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Some calculations of $\mathrm{Lie}(n)^{\max}$ for low n

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ABSTRACT

In a previous paper, the authors gave the finest functorial decomposition of the loop suspension of a p-torsion suspension. In order to determine the decomposition, one must know the maximum projective submodule of $\mathrm{Lie}(n)$, which we label $\mathrm{Lie}^{\max}(n)$. The purpose of this paper is to give some sample calculations of $\mathrm{Lie}^{\max}(n)$ for low n when $p = 2$.

1. Introduction

In the work of Cohen [3], Arone–Kankaanrinta [1], Arone–Mahowald [2], Dwyer, and other topologists, an important module labelled $\mathrm{Lie}(n)$ over the symmetric group S_n has arisen as a homology group in spaces related to braid groups [4], configuration spaces, and Goodwillie towers. It also appears as the kernels in the Cohen pro-group [3], which he is using in his attack on Barratt's conjecture through combinatorial group theory. The module is also part of the classical algebraic literature, having appeared in the works of Witt [16], Weber [15], and others. In characteristic 0, several of its properties are given in Reutenauer [11]. It has also been studied by algebraists like Donkin and Erdmann [6] who are interested in the modular theory of the symmetric group.

In [13], the problem of finding the finest natural decomposition of the loop suspension of a p-torsion suspension (for a prime p) was solved in two stages. The first step was to reduce the problem to algebra by showing that natural coalgebra decompositions of the mod p homology of such a space (a tensor algebra) could be geometrically realized and the second step was to construct the finest natural coalgebra decomposition of a tensor algebra. The solution appears in [13], labelled the "Functorial Poincaré–Birkhoff–Witt Theorem" (Theorem 6.5). Of special interest is the factor in the decomposition of the tensor algebra $T(V)$ which contains the space V itself. This factor was labelled $A^{\min}(V)$. While the existence of $A^{\min}(V)$ is established in [13], it appears however that computing the space $A^{\min}(V)$ and the other factors in the decomposition is not easy. It seems to depend heavily on properties of the module $\mathrm{Lie}(n)$, and is related to unsolved problems in the modular representation theory of the symmetric group, including that of computing the mod p decomposition matrices. Even if the modules cannot be computed precisely, it would be interesting to know how quickly their dimensions grow, and especially whether or not the growth rate is exponential. In order to calculate the factors of this functorial Poincaré–Birkhoff–Witt decomposition, one must know the maximum projective submodule of $\mathrm{Lie}(n)$, which we label $\mathrm{Lie}^{\max}(n)$. It is well known that if n is prime to p, then $\mathrm{Lie}(n)$ is itself projective, so that $\mathrm{Lie}^{\max}(n) = \mathrm{Lie}(n)$ in this case. The purpose of this paper is to give some sample calculations of $\mathrm{Lie}^{\max}(n)$ for low n for $p = 2$. The calculations given depend on knowledge of the decomposition

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matrices which are known in these low cases. The evidence so far from these calculations shows that $\mathrm{Lie}^{\max}(n)$ is a relatively large part of $\mathrm{Lie}(n)$ and this corresponds to $A^{\min}(V)$ being relatively small.

Since {*projective modules*} = {*injective modules*} in the group ring of a finite group, $\mathrm{Lie}^{\max}(n)$ is also injective and thus there is a direct sum decomposition $\mathrm{Lie}(n) = \mathrm{Lie}^{\max}(n) \oplus \mathrm{Lie}(n)/\mathrm{Lie}^{\max}(n)$. As noted above, one application of $\mathrm{Lie}(n)$ is to the study of Goodwillie calculus. In [1], Arone and Kankaanrinta calculated $H_*(S_n; \mathrm{Lie}(n))$ and found that it was 0 unless $n = p^k$ and that $H_*\left(S_{p^k}; \mathrm{Lie}(p^k)\right)$ has a basis corresponding to admissible sequences of Steenrod operations of length k (or equivalently, to "completely unadmissible" sequences of Dyer–Lashof operations). Of course, this cohomology of S_n with coefficients in $\mathrm{Lie}(n)$ comes entirely from the complementary factor $\mathrm{Lie}(n)/\mathrm{Lie}^{\max}(n)$. In the group ring of a p-group over a field of characteristic p, {*projective modules*} = {*free modules*} (cf. [12] Corollaries on pages 64 and 118). In particular when $\mathbf{k}$ has characteristic p, for a finite group G the order of any projective $\mathbf{k}(G)$-module must be divisible by the order of the Sylow p-subgroup of G. Since the dimension of $\mathrm{Lie}(n)$ is $(n-1)!$, the complementary factor $\mathrm{Lie}(n)/\mathrm{Lie}^{\max}(n)$ is nontrivial whenever p divides n, in spite of $H_*\left(S_n; \mathrm{Lie}(n)/\mathrm{Lie}^{\max}(n)\right)$ being trivial when n is not a power of p.

In this paper we compute $\mathrm{Lie}^{\max}(n)$ (for characteristic 2) when $n < 10$, the difficult cases being $n = 6$ and $n = 8$. As noted above, $\mathrm{Lie}^{\max}(n) = 0$ when n is odd, and the fact that the dimension of a projective module must be divisible by the order of the Sylow 2-subgroup shows that $\mathrm{Lie}^{\max}(2) = \mathrm{Lie}^{\max}(4) = 0$. When $n = 6$, the order of the Sylow 2-subgroup of S_n is 16 and we find that the dimension of $\mathrm{Lie}(6)^{\max}$ is 96 with a corresponding 24-dimensional complementary module C_6 for which $H_*(S_6; C_6) = 0$. When $n = 8$, the order of the Sylow 2-subgroup of S_n is 128 and we find that the dimension of $\mathrm{Lie}(8)^{\max}$ is 4224 with a corresponding 816-dimensional complementary module C_8. Thus according to [2], $H_*(S_8; C_8)$ corresponds to sequences of admissible Steenrod operations of length 3. In each case the factors of $\mathrm{Lie}^{\max}$ are computed in terms of the indecomposable projectives within the corresponding group ring.

The layout of this paper is as follows. Section 2 contains definitions, including that of $\mathrm{Lie}(n)$ itself, along with the introduction of notation and review of relevant material from [13]. Section 3 contains the correspondence between representations of the symmetric group and symmetric polynomials. This is applied in Section 4 to the case of Lie modules. Section 5 contains the calculation of $\mathrm{Lie}^{\max}(6)$. The most difficult part of the calculation of $\mathrm{Lie}^{\max}(8)$ is the determination of the multiplicity of P^{431}, a particular indecomposable projective. As preparation for this, detailed calculations of Steenrod operations within $P^{431}(V)$ (the application of the functor corresponding to P^{431}) are given in Section 6. The calculation of $\mathrm{Lie}^{\max}(8)$ appears in Section 7.

2. Notation and review of previous results

As in [13], we use the following notation. Let $\mathbf{k}$ be a field. Given a vector space V, we write $|V|$ for the dimension of V. In the tensor algebra $T(V)$, let $[x, y]$ denote the commutator $xy - yx$. We often write $[[v_1, v_2, v_3, \ldots, v_n]$ for the iterated commutator $[\ldots[[v_1, v_2], v_3], \ldots v_n]$. Define $\beta_n : V^{\otimes n} \to V^{\otimes n}$ by $\beta_n(v_1 v_2 v_3 \cdots v_n) = [[v_1, v_2, v_3, \ldots, v_n]$. When there is no possibility of confusion we will sometimes write simply β for β_n. The image $\beta_n(V)$ is denoted $L_n(V)$. For later use, observe that

$$\beta_n \circ (V^{\otimes n-k} \otimes \beta_k)(v_1, \ldots, v_n) = -[\beta_{n-k}(v_1, \ldots, v_{n-k}), \beta_k(v_{n-k+1}, \ldots v_n)]$$

where we use the Milnor–Moore convention under which the name of a space also denotes the identity map of that space. In particular, $\mathrm{Im}[\beta_j, \beta_k]$ is contained in $L_{j+k}(V)$.

Let $\bar{V} = \langle x_1, x_2, \ldots, x_n \rangle$ be an n-dimensional vector space over $\mathbf{k}$. The (left) action of the symmetric group S_n on $\bar{V}$ given by permuting the basis of $\bar{V}$ extends to what we call the "internal action" on any functorial image of $\bar{V}$. We write γ_n for the (vector) subspace $\langle x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)} \rangle_{\sigma \in S_n}$ of $\bar{V}^{\otimes n}$, made into a (left) $\mathbf{k}(S_n)$-module by the internal action. As a $\mathbf{k}(S_n)$-module, γ is isomorphic to the group ring $\mathbf{k}(S_n)$, with S_n acting by multiplication on the left (using left-to-right multiplication of cycles within S_n). The $\mathbf{k}(S_n)$-module $\mathrm{Lie}(n)$ is defined as $L_n(\bar{V}) \cap \gamma$. Explicitly, $\mathrm{Lie}(n)$ is the linear span within $\bar{V}^{\otimes n}$ of the elements $[[x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}]_{\sigma \in S_n}$. That is, $\mathrm{Lie}(n)$ consists of n-fold brackets in which each basis element x_i appears exactly once. The (left) action of the symmetric group is induced from permutation of the basis elements.

By iterated use of the Jacobi identity, one can show that the elements of the form $[[x_1, x_{\sigma(2)}, \ldots, x_{\sigma(n)}]$ as σ runs through the permutations of $\{2, \ldots, n\}$, form a basis for $\mathrm{Lie}(n)$. In particular, it follows that $|\mathrm{Lie}(n)| = (n-1)!$ and that the restriction of $\mathrm{Lie}(n)$ to a $\mathbf{k}(S_{n-1})$-module is isomorphic to the group ring $\mathbf{k}(S_{n-1})$.

We also use the notation $L_i L_j$ to denote $\left(L_i(\bar{V}) \otimes L_j(\bar{V})\right) \cap \gamma$ and similarly we write $[L_i, L_j]$ for $[L_i(\bar{V}), L_j(\bar{V})] \cap \gamma$. Observe that $L_{k-1}L_1$ is the module induced from L_{k-1} under the inclusion $\mathbf{k}(S_{k-1}) \hookrightarrow \mathbf{k}(S_k)$.

Next we introduce some notation from representation theory. For a vector space V, there is a (right) "position" action of S_n on $V^{\otimes n}$. This gives a correspondence between elements of $\mathbf{k}(S_n)$ and natural transformations from the functor $V \mapsto V^{\otimes n}$ to itself. On $\gamma \cong \mathbf{k}(S_n)$ we have both actions, and since they commute each element of $\mathbf{k}(S_n)$ gives, by its position action, an endomorphism of γ with respect to the internal action. If we let σ_k be the k-cycle $(1\ 2\ \cdots\ k) \in \mathbf{k}(S_n)$ then $\beta_n = (1 - \sigma_n)(1 - \sigma_{n-1}) \cdots (1 - \sigma_2)$. Since β_n makes sense over the integers, $\mathrm{Lie}(n)$ is defined with any coefficients.

More generally, for any element $f \in \mathbf{k}(S_n)$, the natural transformation η_f corresponding to the position action of f gives a subspace $\eta_f\left(V^{\otimes n}\right) = V^{\otimes n} \cdot f \subset V^{\otimes n}$ which becomes a $\mathbf{k}\left(S_{|V|}\right)$-module under the internal action. In particular we get a $\mathbf{k}(S_n)$-module $\eta_f\left(\bar{V}^{\otimes n}\right) \cap \gamma \cong \mathbf{k}(S_n)f$. If f is an idempotent, then $\mathbf{k}(S_n)f$ is projective. Conversely, for each projective submodule P of $\mathbf{k}(S_n)$ there is an idempotent $f \in \mathbf{k}(S_n)$, and thus we get a $\mathbf{k}\left(S_{|V|}\right)$-submodule $V^{\otimes n} \cdot f \subset V^{\otimes n}$ which we write as $P(V)$.

Since $\mathbf{k}(S_n)$ is Artinian, any f yields, by iteration, an idempotent f^N for some N, which we call a stable idempotent obtained from f. The resulting projective $\mathbf{k}(S_n)f^N$ is isomorphic to the colimit of $\mathbf{k}(S_n)$ under iterated multiplication by f.

Unless stated otherwise, a partition of n shall refer to an ordered partition into positive integers, where $\lambda = (\lambda_1, \ldots, \lambda_r)$ is an ordered partition if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. In this case we set $\mathrm{Len}(\lambda) = r$. We write $\lambda \longrightarrow\!\!| \ n$ to mean "λ is a partition of n". If λ is a partition, we sometimes write $|\lambda|$ for $\lambda_1 + \cdots + \lambda_r$. We will also sometimes use the other standard notation for partitions in which one writes $\lambda = k_1^{q_1} \cdots k_m^{q_m}$, meaning that k_j appears q_j times, so that $n = q_1k_1 + \cdots + q_mk_m$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r) = k_1^{q_1} \cdots k_m^{q_m}$, in a polynomial algebra with ordered variables $x_1, \ldots, x_s, \ldots$, we write x^λ for the monomial $x_1^{\lambda_1} \cdots x_r^{\lambda_r}$ and x_λ for the monomial $x_{k_1}^{q_1} \cdots x_{k_m}^{q_m}$.

As is well known, a property of the group S_n is that isomorphism classes of projective $\mathbf{k}(S_n)$-modules depend only upon the characteristic of the field $\mathbf{k}$. (See [7].) If char $\mathbf{k} = 0$ they are in 1–1 correspondence with partitions of n and if char $\mathbf{k} = p$ they are in 1–1 correspondence with p-regular partitions of n, where a partition of n into positive integers is called p-regular if no integer occurs more than $p - 1$ times.

Let $\mathbb{Q}_p$ denote the p-adic field and $\mathbb{Z}_p$ the p-adic integers. For any finite group G, each projective $\mathbb{F}_p(G)$-module P lifts uniquely to a projective $\mathbb{Z}_p$-module $\hat{P}$. (c.f. [8,5], or [12].) For a p-regular partition λ we write P^λ for the projective $\mathbb{F}_p(S_n)$-module corresponding to λ, and $\hat{P}^\lambda$ for its lift to $\mathbb{Z}_p(S_n)$. In characteristic 0, indecomposable $\mathbf{k}(G)$-modules are irreducible and we write α^λ for the irreducible representation of $\mathbf{k}(S_n)$ corresponding to partition λ. The decomposition of $\hat{P}^\lambda \otimes \mathbb{Q}_p$ into irreducible factors is given by what is called the "decomposition matrix", although these matrices have been calculated only for low values of n. For $p = 2$ and 3, decomposition matrices up to $n = 13$ can be found in [7, pp 213–216].

In [13] the authors constructed a natural coalgebra decomposition

$$T(V) \cong A^{\min}(V) \otimes B^{\max}(V)$$

where $A^{\min}(V)$ is the minimum natural coalgebra retract of $T(V)$ which contains V. This led to two natural coalgebra decompositions of $T(V)$. The first decomposition, which we have labelled the "Functorial Poincaré–Birkhoff–Witt Theorem", [13, Thm. 6.5], is the finest natural decomposition. We will also refer to this as the "minimal" decomposition. It leads to the construction of a $\mathbf{k}(S_n)$-submodule $\mathrm{Lie}^{\max}(n)$ of $\mathrm{Lie}(n)$, which was shown to be the maximum projective submodule of $\mathrm{Lie}(n)$. The second decomposition, which we will call the "tensor decomposition", [13, Thm. 10.7 applied with $n = 1$] writes the coalgebra $B^{\max}(V)$ as a tensor product of tensor algebras rather than as a product of minimal factors. It leads to the construction of another projective $\mathbf{k}(S_n)$-submodule of $\mathrm{Lie}(n)$, labelled $\mathrm{Lie}'(n)$.

Thus for each n we have the inclusions

$$0 \subset \mathrm{Lie}'(n) \subset \mathrm{Lie}^{\max}(n) \subset \mathrm{Lie}(n)$$

of $\mathbf{k}(S_n)$-modules, where all but the last are projective. Although it arises naturally, the module $\mathrm{Lie}^{\max}(n)$ does not appear to be easy to calculate. In this paper we calculate this module for $n \leq 9$ when char $\mathbf{k} = 2$. In some cases the submodule $\mathrm{Lie}'(n)$ is easier to calculate and assists in calculating $\mathrm{Lie}^{\max}(n)$.

In [13, Section 11.4], we noted the following facts which are useful in calculating $\mathrm{Lie}^{\max}(n)$.

(1) $\beta_n\beta_n = n\beta_n$, and so β_n/n is an idempotent when n is prime to p. Thus $\mathrm{Lie}(n)$ is projective when n is prime to p.
(2) For a finite group G, the order of any projective $\mathbf{k}(G)$-module must be divisible by the order of the Sylow p-subgroup of G.
(3) All nonzero primitives in $A^{\min}(V)$ lie in degrees which are powers of p.

From now on suppose that $p = 2$. Although we shall sometimes need to consider other coefficients, unless stated otherwise L_n shall denote the mod 2 reduction $L_n \otimes \mathbb{F}_2$.

From (1) and (2) and the definitions in [13] of $\mathrm{Lie}'(n)$ and $\mathrm{Lie}^{\max}(n)$ we see that

$$\begin{aligned}
&0 \subsetneq \mathrm{Lie}'(1) = \mathrm{Lie}^{\max}(1) = \mathrm{Lie}(1),\\
&0 = \mathrm{Lie}'(2) = \mathrm{Lie}^{\max}(2) \subsetneq \mathrm{Lie}(2),\\
&0 \subsetneq \mathrm{Lie}'(3) = \mathrm{Lie}^{\max}(3) = \mathrm{Lie}(3),\\
&0 = \mathrm{Lie}'(4) = \mathrm{Lie}^{\max}(4) \subsetneq \mathrm{Lie}(4),\\
&0 \subsetneq \mathrm{Lie}'(5) = \mathrm{Lie}^{\max}(5) = \mathrm{Lie}(5),\\
&0 \subsetneq \mathrm{Lie}'(7) = \mathrm{Lie}^{\max}(7) = \mathrm{Lie}(7),\\
&0 \subsetneq \mathrm{Lie}'(9) \subset \mathrm{Lie}^{\max}(9) = \mathrm{Lie}(9).
\end{aligned}$$

We shall calculate $\mathrm{Lie}^{\max}(6)$ and $\mathrm{Lie}^{\max}(8)$. The results can be summarized as follows.

Theorem. $\mathrm{Lie}(6)^{\max}$ *has dimension* 96 *and is given explicitly by*

$$\mathrm{Lie}(6)^{\max} \cong P^{51} \oplus 3P^{321}$$

where P^{51} and P^{321} are indecomposable projective modules with $|P^{51}| = 48$, and $|P^{321}| = 16$. The complementary module $\mathrm{Lie}(6)/\mathrm{Lie}^{\max}(6)$ is a 24-dimensional module with no projective submodules and has the property that

$$H_*\left(S_n; \mathrm{Lie}(6)/\mathrm{Lie}^{\max}(6)\right) = H_*\left(S_n; \mathrm{Lie}(6)\right) = 0.$$

Theorem. $\mathrm{Lie}(8)^{\max}$ *has dimension* 4224 *and is given explicitly by*

$$\mathrm{Lie}^{\max}(8) \cong 2P^{62} \oplus P^{53} \oplus 8P^{521} \oplus 4P^{431}$$

where P^{62}, P^{53}, P^{521}, *and* P^{431} *are indecomposable projective modules with* $|P^{62}| = 640$, $|P^{53}| = 384$, $|P^{521}| = 128$, *and* $|P^{431}| = 384$. *The complementary module* $\mathrm{Lie}(8)/\mathrm{Lie}^{\max}(8)$ *is an 816-dimensional module with no projective submodules and has the property that* $H_*\left(S_n; \mathrm{Lie}(6)/\mathrm{Lie}^{\max}(8)\right) = H_*\left(S_n; \mathrm{Lie}(8)\right)$ *has a basis consisting of admissible sequences of Steenrod operations of length* 3.

3. Characteristic polynomials and representations of S_n

Let $V = \langle x_1, \ldots, x_r\rangle$ be an r-dimensional vector space. For an unordered partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n into r nonnegative integers let V^λ denote the $\mathbf{k}(S_n)$-submodule of $V^{\otimes n}$ (under the position action) generated by $x_1^{\otimes\lambda_1} \otimes \cdots \otimes x_r^{\otimes\lambda r} \in V^{\otimes n}$. Given a homogeneous vector subspace A of $V^{\otimes n}$ define the *characteristic polynomial* of A by $\mathrm{ch}(A) = \sum_{\lambda \dashv n} d_\lambda x^\lambda$, where d_λ is the dimension of $A \cap V^\lambda$. It is clear from the definition that $\mathrm{ch}(A \oplus B) = \mathrm{ch}(A) + \mathrm{ch}(B)$ and $\mathrm{ch}(A \otimes B) = \mathrm{ch}(A)\mathrm{ch}(B)$.

If A is a $\mathbf{k}(S_r)$-submodule of $V^{\otimes n}$ (under the internal action) then $\mathrm{ch}(A)$ is a symmetric polynomial in $x_1, \ldots, x_r$. A symmetric polynomial can be written in terms of the power functions $\psi_j = x_1^j + \cdots + x_r^j$. For a natural submodule $M(V) \subset V^{\otimes n}$, when written in power functions the polynomial $\mathrm{ch}\,(M(V))$ is independent of the dimension r of V provided that r is sufficiently large. ($r \geq n$ always suffices.) We write simply $\mathrm{ch}(M)$ for this polynomial.

In characteristic 0, the characteristic polynomial of a natural submodule of $V^{\otimes n}$ can be computed via characters, hence the name.

Theorem. *Let* f *belong to* $\mathbf{k}(S_n)$ *where* $\mathrm{char}\,\mathbf{k} = 0$ *and let* $M = \mathbf{k}(S_n)f$ *be the left* $\mathbf{k}(S_n)$*-module obtained from right multiplication by* f. *Then for the natural submodule* $M(V) = V^{\otimes n} \cdot f$ *obtained from* f *we have*

$$\mathrm{ch}(M) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_M(\sigma) \psi_{\lambda(\sigma)},$$

where $\lambda(\sigma)$ *denotes the partition of* n *corresponding to the cycle decomposition of* σ, *and* $\chi_M(\sigma)$ *is the character of* σ *in the representation* M.

See [10, Chapter I, A7]. $\square$

More generally, if $\mathrm{char}\,\mathbf{k} = 0$, the characteristic polynomial of any $\mathbf{k}(S_n)$-module M is defined by

$$\mathrm{ch}(M) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_M(\sigma) \psi_{\lambda(\sigma)},$$

and according to the preceding theorem, the two definitions of $\mathrm{ch}(M)$ agree when both make sense.

For a subgroup G of S_n, corresponding to any representation M of G there is an induced representation $M_G^{S^n}$ defined by $M_G^{S^n} = \mathbf{k}(S^n) \otimes_{\mathbf{k}(G)} M$. As in [7] we let IG denote the trivial 1-dimensional representation of G. We write simply $\mathrm{ch}(G)$ for $\mathrm{ch}\left((IG)_G^{S^n}\right)$. From the definition we get

$$\mathrm{ch}(G) = \frac{1}{|G|} \sum_{\lambda \dashv n} (\#\text{ of elements of } G \text{ having cycle type } \lambda) \psi_\lambda.$$

In particular, $\mathrm{ch}(S_n) = \frac{1}{n!} \sum_{\lambda \dashv n} c_\lambda \psi_\lambda$, where for a partition $\lambda = k_1^{r_1} \cdots k_m^{r_m}$ (meaning $n = r_1k_1 + \cdots + r_mk_m$), $c_\lambda = n!/(r_1! \ldots r_m! k_1 \ldots k_m)$ is number of elements in S_n having cycle type λ.

Let f belong to $\mathbf{k}(S_m)$, where $\mathrm{char}\,\mathbf{k} = 0$. Let $M = \mathbf{k}(S_m)f$ be the representation of S_m corresponding to f and let $M(V) = V^{\otimes m} \cdot f$ be the corresponding functor. Let $S_k[M(V)]$ denote the invariants of $M(V)^{\otimes k}$ under the (position) action of S_k. Thus $S_k[M(V)]$, the space of symmetric tensors of length k, is isomorphic to the subspace of degree k polynomials in the symmetric algebra $S[M(V)]$. Also $S_k[M(V)] = M(V)^{\otimes k} \cdot N$, where $N = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \in \mathbf{k}(S_k)$. The corresponding representation $\mathrm{Im}\, N \circ f^{\otimes k}$ of S_{km} has been called a *plethysm*, written as $S_k[M]$ or $IS_k \odot M$. (See [9].) It is the representation of S_{km} induced from the representation of the wreath product $S_m \mathrm{wr}\, S_k$ which comes from the representation M of S_m and the trivial representation of S_k.

If $g(\psi) = \sum a_\lambda \psi_\lambda$ and q is a positive integer, let $g(q\psi)$ denote the polynomial $\sum a_\lambda \psi_{q\lambda}$, where $q\lambda$ is the partition $(q\lambda_1, q\lambda_2, \ldots, q\lambda_n)$ of $q|\lambda|$ The formula for computing the characteristic polynomials of plethysms together with that for $\mathrm{ch}(S_n)$ gives

$$\mathrm{ch}\,(S_k[M]) = \frac{1}{n!} \sum_{\lambda \dashv n} c_\lambda \left(\mathrm{ch}\, M\, (k(\lambda)_1 \psi)\right)^{r(\lambda)_1} \cdots \left(\mathrm{ch}\, M\, (k(\lambda)_m \psi)\right)^{r(\lambda)_m},$$

where $\lambda = k(\lambda)_1^{r(\lambda)_1} \cdots k(\lambda)_m^{r(\lambda)_m}$ (see [10]). For example, $\mathrm{ch}(S_3) = \frac{1}{6}(\psi_1^3 + 3\psi_1\psi_2 + 2\psi_3)$, so if $\mathrm{ch}(M)(\psi) = \sum a_\lambda \psi_\lambda$, then

$$\mathrm{ch}\,(S_3[M]) = \frac{1}{6}\left(\left(\sum a_\lambda \psi_\lambda\right)^3 + 3\left(\sum a_\lambda \psi_\lambda\right)\left(\sum a_\lambda \psi_{2\lambda}\right) + 2\left(\sum a_\lambda \psi_{3\lambda}\right)\right).$$

It is well known that if P is a projective $\mathbf{k}(S_n)$-module then the isomorphism class of P is uniquely determined by its characteristic polynomial. More precisely, if P and Q are projectives such that $\mathrm{ch}(P) = \mathrm{ch}(Q)$ then $P \cong Q$. (See [10,12].) In particular, in characteristic 0, the isomorphism class of a module is uniquely determined by its characteristic polynomial. However in characteristic p, it is possible that $\mathrm{ch}(M) = \mathrm{ch}(N)$ without $M \cong N$, and it could even happen that one of M, N is projective and the other is not. For a projective $\mathbb{F}_p(S_n)$-module P, $\mathrm{ch}(P) = \mathrm{ch}\left(\hat{P} \otimes \mathbb{Q}_p\right)$ and so calculating $\mathrm{ch}(P)$ is equivalent to calculating the row of the decomposition matrix corresponding to P.

For an irreducible representation α^λ of $\mathbf{k}(S_n)$ in characteristic 0, the characteristic polynomial $\mathrm{ch}\left(\alpha^\lambda(V)\right)$ is the Schur polynomial F^λ, given as the quotient of determinants

$$\begin{vmatrix} x_1^{i_1+r-1} & x_1^{i_2+r-2} & \cdots & x_1^{i_r} \\ x_2^{i_1+r-1} & x_2^{i_2+r-2} & \cdots & x_2^{i_r} \\ \vdots & \vdots & \ddots & \vdots \\ x_r^{i_1+r-1} & x_2^{i_2+r-2} & \cdots & x_r^{i_r} \end{vmatrix} \Bigg/ \begin{vmatrix} x_1^{r-1} & x_1^{r-2} & \cdots & x_1 & 1 \\ x_2^{r-1} & x_2^{r-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_r^{r-1} & x_2^{r-2} & \cdots & x_r & 1 \end{vmatrix}.$$

In particular, if $|V| < \mathrm{Len}(\lambda)$ then $\alpha^\lambda(V) = 0$. From the definition of the dominance order $\prec$ on partitions [7, p. 23], it follows that if $\lambda \prec \mu$ then $\mathrm{Len}(\lambda) \geq \mathrm{Len}(\mu)$. For the projective indecomposable $\mathbb{F}_2(S_n)$-module P^λ, the preceding, together with [7, 6.3.51] and [7, 2.1.10] therefore implies that $P^\lambda(V) = 0$ whenever $\mathrm{Len}(\lambda) > |V|$. Thus when comparing characteristic polynomials of projectives in $\mathbb{F}_2(S_n)$, it suffices to consider $\mathrm{ch}\,(P(V))$ for $|V|$ equal to the length of the longest p-regular partition of n. As noted earlier, this does not suffice when considering arbitrary $\mathbb{F}_2(S_n)$-modules. For $p = 2$, the length of the longest 2-regular partition of n is the maximum k such that $(k^2 + k)/2 \leq n$.

4. Characteristic polynomials of Lie modules

We now calculate $\mathrm{ch}(L_n)$.

Let $\mu(q)$ denote the Möbius function defined for positive integers by

$$\mu(q) = \begin{cases} \mu(1) = 1 & \\ \mu(q) = 0 & \text{unless } q \text{ is squarefree;} \\ \mu(q) = (-1)^k & \text{if } q \text{ is a product of } k \text{ distinct primes.} \end{cases}$$

Theorem (*Witt*). $\mathrm{ch}(L_n) = \frac{1}{n}\sum_{q|n} \mu(q)\psi_q^{n/q}$.

Proof. Throughout this proof, partitions refer to unordered partitions into nonnegative integers. For a partition λ, let z_λ denote the multinomial coefficient $|\lambda|!/(\lambda_1! \cdots \lambda_r!)$. Since the dimensions are independent of the field, we can use rational coefficients in the calculation. Over $\mathbb{Q}$, the Poincaré–Birkhoff–Witt Theorem (either standard version or our functorial version) gives the (coalgebra) isomorphism

$$T(V) \cong \bigotimes_{k=1}^{\infty} S\left[L_k(V)\right], \tag{$*$}$$

where $S[\ ]$ denotes the symmetric (polynomial) algebra. Let $\mathrm{ch}\,(L_k(V)) = \sum_\lambda d_\lambda x^\lambda$. Then $\mathrm{ch}\,(S[L_k(V)]) = \prod_\lambda 1/(1-x^\lambda)^{d_\lambda}$. We wish to find d_λ. Applying ch to $(*)$ gives $1/(1-\psi_1) = \prod_\lambda 1/(1-x^\lambda)^{d_\lambda}$ and so $\log(1-\psi_1) = \sum_\lambda d_\lambda \log(1-x^\lambda)$. Therefore

$$\sum_n \psi_1^n/n = \sum_\lambda \sum_n d_\lambda x^{n\lambda}/n = \sum_n \sum_\tau d_{\tau/n} x^\tau/n,$$

where by convention $d_{\tau/n} = 0$ unless all entries of τ are divisible by n. Equating coefficients of x^λ gives

$$z_\lambda/|\lambda| = \sum_{\tau=\lambda/k} d_\tau/k. \tag{$**$}$$

Given λ, write $\lambda = n\lambda'$, where n is the greatest common divisor of the entries of λ. Setting $f(k) = kd_{k\lambda'}$ and $g(k) = z_{k\lambda'}/|\lambda'|$, $(**)$ reads $g(n) = \sum_{q|n} f(q)$, so the Möbius Inversion Formula [14, p. 514] gives $d_\lambda = \frac{1}{n}\sum_{q|n}\mu(q)z_{(n/q)\lambda'}/|\lambda'| = \frac{1}{\lambda}\sum_{q|n}\mu(q)z_{\lambda/q}$. Therefore

$$\mathrm{ch}(L_n) = \sum_{|\lambda|=n} d_\lambda x^\lambda = \sum_{|\lambda|=n}\sum_{q|n}\frac{1}{n}\mu(q)z_{\lambda/q}x^\lambda = \frac{1}{n}\sum_{q|n}\mu(q)\sum_{|\lambda|=n} z_\lambda x^\lambda = \frac{1}{n}\sum_{q|n}\mu(q)\psi_q^{n/q}. \quad \square$$

See [11, Chapter 8] for some other proofs of this theorem.

5. $n = 6$

For $p = 2$, the tensor decomposition [13, Theorem 10.7] gives

$$V^{\otimes 6} \cong A^{\min}(V)_6 \oplus \left(A^{\min}(V)_3 \otimes L_3(V)\right) \oplus \left(A^{\min}(V)_1 \otimes L_5(V)\right) \oplus T\,(L_3(V))_6 \oplus L_6'(V)$$

and passing to primitives gives $L_6(V) \cong [L_3(V), L_3(V)] \oplus L_6'(V)$. Therefore, as noted in [13, Section 11.4], $\mathrm{Lie}(6) \cong \mathrm{Lie}'(6) \oplus [L_3, L_3]$. To calculate $\mathrm{Lie}^{\max}(6)$ we must calculate $\mathrm{Lie}'(6)$ and we must find the maximum projective submodule of $[L_3, L_3]$.

With any coefficients there is a short exact sequence

$$0 \to S_2[L_3(V)] \to L_3(V) \otimes L_3(V) \to [L_3(V), L_3(V)] \to 0.$$

Since the dimensions are independent of the field, we can use rational coefficients to calculate the characteristic polynomials. From Section 3 we get $\mathrm{ch}(L_3) = (\psi_1^3 - \psi_3)/3$ and

$$\mathrm{ch}\,(S_2[L_3]) = \frac{1}{2}\left(\left(\frac{\psi_1^3 - \psi_3}{3}\right)^2 + \frac{\psi_2^3 - \psi_6}{3}\right).$$

Therefore

$$\mathrm{ch}\,([L_3, L_3]) = \left(\frac{\psi_1^3 - \psi_3}{3}\right)^2 - \frac{1}{2}\left(\left(\frac{\psi_1^3 - \psi_3}{3}\right)^2 + \frac{\psi_2^3 - \psi_6}{3}\right).$$

Since $\mathrm{ch}\,(\mathrm{Lie}(6)) = (\psi_1^6 - \psi_2^3 - \psi_3^2 + \psi_6)/6$, we get $\mathrm{ch}\left(\mathrm{Lie}'(6)\right) = (\psi_1^6 + \psi_1^3\psi_3 - 2\psi_3^2)/9$.

Suppose $|W| = 3$ and write $W = \langle x, y, z\rangle$. Expanding gives

$$\mathrm{ch}\left(\mathrm{Lie}'(6)(W)\right) = x^5y + 2x^4y^2 + 4x^4yz + 2x^3y^3 + 7x^3y^2z + 10x^2y^2z^2 + \text{ symmetrical terms.}$$

Using the known decomposition matrix for $n = 6$, [7, p. 414], we get

$$\mathrm{ch}\left(P^{6}(W)\right) = x^6 + 2x^5y + 3x^4y^2 + 6x^4yz + 4x^3y^3 + 8x^3y^2z + 10x^2y^2z^2 + \text{ symmetrical terms,}$$
$$\mathrm{ch}\left(P^{51}(W)\right) = x^5y + 2x^4y^2 + 4x^4yz + 2x^3y^3 + 5x^3y^2z + 6x^2y^2z^2 + \text{ symmetrical terms,}$$
$$\mathrm{ch}\left(P^{42}(W)\right) = x^4y^2 + 2x^4yz + 2x^3y^3 + 4x^3y^2z + 6x^2y^2z^2 + \text{ symmetrical terms,}$$
$$\mathrm{ch}\left(P^{321}(W)\right) = x^3y^2z + 2x^2y^2z^2 + \text{ symmetrical terms.}$$

Therefore $\mathrm{Lie}'(6) \cong P^{51} + 2P^{321}$.

Using the results of Section 3, the polynomial for $[L_3, L_3]$ tells us that

$$[L_3, L_3] \otimes \mathbb{Q} \cong \alpha^{411} + \alpha^{321} + \alpha^{33} + \alpha^{2211}$$

in characteristic 0. Examination of the decomposition matrix now shows that over $\mathbb{F}_2$ the only possible projective summand of $[L_3, L_3]$ is P^{321}. In fact, since α^{321} is the only summand in block 2, the rational decomposition already implies that P^{321} is indeed a summand of $[L_3, L_3]$, but for future use we need more specific information, so we appeal instead to the argument given in [13]. Namely, set

$$A = [\beta(zxx), \beta(yxy)] + [\beta(zxy), \beta(yxx)] \in [L_3(W), L_3(W)]$$

and let $\phi = [\beta_3, \beta_3] \circ T_{2,4} \in \mathbf{k}(S_6)$, where $T_{i,j}$ denotes the transposition $(i\,j)$. That is, ϕ acts on $V^{\otimes 6}$ by $\phi(v_1v_2\cdots v_6) = [\beta(v_1v_4v_3), \beta(v_2v_5v_6)]$. Let $\tilde{\phi}$ be a stable idempotent formed by iterating ϕ and let P be the projective $\mathbf{k}(S_n)$-module $\mathrm{Im}\,\tilde{\phi}$. Direct computation shows that $\phi(A) = A$, so A belongs to $P(W)$. In particular, $P \neq 0$ so $P \cong P^{321}$. Thus the maximum projective submodule of $[L_3, L_3]$ is P^{321}, and so we get

$$\mathrm{Lie}(6)^{\max} \cong \mathrm{Lie}'(6) \oplus P \cong P^{51} + 3P^{321}.$$

Since, as determined from the decomposition matrix, $|P^{51}| = 48$ and $|P^{321}| = 16$ this has dimension 96 as claimed in [13].

Notice that the general theory says that since the length of the largest partition of 6 is 3, the projective modules over $\mathbf{k}(S_6)$ are determined by restriction to 3-dimensional vector spaces, and we were indeed able to obtain the information we needed about $\tilde{\phi}$ from calculations in 3 variables. An explicit formula for $\tilde{\phi}$ would involve 6 variables and looks extremely complicated.

6. The Steenrod module $P^{431}(V)$

In this section $\mathbf{k} = \mathbb{F}_2$.

Since the longest 2-regular partition of 8 is 3, projective $\mathbf{k}(S_8)$-modules are determined by their characteristic series in 3 variables. Therefore, as before let $W = \langle x, y, z\rangle$. Using the known decomposition matrix for $n = 8$, we compute

$$\begin{aligned}
\mathrm{ch}\left(P^{8}(W)\right) &= x^8 + 2x^7y + 2x^6y^2 + 4x^6yz + 2x^5y^3 + 4x^5y^2z + 2x^4y^4 + 6x^4y^3z \\
&\quad + 8x^4y^2z^2 + 12x^3y^3z^2 + \text{ symmetrical terms}, \\
\mathrm{ch}\left(P^{71}(W)\right) &= x^7y + 2x^6y^2 + 4x^6yz + 3x^5y^3 + 6x^5y^2z + 4x^4y^4 + 9x^4y^3z + 10x^4y^2z^2 \\
&\quad + 12x^3y^3z^2 + \text{ symmetrical terms}, \\
\mathrm{ch}\left(P^{62}(W)\right) &= x^6y^2 + 2x^6yz + 2x^5y^3 + 4x^5y^2z + 2x^4y^4 + 6x^4y^3z + 8x^4y^2z^2 \\
&\quad + 10x^3y^3z^2 + \text{ symmetrical terms}, \\
\mathrm{ch}\left(P^{53}(W)\right) &= x^5y^3 + x^5y^2z + 2x^4y^4 + 4x^4y^3z + 8x^4y^2z^2 + 6x^3y^3z^2 \\
&\quad + \text{ symmetrical terms}, \\
\mathrm{ch}\left(P^{521}(W)\right) &= x^5y^2z + x^4y^3z + 2x^4y^2z^2 + 2x^3y^3z^2 + \text{ symmetrical terms}, \\
\mathrm{ch}\left(P^{431}(W)\right) &= x^4y^3z + 2x^4y^2z^2 + 4x^3y^3z^2 + \text{ symmetrical terms}.
\end{aligned}$$

Let P be the projective summand of $[L_3, L_3]$ found in Section 5, and let P' be the induced projective $\mathbf{k}(S_8)$-module $P_{S_6}^{S_8}$. Comparing

$$\begin{aligned}
\mathrm{ch}\left(P'(W)\right) &= \mathrm{ch}\left(P^{321}(W) \otimes W \otimes W\right) = \mathrm{ch}\left((x+y+z)^2 P^{321}(W)\right) \\
&= x^5y^2z + 3x^4y^3z + 6x^4y^2z^2 + 10x^3y^3z^2 + \text{ symmetrical terms},
\end{aligned}$$

to the polynomials for the indecomposable projectives shows that $P' \cong P^{521} + 2P^{431}$.

As before set $A = [\beta(zxx), \beta(yxy)] + [\beta(zxy), \beta(yxx)] \in [L_3(W), L_3(W)]$ and let $A' = T_{xy}A$, where T_{xy} interchanges x and y. Symmetric polynomials in $\{x, y, z\}$ of a given degree are in fact multi-graded according to the exponents of the various factors. Strictly speaking, this grading is indexed by $(\mathbb{Z}^3/\Sigma_3)$; we refer to it as grading by tridegrees. According to the characteristic polynomial, the dimensions of tridegrees 521 and 431 in $P(W) \otimes W \otimes W$ are 1 and 3 respectively. In tridegree 521 we have Axx and a basis in tridegree 431 consists of Axy, Ayx, and $A'xx$.

Write $P' = P_1 \oplus P_2 \oplus P_3$ where $P_1 \cong P^{521}$, and $P_2 \cong P_3 \cong P^{431}$. Since $P_2(W)$ and $P_3(W)$ are zero in tridegree 521, we know that $Axy + Ayx$, and $Axy + A'xx$ form a basis for $P_2 \oplus P_3$ in tridegree 431. The element $Axy + Ayx$ lies in at most one of P_2, P_3 say $Axy + Ayx \notin P_3$.

Define $r : P' \to P'$ by $r(Yuv) = Yuv + Yvu$. That is, $r = 1 \otimes \beta_2 = 1 - T_{78}$. Observe that in tridegree 431 of $P'(W)$ we have $\operatorname{Ker} r = \operatorname{Im} r = \langle Axy + Ayx\rangle$. Let ϕ be the composite $P_2 \underset{\cong}{\xrightarrow{g}} P_3 \xrightarrow{r} P' \xrightarrow{\text{projection}} P_2$. Because $Axy + Ayx \notin P_3(W)$, in tridegree 431 we have $\operatorname{Im}(r \circ g)$ is not zero so $\operatorname{Im}(r \circ g) = \langle Axy + Ayx\rangle$. However the projection of $Axy + Ayx$ to $P_2(W)$ is nonzero, since $Axy + Ayx$ is not in $P_1 \oplus P_3$, and so it follows that $\phi(W)$ is an isomorphism in tridegree 431. Therefore $\phi \neq 0$, and since P_2 is an indecomposable projective, any idempotent obtained by iterating ϕ is an isomorphism and so ϕ is an isomorphism. In particular, $r : P_3 \to r(P_3)$ is an injection and thus an isomorphism. Since $r^2 = 0$, the fact that the restriction of r to P_3 is an injection implies that $P_3 \cap r(P_3) = 0$. Also $P_1 \cap r(P_3) = 0$ since P^{521} and P^{431} are in different blocks. Therefore, replacing P_2 by the isomorphic module $r(P_3)$ we may assume that $P_2 = r(P_3)$ in the decomposition $P' \cong P_1 \oplus P_2 \oplus P_3$.

In the remainder of this section we will need to consider some modules which are not projective (or at least not initially known to be projective). Since such modules are not uniquely determined by their characteristic polynomial in 3 variables we shall sometimes need to consider more variables. Let $X = \langle x, y, z, w\rangle$ have dimension 4. Then we compute

$$\begin{aligned}
\mathrm{ch}\left(P^{521}(X)\right) &= x^5y^2z + x^4y^3z + 2x^4y^2z^2 + 2x^3y^3z^2 + 4x^4y^2zw + 4x^3y^3zw \\
&\quad + 6x^3y^2z^2w + 8x^2y^2z^2w^2 + \text{ symmetrical terms}, \\
\mathrm{ch}\left(P^{431}(X)\right) &= x^4y^3z + 2x^4y^2z^2 + 4x^3y^3z^2 + 4x^4y^2zw + 8x^3y^3zw + 14x^3y^2z^2w \\
&\quad + 24x^2y^2z^2w^2 + \text{ symmetrical terms}. \\
\mathrm{ch}\left(P^{321}(X)\right) &= x^3y^2z + 2x^2y^2z^2 + 2x^3yzw + 4x^2y^2zw + \text{ symmetrical terms}.
\end{aligned}$$

We will consider some natural $\mathbf{k}(S_8)$ submodules of $X^{\otimes 8}$. Besides being $\mathbf{k}(S_8)$-submodules they are closed under the action of $GL(X)$. In particular, they are closed under "formal Steenrod operations". By Sq_u^v we will mean a formal Steenrod operation which takes u to v and is zero on basis elements of X other than u. The induced action on $T(X)$ is a derivation which satisfies $\mathrm{Sq}_u^v \circ \mathrm{Sq}_u^v = 0$. Commutativity with the operation Sq_u^v is a way of expressing the information given by commutativity with the element of $GL(X)$ which takes u to $u + v$, and other basis elements to themselves. T_{uv} will denote the element of $GL(X)$ which switches u and v.

We will need to compute Steenrod operations in $P^{431}(X)$. Our multi-grading by exponents becomes a $(\mathbb{Z}^4/\Sigma_4)$ grading which we refer to it as grading by quaddegrees. In quaddegree 5210 of $P'(X)$ we have $r(Axx) = 0$. Since the decomposition matrix tells us that $\hat{P}^{521} \otimes \mathbb{Q} \cong \alpha^{521} + \alpha^{32111}$, the fact that r is zero in degree 5210 implies that $r(P_1)$ is either zero or is a module whose characteristic polynomial is the same as $\mathrm{ch}(\alpha^{32111})$. In either case, applied to a vector space of dimension 4 we get $r\,(P_1(X)) = 0$. Therefore, since $r\,(P_1(X)) = 0$ and $r\,(P_2(X)) = r^2\,(P_3(X)) = 0$ we have $r\left(P'(x)\right) = r\,(P_3(X)) = P_2(X) \cong P^{431}(X)$. Thus we can easily identify the elements of $P'(X)$ which lie in $P_2(X)$, and can use this to compute Steenrod operations in the abstract module $P^{431}(X)$.

To describe explicit bases for $P_2(X)$ we will need the following elements of $P(X)$ which constitute bases in the specified quaddegrees. Quaddegrees are on the left and the notation ":=" indicates a definition. A and A' are as before.

3210	A
2310	A'
3120	$A'' := T_{yz}A$
3201	$A''' := T_{zw}A$
2220	$B := \mathrm{Sq}_x^z A$
	$B' := T_{xy}B$
3111	$C := \mathrm{Sq}_y^w A$
	$C' := T_{yz}C$
2211	$D := \mathrm{Sq}_y^w A'$
	$D' := T_{zw}D$
	$E := \mathrm{Sq}_x^y C$
	$F := \mathrm{Sq}_z^w B$

Then bases of $P_2(X)$ in various quaddegrees are as follows:

4310	$A[xy]$	
4220	$A[xz]$	$A''[xy]$
3320	$B[xy]$	$B'[xy]$
	$A[yz]$	$A'[xz]$
4211	$C[xy]$	$C'[xy]$
	$A[xw]$	$A'''[xz]$
3311	$A[yw]$	$A'[xw]$
	$A'''[yz]$	$(T_{wz}A')[xz]$
	$D[xy]$	$D'[xy]$
	$E[xy]$	$F[xy]$
3221	$A[zw]$	$A''[yw]$
	$B[xw]$	$B'[xw]$
	$C[yz]$	$C'[yz]$
	$D[xz]$	$D'[xz]$
	$E[xz]$	$F[xz]$
	$(T_{yz}D)[xy]$	$(T_{yz}D')[xy]$
	$(T_{yz}E)[xy]$	$(T_{yz}F)[xy]$

Lemma. *Let $\hat{M}$ be a sub $\mathbb{Z}_2(S_8)$-lattice of $\hat{P}^{431}$. (That is, $\hat{M}$ is a submodule of $\hat{P}^{431}$ which is free as a $\mathbb{Z}_2$-module.) Let X be a 4-dimensional vector space over $\mathbf{k}$ and let M be the reduction of $\hat{M}$ modulo 2. If $M(X)$ is zero in quaddegrees 3320 and 4211 then $\hat{M} = 0$.*

Proof. Since the only partitions occurring in the decomposition of $\hat{P}^{431} \otimes \mathbb{Q}$ are α^{431}, α^{442}, α^{332}, α^{4211}, α^{3111}, and α^{3221}, we need to show that $M(X)$ is zero in quaddegrees 4310, 4220, 3320, 4211, 3311, and 3221. Given the hypotheses, to do this it suffices to show that every nonzero element of $P^{431}(X)$ has a nonzero image under some operation into one of quaddegree 3320 or 4211. The operations in the abstract module $P^{431}(X)$ can be computed from the isomorphic $P_2(X)$.

Using the formulas (computed from the definitions)

$\mathrm{Sq}^y_x A = A'$	$\mathrm{Sq}^y_x A'' = B + B'$
$\mathrm{Sq}^z_w A''' = A$	$\mathrm{Sq}^z_w T_{zw} A' = A'$
$\mathrm{Sq}^z_w D = B'$	$\mathrm{Sq}^z_w D' = B'$
$\mathrm{Sq}^z_w E = B + B'$	$\mathrm{Sq}^z_w F = 0$
$\mathrm{Sq}^x_y A' = A$	$\mathrm{Sq}^x_y T_{zw} A' = A'''$
$\mathrm{Sq}^x_y D = C$	$\mathrm{Sq}^x_y D' = C + C'$
$\mathrm{Sq}^x_y E = 0$	$\mathrm{Sq}^x_y F = C'$
$\mathrm{Sq}^y_w C = 0$	$\mathrm{Sq}^y_w C' = A$
$\mathrm{Sq}^y_w D = A'$	$\mathrm{Sq}^y_w D' = A'$
$\mathrm{Sq}^y_w E = 0$	$\mathrm{Sq}^y_w F = A'$
$\mathrm{Sq}^x_w D = A$	$\mathrm{Sq}^x_w D' = 0$
$\mathrm{Sq}^x_w E = 0$	$\mathrm{Sq}^x_w F = 0$
$\mathrm{Sq}^y_z A = 0$	$\mathrm{Sq}^y_z A'' = A$
$\mathrm{Sq}^y_z B = A'$	$\mathrm{Sq}^y_z B' = 0$
$\mathrm{Sq}^y_z C = A'''$	$\mathrm{Sq}^y_z C' = A'''$
$\mathrm{Sq}^y_z D = T_{zw} A'$	$\mathrm{Sq}^y_z D' = T_{zw} A'$
$\mathrm{Sq}^y_z E = T_{zw} A'$	$\mathrm{Sq}^y_z F = T_{zw} A'$
$\mathrm{Sq}^y_z T_{yz} D = D + D'$	$\mathrm{Sq}^y_z T_{yz} D' = 0$
$\mathrm{Sq}^y_z T_{yz} E = D' + E$	$\mathrm{Sq}^y_z T_{yz} F = D' + F$
$\mathrm{Sq}^x_z B = 0$	$\mathrm{Sq}^x_z B' = A$

(and the fact that operations landing in degrees not occurring in P^{321} are zero) we find

4310	$\mathrm{Ker}\ \mathrm{Sq}^z_x = 0$
4220	$\mathrm{Ker}\ \mathrm{Sq}^y_x = 0$
3311	$\mathrm{Ker}\ \mathrm{Sq}^z_w \cap \mathrm{Ker}\ \mathrm{Sq}^x_y \cap \mathrm{Ker}\ \mathrm{Sq}^y_w \cap \mathrm{Ker}\ \mathrm{Sq}^x_w \circ T_{xz} = 0$
3221	$\mathrm{Ker}\ \mathrm{Sq}^y_z \cap \mathrm{Ker}\ \mathrm{Sq}^y_w \cap \mathrm{Ker}\ \mathrm{Sq}^z_w \cap \mathrm{Ker}\ \mathrm{Sq}^x_w \cap \mathrm{Ker}\ \mathrm{Sq}^x_z = 0$

In each row, the images of the specified operations are (up to symmetry) in quaddegree 3320, 4211, or one of the previous rows. Therefore every nonzero element of $P^{431}(X)$ has a nonzero image under some (composite) operation into one of quaddegree 3320 or 4421. $\square$

Corollary. *Let $\hat{f} : \widehat{P^{431}} \to \hat{N}$ be a map of $\mathbb{Z}_2$-modules and let $f_X : P^{431}(X) \to N(X)$ be the induced map between the mod 2 reductions. If $\mathrm{Im} f_X$ has dimension 4 in quaddegrees 3320 and 4421, then $\mathrm{Im}\,\hat{f}$ is isomorphic to P^{431}.*

Proof. Apply the preceding Lemma to $\mathrm{Ker}\,\hat{f}$. $\square$

7. $\mathrm{Lie}^{\max}(8)$

Once again, $\mathbf{k} = \mathbb{F}_2$ in this section.

Since $\mathrm{Lie}'(2) = 0$ and $\mathrm{Lie}'(4) = 0$ for dimensional reasons, (their dimensions are less than the size of the corresponding Sylow 2-subgroups,) the tensor decomposition gives $\mathrm{Lie}'(8) = \mathrm{Lie}^{\max}(8)$. The map $x \otimes y \mapsto [x, y]$ gives an isomorphism from the projective/injective module $L_3 \otimes L_5$ to the submodule $[L_3, L_5]$ of $\mathrm{Lie}(8)$. Therefore $[L_3, L_5]$ is a summand of $\mathrm{Lie}^{\max}(8)$.

Since

$$\mathrm{ch}\,([L_3, L_5]) = \mathrm{ch}\,([L_3 \otimes L_5]) = \left(\frac{\psi_1^3 - \psi_3}{3}\right)\left(\frac{\psi_1^5 - \psi_5}{5}\right),$$

expanding gives

$$\begin{aligned}\mathrm{ch}\,([L_3(W), L_5(W)]) &= x^6y^2 + 2x^6yz + 3x^5y^3 + 9x^5y^2z + 4x^4y^4 + 17x^4y^3z \\ &\quad + 26x^4y^2z^2 + 36x^3y^3z^2 + \text{ symmetrical terms,}\end{aligned}$$

and we conclude that $[L_3, L_5] \cong P^{62} + P^{53} + 4P^{521} + 3P^{431}$.

Using $\mathrm{ch}\,(\mathrm{Lie}(8)) = (\psi_1^8 - \psi_2^4)/8$ we can calculate that in characteristic 0,

$$(\mathrm{Lie}(8)/[L_3, L_5]) \otimes \mathbb{Q} \cong \alpha^{71} + \alpha^{62} + 2\alpha^{611} + 2\alpha^{53} + 4\alpha^{521} + 2\alpha^{5111} + 4\alpha^{431} + 2\alpha^{422} + 6\alpha^{4211} + 2\alpha^{41111} + 3\alpha^{332} + 2\alpha^{3311} + 4\alpha^{3221} + 4\alpha^{32111} + 2\alpha^{311111} + 2\alpha^{22211} + \alpha^{221111} + \alpha^{2111111}.$$

Comparing this with the summands of $\hat{P} \otimes \mathbb{Q}_p$ given by the decomposition matrix for each of the various indecomposable projectives, gives the upper bound

$$\mathrm{Lie}^{\max}(8) \cong [L_3, L_5] + aP^{62} + bP^{521} + cP^{431}$$

where $0 \le a \le 1$, $0 \le b \le 4$, $0 \le c \le 2$, and $a + c \le 2$.

To find b we use:

Lemma. *Let X be a $\mathbb{Z}_2(S_8)$-module such that $\tilde{P}^{521} \otimes \mathbb{Q}$ is a summand of $X \otimes \mathbb{Q}$. Suppose that X restricted to $\mathbb{Z}_2(S_7)$ is projective. Then P^{521} is a summand of $X \otimes \mathbf{k}$.*

Proof. This follows from the fact that from the known decomposition matrices for S_8 and S_7 we can work out that the indecomposable projective $\mathbf{k}$-module P^{521} remains indecomposable when thought of as a $\mathbf{k}(S_7)$-module. (Its restriction is P^{52}.) $\square$

Since Lie(8) restricted to an S_7 module becomes the free module $\mathbf{k}(S_7)$, the Lemma plus induction gives $b = 4$.

Next we apply the same method used in Section 5 to find a. By direct computation we check that $([\beta_2, \beta_6] \circ T_{2,3,5})([\beta(yx), \beta(yxxxxx)]) = [\beta(yx), \beta(yxxxxx)]$. This implies that $\mathrm{Lie}(8)/[L_3, L_5]$ has a projective summand P such that $P(V) \neq 0$ for a 2-dimensional vector space V. Among the possible summands of $\mathrm{Lie}(8)/[L_3, L_5]$, only P^{62} has this property, so $a = 1$. Therefore $c \le 1$.

To finish the computation, we show that P^{431} is a summand of $\mathrm{Lie}(8)/[L_3, L_5]$.

Let $\mathrm{ad} : \mathrm{Lie}(k-1) \to \mathrm{Lie}(k)$ be the adjoint map. That is, ad is given on γ by $\mathrm{ad}([[v_1, \ldots, v_{k-1}]v_k) = [[v_1, \ldots, v_k]$. Equivalently, if we let $\sigma_k = (1\,2\,3\,\cdots\,k)$ in $\mathbf{k}(S_8)$ then $\mathrm{ad} = 1 - \sigma_k$. Let Ad denote the composition

$$[L_3(V), L_2(V)] \otimes V \xrightarrow{\mathrm{ad}} [L_3(V), L_3(V)] + [L_4(V), L_2(V)] \twoheadrightarrow [L_3(V), L_3(V)].$$

Observe that $\mathrm{Ad} \circ [\beta_3, \beta_2] = [\beta_3, \beta_3]$.

Let f be the composite

$$[L_3(V), L_3(V)] \hookrightarrow V^{\otimes 6} \xrightarrow{T_{24}} V^{\otimes 6} \xrightarrow{[\beta_3, \beta_2]} [L_3(V), L_2(V)] \otimes V.$$

Let ϕ, A, and P be as in Section 5. Then $\mathrm{Ad} \circ f(A) = \phi(A) = A$ and so $f(P)$ is a projective summand of $[L_3, L_2]L_1$ which is isomorphic to $P \cong P^{321}$. Set $\tilde{P}$ equal to $f(P)$. We write $\tilde{P'} = \tilde{P}L_1L_1$ for the $\mathbf{k}(S_8)$-module induced by $\tilde{P}$, and we write simply f for the isomorphism $P' \cong \tilde{P}'$ induced by $f : P \cong \tilde{P}$. Elements of P will be denoted by the letters introduced in Section 6, and their images under the isomorphism f will be written with a tilde over the same letter. We also write $\tilde{P}_2$ for $f(P_2)$.

Direct calculation shows that

$$\tilde{A} = [\beta(zyx), \beta(yx)]x + [\beta(zxx), \beta(yx)]y + [\beta(yxx), \beta(yx)]z.$$

Applying the operations gives formulas for the images of the other elements described in Section 6. Explicitly, we find

$$\tilde{C} = [\beta(zwx), \beta(yx)]x + [\beta(zyx), \beta(wx)]x + [\beta(zxx), \beta(wx)]y + [\beta(zxx), \beta(yx)]w + [\beta(wxx), \beta(yx)]z + [\beta(yxx), \beta(wx)]z$$

$$\tilde{C}' = [\beta(zyx), \beta(wx)]x + [\beta(wyx), \beta(zx)]x + [\beta(zxx), \beta(wx)]y + [\beta(wxx), \beta(zx)]y + [\beta(yxx), \beta(wx)]z + [\beta(yxx), \beta(zx)]w$$

$$\tilde{A}''' = [\beta(zyx), \beta(zx)]x + [\beta(zxx), \beta(zx)]y + [\beta(yxx), \beta(zx)]z$$

$$\tilde{B} = [\beta(zyx), \beta(zy)]x + [\beta(zyx), \beta(zy)]x + [\beta(zxz), \beta(yx)]y + [\beta(zxx), \beta(zy)]y + [\beta(yxx), \beta(zy)]z + [\beta(yxz), \beta(yx)]z$$

$$\tilde{B'} = [\beta(zyz), \beta(yx)]x + [\beta(zyy), \beta(zx)]x + [\beta(zxy), \beta(zx)]y + [\beta(zxz), \beta(yx)]y + [\beta(yxz), \beta(yx)]z + [\beta(yxy), \beta(zx)]z.$$

Let g denote the composite

$$[L_3(V), L_2]V \otimes V \otimes V \xrightarrow{[L_3(V), L_2(V)] \otimes \beta_2 \otimes V} [L_3(V), L_2(V)] \otimes L_2(V) \otimes V \xrightarrow{\mathrm{ad} \otimes V} [[L_3(V), L_2(V)], L_2(V)] \otimes V \xrightarrow{\mathrm{ad}} [[L_4(V), L_2(V)], L_2(V)] + [L_5, L_3] \twoheadrightarrow [[L_4(V), L_2(V)], L_2(V)].$$

Notice that $[[L_4(V), L_2(V)], L_2(V)] \cap [L_5, L_3] = 0$.

Calculating in quaddegree 3320 from the definition of g gives

$$\begin{aligned}
g(\tilde{B}[xy]) = {} & [[\beta(yxxx), \beta(zy), \beta(zy)] + [[\beta(yxxy), \beta(zx), \beta(zy)] + [[\beta(yxyz), \beta(yx), \beta(zx)] \\
& + [[\beta(yxzz), \beta(yx), \beta(yx)] + [[\beta(zx), \beta(yx), \beta(yx), \beta(zy)] \\
& + [[\beta(zy), \beta(yx), \beta(yx), \beta(zx)] + [[\beta(zy), \beta(yx), \beta(yxxy)]
\end{aligned}$$

$$\begin{aligned}
g(\tilde{B}'[xy]) = {} & [[\beta(yxxy), \beta(zx), \beta(zy)] + [[\beta(yxxz), \beta(yx), \beta(zy)] + [[\beta(yxyy), \beta(zx), \beta(zx)] \\
& + [[\beta(yxzz), \beta(yx), \beta(yx)] + [[\beta(zx), \beta(yx), \beta(yx), \beta(zy)] \\
& + [[\beta(zy), \beta(yx), \beta(yx), \beta(zx)] + [[\beta(zx), \beta(yx), \beta(yxyz)]
\end{aligned}$$

$$\begin{aligned}
g(\tilde{A}'[xz]) = {} & [[\beta(zxxy), \beta(yx), \beta(zy)] + [[\beta(zxyz), \beta(yx), \beta(yx)] + [[\beta(zxyy), \beta(yx), \beta(zx)] \\
& + [[\beta(zy), \beta(yx), \beta(zxxy)]
\end{aligned}$$

$$\begin{aligned}
g(\tilde{A}[yz]) = {} & [[\beta(yxxz), \beta(yx), \beta(zy)] + [[\beta(yxyz), \beta(yx), \beta(zx)] + [[\beta(yxzz), \beta(yx), \beta(yx)] \\
& + [[\beta(zxxy), \beta(yx), \beta(zy)] + [[\beta(zxyy), \beta(yx), \beta(zx)] \\
& + [[\beta(zxyz), \beta(yx), \beta(yx)] + [[\beta(zy), \beta(yx), \beta(yx), \beta(zx)]
\end{aligned}$$

which are linearly independent. (Note that the right-hand sides are written in a Hall basis.) Similarly in quaddegree 4211 we find

$$\begin{aligned}
g(\tilde{C}[xy]) = {} & [[\beta(zxxw), \beta(yx), \beta(yx)] + [[\beta(wxxz), \beta(yx), \beta(yx)] + [[\beta(yxxz), \beta(yx), \beta(wx)] \\
& + [[\beta(zxxx), \beta(yx), \beta(wy)] + [[\beta(zxxy), \beta(yx), \beta(wx)] + [[\beta(wxxx), \beta(yx), \beta(zy)] \\
& + [[\beta(wxxy), \beta(yx), \beta(zx)] + [[\beta(yxxx), \beta(zy), \beta(wx)] + [[\beta(yxxy), \beta(zx), \beta(wx)] \\
& + [[\beta(wx), \beta(zy), \beta(yxxx)] + [[\beta(wx), \beta(zx), \beta(yxxy)] + [[\beta(wx), \beta(yx), \beta(yxxz)]
\end{aligned}$$

$$\begin{aligned}
g(\tilde{C}'[xy]) = {} & [[\beta(yxxx), \beta(zy), \beta(wx)] + [[\beta(yxxx), \beta(zx), \beta(wy)] + [[\beta(zxxx), \beta(yx), \beta(wy)] \\
& + [[\beta(zxxy), \beta(yx), \beta(wx)] + [[\beta(wxxy), \beta(yx), \beta(zx)] + [[\beta(zx), \beta(yx), \beta(wxxy)] \\
& + [[\beta(wx), \beta(zx), \beta(yxxy)] + [[\beta(zx), \beta(yx), \beta(yx), \beta(wx)] \\
& + [[\beta(wx), \beta(yx), \beta(yx), \beta(zx)] + [[\beta(wx), \beta(yx), \beta(zxxy)]
\end{aligned}$$

$$\begin{aligned}
g(\tilde{A}[xw]) = {} & [[\beta(zxxy), \beta(yx), \beta(wx)] + [[\beta(yxxz), \beta(yx), \beta(wx)] + [[\beta(zxxx), \beta(yx), \beta(wy)] \\
& + [[\beta(zxxw), \beta(yx), \beta(yx)] + [[\beta(yxxx), \beta(yx), \beta(wz)] + [[\beta(yxxw), \beta(yx), \beta(zx)]
\end{aligned}$$

$$\begin{aligned}
g(\tilde{A'''}[xz]) = {} & [[\beta(wxxy), \beta(yx), \beta(zx)] + [[\beta(yxxw), \beta(yx), \beta(zx)] + [[\beta(wxxx), \beta(yx), \beta(zy)] \\
& + [[\beta(wxxz), \beta(yx), \beta(yx)] + [[\beta(yxxx), \beta(yx), \beta(wz)] + [[\beta(yxxz), \beta(yx), \beta(wx)]
\end{aligned}$$

which are also linearly independent. Therefore, according to the Corollary of Section 6, the submodule $[[L_4, L_2], L_2]$ of $\mathrm{Lie}(8)/[L_3, L_5]$ contains a submodule isomorphic to P^{431}, and so $c = 1$.

Thus we have shown that

$$\mathrm{Lie}^{\max}(8) \cong [L_3, L_5] + P^{62} + 4P^{521} + P^{431} \cong 2P^{62} + P^{53} + 8P^{521} + 4P^{431}.$$

From the decomposition matrix, we know that $|P^{62}| = 640$, $|P^{53}| = 384$, $|P^{521}| = 128$, and $|P^{431}| = 384$ so its dimension is 4224.

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