1).

Denote $R_n(z) = q^{-n}P_n(qz)$, the $q$-Appel condition (7.1) is then rewritten as

$$R_n(z) = V_nP_n(z) + z(1 - V_n)P_{n-1}(z) \tag{7.3}$$

with $V_n = q^{-n}$. In view of Proposition 2, (7.3) leads to the reciprocal relation

$$P_n(z) = \frac{R_{n+1}(z) - U_nR_n(z)}{z - v} \tag{7.4}$$

with some constant $v$. This last relation (7.4) coincides with closure condition (6.1) (with $v = q\mu$), the $q$-Appel LBP are thus given by (7.10).

This observation allows one to get explicit expression of $q$-Appel LBP. Indeed, one can express

$$P_n(z) = \sum_{k=0}^{n} A_k^{(n)} z^k \tag{7.5}$$

with some expansion coefficients $A_k^{(n)}$. From (7.1), (6.1) and (6.6) we then get two relations for the coefficients $A_k^{(n)}$:

$$q^k A_k^{(n+1)} - A_k^{(n+1)} = (q^{n+1} - 1) A_{k-1}^{(n)},$$

$$A_k^{(n+1)} + \beta A_k^{(n)} = q^{n+1-k} A_{k-1}^{(n)} - \mu q^{n-k} A_k^{(n)} \tag{7.6}$$

with the condition

$$A_n^{(n)} = 1 \tag{7.7}$$

(because LBP $P_n(z)$ are monic). From (7.6) we can exclude $A_k^{(n+1)}$ thus leading to the two-term recurrence relation

$$\frac{A_k^{(n)}}{A_{k-1}^{(n)}} = \frac{1 - q^{k-n-1}}{(1 - q^k)(\mu + \beta q^{k-n})}. \tag{7.8}$$

Taking into account condition (7.7) we obtain from (7.8) explicit expression

$$A_k^{(n)} = (-\beta)^n (q/\mu)^k (-\mu/\beta; q)_n \frac{(q^{-n}; q)_k}{(q; q)_k (-\beta/\mu q^{1-n})_k}, \tag{7.9}$$

whence we get the expression of $P_n(z)$ in terms of basic hypergeometric function

$$P_n(z) = (-\beta)^n (-\mu/\beta; q)_n \, {}_2\phi_1 \left( \begin{matrix} q^{-n}, 0 \\ \frac{-\beta q^{1-n}}{\mu} \end{matrix} ; \frac{zq}{\mu} \right). \tag{7.10}$$

(The notation for the basic hypergeometric functions is the standardized one of Ref. [2].)

In the limiting case $\mu = 0$, we have for the coefficients $b_n = \beta(1 - q^n)$, $d_n = \beta$, and the explicit expression for the polynomials reads

$$P_n(z) = (-\beta)^n \sum_{k=0}^{n} \frac{(q^{-n})_k}{(q)_k} q^{-k(k-1)/2} (zq^n/\beta)^k. \tag{7.11}$$

The corresponding moments are

$$c_n = \beta^n q^{n(n-1)/2}. \tag{7.12}$$

It is interesting to note that for $\beta = q^{1/2}$ we get a *symmetric* moment problem: $c_n = c_{-n} = q^{n^2}$. When $0 < q < 1$, the corresponding polynomials coincide with those found by Szegő [13]. The Szegő polynomials are orthogonal on the unit circle.

Let us consider the limit $q \to 1$ of the $q$-Appel polynomials.

Using recurrence relation (1.9) we see that for $\mu \to \infty$ the Christoffel transformation becomes (to within terms $O(1/\mu)$)

$$\mathscr{C}(\mu)\{P_n(z)\} = P_n(z) + \frac{b_n(zP_{n-1}(z) - P_n(z))}{\mu} + O(1/\mu^2). \tag{7.13}$$

Thus for $\mu \to \infty$ the Christoffel transformation differs from the identity transformation (i.e. $P_n(z) \to P_n(z)$) only by terms of the order of $\mathrm{O}(1/\mu)$.

On the other hand, for $q = 1 + \epsilon + O(\epsilon^2)$ we have

$$q^n P_n(z/q) = P_n(z) + \epsilon(nP_n(z) - zP_n'(z)) + \mathrm{O}(\epsilon^2). \tag{7.14}$$

Putting

$$\epsilon = -1/\mu, \tag{7.15}$$

we get from (6.1), in the limit $\epsilon \to 0$, the following condition:

$$nP_n(z) - zP_n'(z) = b_n(zP_{n-1}(z) - P_n(z)). \tag{7.16}$$

Equating, in (7.16), the coefficients in front of $z^{n-1}$, we find that $b_n = -n$ and hence (7.16) becomes

$$P_n'(z) = nP_{n-1}(z). \tag{7.17}$$

Condition (7.17) defines the Appel polynomials. They have the recurrence coefficients (see, e.g., [15]) $b_n = \beta n$, $d_n = \beta(n+\gamma)$ where $\beta$, $\gamma$ are arbitrary constants. In our case $\beta = -1$ according to the choice (7.15). (Obviously the constant $\beta$ plays the role of the scaling parameter.) The explicit expression for these polynomials is

$$P_n(z) = (\gamma)_n\, {}_1F_1\left(\begin{matrix} -n \\ 1-n-\gamma \end{matrix}; z\right), \tag{7.18}$$

where $(a)_n$ is the Pochhammer symbol and ${}_1F_1$ the ordinary hypergeometric function. The Appel property (7.17) is an elemenatry consequence of the relation

$$\frac{\mathrm{d}}{\mathrm{d}z}\, {}_1F_1\left(\begin{matrix} a \\ b \end{matrix}; z\right) = (a/b)\, {}_1F_1\left(\begin{matrix} a+1 \\ b+1 \end{matrix}; z\right), \tag{7.19}$$

where $a, b$ are arbitrary parameters.

## 8. Operator algebra corresponding to the *q*-Appel LBP

In this section we consider the operator algebra corresponding to closure condition (5.2). Here the operators $J, K$ have the simple form

$$J|n\rangle = |n-1\rangle + \beta|n\rangle, \quad K|n\rangle = |n-1\rangle + q\beta|n\rangle \tag{8.1}$$

and the relation between these operators becomes

$$K = qQJQ^{-1}. \tag{8.2}$$

We can thus eliminate the operator $K$ using this relation. We then get from (5.4) the following closure conditions for $N = 1$:

$$AQ^{-1}J = JQ^{-1}A, \tag{8.3}$$

$$BQ^{-1}J = qJQ^{-1}B. \tag{8.4}$$

Moreover, definition (8.1) entails an additional condition for the operator $Q$ and $J$:

$$Q^{-1}J - qJQ^{-1} = \beta(1-q)Q^{-1}. \tag{8.5}$$

We thus have operator algebra generated by three relations (8.3)–(8.5) for four operators $A, B, J, Q$. Note that this algebra seems to be more complicated than the simple $q$-oscillator algebra obtained in the case of the ordinary OP [10,11] for $N = 1$ closure.

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