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Implementing conditional term rewriting by graph rewriting

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Abstract

For reasons of efficiency, term rewriting is usually implemented by graph rewriting. Barendregt et al. showed that graph rewriting is a sound and complete implementation of (almost) orthogonal term rewriting systems. Their result was strengthened by Kennaway et al. who showed that graph rewriting is adequate for simulating term rewriting. In this paper, we extend the aforementioned results to a class of conditional term rewriting systems which plays a key role in the integration of functional and logic programming. In these systems extra variables are allowed in conditions and right-hand sides of rules. Moreover, it is shown that orthogonal conditional rules give rise to a subcommutative conditional graph rewrite relation. This implies that the graph rewrite relation is level-confluent. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Attempts to combine the functional and logic programming paradigms have recently been receiving increasing attention; see [8] for an overview of the field. It has been argued in [9] that *strict equality* is the only sensible notion of equality for possibly non-terminating programs. In this paper, we adopt this point of view; so every functional-logic program is regarded as an (almost) orthogonal conditional term rewriting system (CTRS) with strict equality. The standard operational semantics for functional-logic programming is conditional narrowing. It is well-known that extra variables in conditions (not to mention right-hand sides) cause problems because narrowing may become incomplete or confluence may be lost. Therefore, many efforts have been made to characterize classes of confluent functional-logic programs with extra variables for which narrowing is complete; see [9] for details. In [9], new interesting completeness results are provided. However, all of these results are standing on shaky ground. This is because all of them depend on the fact that conditional graph rewriting is a sound and complete implementation (w.r.t. the computation of normal forms) of CTRSs with

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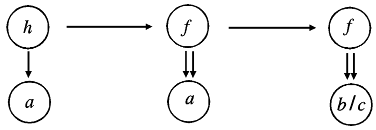
strict equality ([9, Theorem 3.5, p. 676]: "Conditions 1 and 2 are necessary to extend Theorem 3.5 ..."). Informally, soundness ensures that the graph implementation of a CTRS cannot give incorrect results and completeness means that graph rewriting gives all results; see [4]. But the proof of the above-mentioned fact (given in [7, Theorem 3.8]) is incorrect. There is the following counterexample:

$$\mathscr{R} = \begin{cases} a \to x \Leftarrow g(x) == e \\ g(b) \to e, \qquad g(c) \to e \\ h(x) \to f(x,x), \qquad f(b,c) \to d \end{cases}$$

Since $a \to_{\mathscr{R}} b$ and $a \to_{\mathscr{R}} c$ it follows that $h(a) \to_{\mathscr{R}} f(a,a) \to_{\mathscr{R}}^{+} f(b,c) \to_{\mathscr{R}} d$. The term d is the result of this term rewriting computation because it is irreducible. In the corresponding graph rewrite system, however, $h(a)$ does not reduce to d because the two occurrences of a are shared in $f(a,a)$:



So the example shows that conditional graph rewriting is in general not a complete implementation of orthogonal CTRSs with strict equality. However, the example is somehow pathological because the CTRS is not confluent; the term a can be reduced to two different normal forms.

The objective of this paper is to prove that conditional graph rewriting is a sound and complete implementation of a subclass of orthogonal CTRSs with strict equality which we call *almost functional* CTRSs. As a matter of fact, we will prove a stronger statement, viz. that graph rewriting is an *adequate* implementation of almost functional CTRSs (the notion "adequacy" originates from [10]). Moreover, it will be shown that every almost functional CTRS is level-confluent and that its graph implementation benefits from the same property. The former is not a new result but rather a special case of a theorem in [19]. Our proof, however, is simpler than that in [19].

Aside from the reasons already mentioned, our new results are interesting on their own, simply because term rewriting is usually implemented by term graph rewriting (the name "term graph rewriting" was coined by Barendregt et al. [4]). Term graph rewriting is more efficient than term rewriting because the representation of expressions as directed acyclic graphs allows a sharing of common subexpressions and a graph rewrite step corresponds thus to several term rewrite steps. Details on acyclic term graph rewriting can be found in the overview article [16] which was recently published. In this paper, however, we neither follow the approach of [4] nor that of [16]. Instead, we use the term-based model of [12] in which directed acyclic graphs are modeled by well-marked terms.

A preliminary version of this paper appeared in [14]. In this revised paper, the results reported in the preliminary version are improved in several respects: In [14], every rule $l \to r \Leftarrow s_1 == t_1, \ldots, s_k == t_k$ in a CTRS $\mathscr{R}$ had to satisfy:

1. $\mathscr{V}ar(s_i) \subseteq \mathscr{V}ar(l) \cup \bigcup_{j=1}^{i-1} \mathscr{V}ar(t_j)$ for all $1 \leqslant i \leqslant k$.
2. Every t_j, $1 \leqslant j \leqslant k$, is a *linear* constructor term.

In this paper, it is only necessary that

1. $\mathscr{V}ar(s_i) \subseteq \mathscr{V}ar(l) \cup \bigcup_{j=1}^{i-1} \mathscr{V}ar(t_j)$ holds if $\mathscr{V}ar(r) \not\subseteq \mathscr{V}ar(l)$.
2. Every t_j, $1 \leqslant j \leqslant k$, is a constructor term.

Moreover, in [14] it is only shown that conditional term graph rewriting is a sound and complete implementation of (almost) functional CTRSs. By means of a new proof idea, it is shown here that conditional term graph rewriting is even an adequate implementation of that class of CTRSs. The fact that orthogonality can be weakened to "almost orthogonality" is also new.

The paper is organized as follows. In the next section, we recapitulate the basics of conditional term rewriting. In Section 3, almost functional CTRSs are introduced. Unfortunately, almost functional CTRSs do not satisfy the parallel moves lemma. In order to overcome this obstacle, we define a closely related "deterministic" reduction relation in which extra variables are instantiated by ground constructor terms only. We obtain as a consequence that almost functional CTRSs are level-confluent. Section 4 is dedicated to graph rewriting. Its main result states that graph rewriting is an adequate implementation of almost functional CTRSs. The proof is based on the fact that the deterministic reduction relation satisfies the parallel moves lemma. Finally, we will show that the graph rewrite relation associated with an almost functional CTRS is level-confluent.

2. Preliminaries

The reader is assumed to be familiar with the basic concepts of term rewriting which can for instance be found in [3, 11, 6]. Here, we will just recall less common definitions and some basic facts concerning conditional term rewriting.

Let $(\mathscr{F}, \mathscr{R})$ be a term rewriting system (TRS for short). A function symbol $f \in \mathscr{F}$ is called a *defined symbol* if there is a rewrite rule $l \to r \in \mathscr{R}$ such that $l = f(t_1, \ldots, t_k)$ for some terms $t_1, \ldots, t_k$, otherwise it is called a *constructor*. The set of defined symbols is denoted by $\mathscr{D}$ while $\mathscr{C}$ stands for the set of constructors. A *constructor term* is a term consisting of constructors and variables only. A non-overlapping left-linear TRS is called *orthogonal*. In an *almost orthogonal* TRS, the non-overlapping restriction is a bit more relaxed in the sense that it allows trivial overlays.

In a CTRS $(\mathscr{F}, \mathscr{R})$, rules have the form $l \to r \Leftarrow s_1 = t_1, \ldots, s_k = t_k$ with $l, r, s_1, \ldots, s_k, t_1, \ldots, t_k \in \mathscr{T}(\mathscr{F}, \mathscr{V})$. It is required that l is not a variable. We frequently abbreviate the conditional part of the rule, i.e. the sequence $s_1 = t_1, \ldots, s_k = t_k$, by c. If a rewrite rule has no conditions, we write $l \to r$, demand that $\mathscr{V}ar(r) \subseteq \mathscr{V}ar(l)$, and call $l \to r$ an unconditional rule. As in [13], rewrite rules $l \to r \Leftarrow c$ will be classified according to the distribution of variables among l, r, and c, as shown in Table 1. An n-CTRS contains only rewrite rules of type n. For every rule $l \to r \Leftarrow c$, we define the set of

Table 1. Classification of CTRSs

Type	Requirement
1	$\mathcal{V}ar(r) \cup \mathcal{V}ar(c) \subseteq \mathcal{V}ar(l)$
2	$\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$
3	$\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l) \cup \mathcal{V}ar(c)$
4	No restrictions

extra variables by

$$\mathcal{EV}ar(l \to r \Leftarrow c) = (\mathcal{V}ar(r) \cup \mathcal{V}ar(c)) \backslash \mathcal{V}ar(l).$$

Thus a 1-CTRS has no extra variables, a 2-CTRS has no extra variables on the right-hand sides of rules, and a 3-CTRS may contain extra variables on the right-hand sides of rules provided that these also occur in the conditions.

The $=$ symbol in the conditions can be interpreted in different ways which lead to different rewrite relations associated with $\mathcal{R}$. In this paper, we are interested in CTRSs with strict equality. As already mentioned, these systems play a fundamental role in functional-logic programming.

Definition 2.1. In a 3-CTRS $(\mathcal{F}, \mathcal{R})$ *with strict equality* the $=$ symbol in the conditions of the rewrite rules is interpreted as follows: the instantiated terms in the conditions are reducible to a common ground constructor term in $\mathcal{R}$. Formally, the rewrite relation $\to_{\mathcal{R}}$ associated with $(\mathcal{F}, \mathcal{R})$ is defined by $\to_{\mathcal{R}} = \bigcup_{n \in \mathbb{N}} \to_{\mathcal{R}_n}$, where$\to_{\mathcal{R}_0} = \emptyset$ and for $n > 0$ the relation $\to_{\mathcal{R}_n}$ is defined by: $s \to_{\mathcal{R}_n} t$ if there exists a rewrite rule $\rho : l \to r \Leftarrow s_1 = t_1, \ldots, s_k = t_k$ in $\mathcal{R}$, a substitution $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ with $\mathcal{D}(\sigma) = \mathcal{V}ar(\rho)$, a context $C[\ ]$, and ground constructor terms $u_1, \ldots, u_k$ such that $s = C[l\sigma]$, $t = C[r\sigma]$, $s_i\sigma \to^*_{\mathcal{R}_{n-1}} u_i$ and $t_i\sigma \to^*_{\mathcal{R}_{n-1}} u_i$ for all $1 \leqslant i \leqslant k$.

The *depth* of a rewrite step $s \to_{\mathcal{R}} t$ is the minimum n with $s \to_{\mathcal{R}_n} t$. A CTRS $\mathcal{R}$ is called *level-confluent* if every relation $\to_{\mathcal{R}_n}$ is confluent.

The unconditional TRS obtained from a CTRS $\mathcal{R}$ by omitting the conditions in its rewrite rules is denoted by $\mathcal{R}_u$. Note that $(\mathcal{F}, \mathcal{R}_u)$ is an unconditional TRS in the usual sense provided that $(\mathcal{F}, \mathcal{R})$ is a 2-CTRS. This is not true for 3-CTRSs because rules of type 3 may contain variables on the right-hand sides of rules which do not occur on the corresponding left-hand side. For a CTRS $\mathcal{R}$, notions like left-linearity, orthogonality, and constructor term are defined via the system $\mathcal{R}_u$. So a CTRS $\mathcal{R}$ is for instance called orthogonal if $\mathcal{R}_u$ is orthogonal. Since the properties mentioned above solely depend on the left-hand sides of the system $\mathcal{R}_u$, they are well-defined even if $\mathcal{R}_u$ is not a TRS in the usual sense. In contrast to orthogonality, almost orthogonality does depend on the right-hand sides of rules and we have to explain this property for 3-CTRSs. So suppose $\mathcal{R}$ is a 3-CTRS. If the critical pair $\langle s, t \rangle$ originates from two rules $l_1 \to r_1$ and $l_2 \to r_2$ (renamed such that they have no variables in common) in $\mathcal{R}_u$, then it is trivial if $s = t$. Now if one of the rules, say $l_1 \to r_1$, has an extra variable

z on its right-hand side (so $z \in \mathscr{V}ar(r_1)\setminus\mathscr{V}ar(l_1)$), then the critical pair $\langle s,t\rangle$ cannot be trivial because z occurs in t but not in s. In other words, in an almost orthogonal 3-CTRS none of the rules of type 3 may overlap another rule.

If the equality signs $=$ in the conditions are interpreted as reachability ($\rightarrow_{\mathscr{R}}^*$), then we obtain an *oriented* CTRS. A *normal* CTRS $\mathscr{R}$ is an oriented CTRS whose rules $l \rightarrow r \Leftarrow s_1 \rightarrow t_1,\dots,s_k \rightarrow t_k$ are subject to the additional constraint that every t_j is a ground normal form with respect to $\mathscr{R}_u$. According to the next proposition, every CTRS with strict equality can be viewed as a normal CTRS.

Proposition 2.2. *Let $(\mathscr{F},\mathscr{R})$ be a CTRS with strict equality. The following statements are equivalent for all terms s and t in $\mathscr{T}(\mathscr{F},\mathscr{V})$:*
(i) *s and t can be reduced to a common ground constructor term in $(\mathscr{F},\mathscr{R})$,*
(ii) *$s{=}{=}t$ can be evaluated to true in the normal CTRS $(\mathscr{F} \uplus \mathscr{F}_{\text{eq}}, \mathscr{R}' \uplus \mathscr{R}_{\text{eq}})$, where $\mathscr{F}_{\text{eq}}$, $\mathscr{R}_{\text{eq}}$, and $\mathscr{R}'$ are defined as follows:*

1. *$\mathscr{F}_{\text{eq}} = \{==, \wedge, true, false\}$, and $\wedge$ is assumed to be right-associative.*
2. *The TRS $\mathscr{R}_{\text{eq}}$ consists of the rules ($\mathscr{C}$ is the set of constructors in $\mathscr{R}$)*

$$\begin{aligned} c == c &\rightarrow true && \forall 0\text{-}ary\ c \in \mathscr{C}\\ c == d &\rightarrow false && \forall c,d \in \mathscr{C},\ c \neq d\\ c(x_1,\dots,x_n) == c(y_1,\dots,y_n) &\rightarrow \bigwedge_{i=1}^{n}(x_i == y_i) && \forall n\text{-}ary\ c \in \mathscr{C}\\ c(x_1,\dots,x_n) == d(y_1,\dots,y_m) &\rightarrow false && \forall c,d \in \mathscr{C},\ c \neq d\\ true \wedge x &\rightarrow x\\ false \wedge x &\rightarrow false \end{aligned}$$

3. *and $\mathscr{R}' = \{l \rightarrow r \Leftarrow s_1{=}{=}t_1 \rightarrow true,\dots,s_k{=}{=}t_k \rightarrow true \mid l \rightarrow r \Leftarrow s_1 = t_1,\dots,s_k = t_k \in \mathscr{R}\}$*

Proof. Similar to the proof for unconditional TRSs; see [1]. □

From now on rewrite rules of a CTRS with strict equality will be written as $l \rightarrow r \Leftarrow s_1{=}{=}t_1,\dots,s_k{=}{=}t_k$.

In the following, we need a result obtained by Staples [17]. An abstract reduction system (ARS) $\mathscr{A}_2 = (A,\rightarrow_2)$ is called a *refinement* of another ARS $\mathscr{A}_1 = (A,\rightarrow_1)$ if $\rightarrow_1 \subseteq \rightarrow_2^*$. Such a refinement is called *compatible* if for all $a \rightarrow_2^* b$, there is a $c \in A$ such that $a \rightarrow_1^* c$ and $b \rightarrow_1^* c$. Staples' result states that a compatible refinement $\mathscr{A}_2$ of $\mathscr{A}_1$ is confluent if and only if $\mathscr{A}_1$ is confluent. In fact, we also need the following generalization of this result, a proof of which can be found in [15]. Let $\mathscr{A}_1 = (A,\rightarrow_1)$ and $\mathscr{A}_2 = (A,\rightarrow_2)$ be ARSs. Let $\sim$ be an equivalence relation on A such that $\rightarrow_1 \subseteq \rightarrow_2^*$ and, for all $a \rightarrow_2^* b$, there are $c,d \in A$ such that $a \rightarrow_1^* c$, $b \rightarrow_1^* d$, and $c \sim d$. Let $i \in \{1,2\}$. If $\mathscr{A}_i$ is confluent modulo $\sim$ (i.e., for all $c\ {}_i^*\!\!\leftarrow a \sim b \rightarrow_i^* d$, there are $e,f \in A$ such that $c \rightarrow_i^* e \sim f\ {}_i^*\!\!\leftarrow d$) and, for all $a \sim b \rightarrow_{3-i}^* c$, there is a $d \in A$ such that $a \rightarrow_{3-i}^* d \sim c$, then $\mathscr{A}_{3-i}$ is confluent modulo $\sim$.

3. Almost functional CTRSs

We start with a result from Bergstra and Klop [5]. They have shown that orthogonal normal 2-CTRS satisfy the parallel moves lemma. Hence these systems are level-confluent.

Definition 3.1. Let $A: s \rightarrow_{p, l \rightarrow r \Leftarrow c} t$ be a rewrite step in a CTRS $\mathscr{R}$ and let $q \in \mathscr{P}os(s)$. The set $q \backslash A$ of *descendants* of q in t is defined by

$$q \backslash A = \begin{cases} \{q\} & \text{if} q < p \text{ or } q \parallel p, \\ \{p \cdot p_3 \cdot p_2 \mid r_{|p_3} = l_{|p_1}\} & \text{if } q = p \cdot p_1 \cdot p_2 \text{ with } p_1 \in \mathscr{V}\mathscr{P}os(l) \\ \emptyset & \text{otherwise.} \end{cases}$$

If $Q \subseteq \mathscr{P}os(s)$, then $Q \backslash A$ denotes the set $\bigcup_{q \in Q} q \backslash A$. The notion of descendant is extended to rewrite sequences in the obvious way.

Definition 3.2. Let $\mathscr{R}$ be a CTRS. We write $s \mathrel{+\!\!\!\!\!\!\;\rightarrow}_{\mathscr{R}_n} t$ if t can be obtained from s by contracting a set of pairwise disjoint redexes in s by $\rightarrow_{\mathscr{R}_n}$. We write $s \mathrel{+\!\!\!\!\!\!\;\rightarrow} t$ if $s \mathrel{+\!\!\!\!\!\!\;\rightarrow}_{\mathscr{R}_n} t$ for some $n \in \mathbb{N}$. The minimum such n is called the *depth* of $s \mathrel{+\!\!\!\!\!\!\;\rightarrow} t$. The relation $\mathrel{+\!\!\!\!\!\!\;\rightarrow}$ is called *parallel rewriting*.

The parallel moves lemma for orthogonal normal 2-CTRS now reads as follows.

Lemma 3.3. *If $t \mathrel{+\!\!\!\!\!\!\;\rightarrow}_{\mathscr{R}_m} t_1$ and $t \mathrel{+\!\!\!\!\!\!\;\rightarrow}_{\mathscr{R}_n} t_2$, then there is a term t_3 such that $t_1 \mathrel{+\!\!\!\!\!\!\;\rightarrow}_{\mathscr{R}_n} t_3$ and $t_2 \mathrel{+\!\!\!\!\!\!\;\rightarrow}_{\mathscr{R}_m} t_3$. Moreover, the redexes contracted in $t_1 \mathrel{+\!\!\!\!\!\!\;\rightarrow}_{\mathscr{R}_n} t_3$ ($t_2 \mathrel{+\!\!\!\!\!\!\;\rightarrow}_{\mathscr{R}_m} t_3$) are the descendants in t_1 (t_2) of the redexes contracted in $t \mathrel{+\!\!\!\!\!\!\;\rightarrow}_{\mathscr{R}_n} t_2$ ($t \mathrel{+\!\!\!\!\!\!\;\rightarrow}_{\mathscr{R}_m} t_1$).*

It is an immediate corollary to Lemma 3.3 that every orthogonal normal 2-CTRS is level-confluent. It is our next goal to show that almost functional 3-CTRS have the same property. It should be pointed out that oriented systems which satisfy condition (1) in Definition 3.4 were called *properly oriented* in [19].

Definition 3.4. A 3-CTRS $\mathscr{R}$ with strict equality is called *almost functional* if it is orthogonal and every rule $l \rightarrow r \Leftarrow s_1 == t_1, \ldots, s_k == t_k$ in $\mathscr{R}$ satisfies:
1. If $\mathscr{V}ar(r) \not\subseteq \mathscr{V}ar(l)$, then $\mathscr{V}ar(s_i) \subseteq \mathscr{V}ar(l) \cup \bigcup_{j=1}^{i-1} \mathscr{V}ar(t_j)$ for all $1 \leqslant i \leqslant k$.
2. Every t_j, $1 \leqslant j \leqslant k$, is a constructor term.

As an example consider the almost functional CTRS $\mathscr{R}_{fib}$ which computes the Fibonacci numbers.

$$\mathscr{R}_{fib} = \begin{cases} 0 + x & \rightarrow x \\ s(x) + y & \rightarrow s(x + y) \\ fib(0) & \rightarrow \langle 0, s(0) \rangle \\ fib(s(x)) & \rightarrow \langle z, y + z \rangle \Leftarrow fib(x) == \langle y, z \rangle \end{cases}$$

Almost functional CTRSs do not satisfy the parallel moves lemma as seen by the following variant of Example 4.4 in [19].

Example 3.5. Let

$$\mathscr{R} = \begin{cases} f(x) \to y \Leftarrow x == y \\ a \to b \\ b \to c \end{cases}$$

Then $f(a) \mathrel{+\!\!\!\!\!\!\not\!\!\!\!\!\to}_{\mathscr{R}_2} a$ and $f(a) \mathrel{+\!\!\!\!\!\!\not\!\!\!\!\!\to}_{\mathscr{R}_2} c$ but not $a \mathrel{+\!\!\!\!\!\!\not\!\!\!\!\!\to}_{\mathscr{R}_2} c$.

Next, we will introduce a special "deterministic" rewrite relation $\to_{\mathscr{R}^d}$ which is closely related to $\to_{\mathscr{R}}$ (the only difference is that in $\to_{\mathscr{R}^d}$ extra variables on right-hand sides must be instantiated by ground constructors terms). In the rest of the paper, $\mathscr{R}$ denotes an *almost functional* CTRS unless stated otherwise.

Definition 3.6. Let $\to_{\mathscr{R}_0^d} = \emptyset$ and for $n > 0$ define $s \to_{\mathscr{R}_n^d} t$ if there exists a rewrite rule $\rho : l \to r \Leftarrow s_1 == t_1, \ldots, s_k == t_k$ in $\mathscr{R}$, a substitution $\sigma : \mathscr{V} \to \mathscr{T}(\mathscr{F}, \mathscr{V})$ with $\mathscr{D}(\sigma) = \mathscr{V}ar(\rho)$, a context $C[\,]$, and ground constructor terms $u_1, \ldots, u_k$ such that $s = C[l\sigma]$, $t = C[r\sigma]$, $s_i\sigma \to^*_{\mathscr{R}_{n-1}^d} u_i$ and $t_i\sigma \to^*_{\mathscr{R}_{n-1}^d} u_i$ for all $1 \leqslant i \leqslant k$, and if $\mathscr{V}ar(r) \not\subseteq \mathscr{V}ar(l)$, then $x\sigma$ must be a ground constructor term for every $x \in \mathscr{E}\mathscr{V}ar(\rho)$. Finally, define $\to_{\mathscr{R}^d} = \bigcup_{n \in \mathbb{N}} \to_{\mathscr{R}_n^d}$.

For example, in the CTRS $\mathscr{R}$ from Example 3.5 we have $f(a) \to_{\mathscr{R}^d} c$, but neither $f(a) \to_{\mathscr{R}^d} a$ nor $f(a) \to_{\mathscr{R}^d} b$.

It is easy to prove (by induction on the depth n) that $s \to_{\mathscr{R}_n^d} t$ implies $s \to_{\mathscr{R}_n} t$ but not vice versa. The first statement of the next lemma shows that $\to_{\mathscr{R}^d}$ is deterministic in the sense that the contractum of a redex is uniquely determined. Furthermore, in contrast to $\to_{\mathscr{R}}$, the relation $\to_{\mathscr{R}^d}$ satisfies the parallel moves lemma. Because of the first statement of Lemma 3.7, the proof of the second statement bears a strong resemblance to that of Lemma 3.3 given in [5].

Lemma 3.7. *For all $m, n \in \mathbb{N}$, the following holds*:

1. *If $s = l_1\sigma_1 \to_{\mathscr{R}_m^d} r_1\sigma_1$ and $s = l_2\sigma_2 \to_{\mathscr{R}_n^d} r_2\sigma_2$, then $r_1\sigma_1 = r_2\sigma_2$.*
2. *If $t \mathrel{+\!\!\!\!\!\!\not\!\!\!\!\!\to}_{\mathscr{R}_m^d} t_1$ and $t \mathrel{+\!\!\!\!\!\!\not\!\!\!\!\!\to}_{\mathscr{R}_n^d} t_2$, then there is a term t_3 such that $t_1 \mathrel{+\!\!\!\!\!\!\not\!\!\!\!\!\to}_{\mathscr{R}_n^d} t_3$ and $t_2 \mathrel{+\!\!\!\!\!\!\not\!\!\!\!\!\to}_{\mathscr{R}_m^d} t_3$. Moreover, the redexes contracted in $t_1 \mathrel{+\!\!\!\!\!\!\not\!\!\!\!\!\to}_{\mathscr{R}_n^d} t_3$ ($t_2 \mathrel{+\!\!\!\!\!\!\not\!\!\!\!\!\to}_{\mathscr{R}_m^d} t_3$) are the descendants in t_1 (t_2) of the redexes contracted in $t \mathrel{+\!\!\!\!\!\!\not\!\!\!\!\!\to}_{\mathscr{R}_n^d} t_2$ ($t \mathrel{+\!\!\!\!\!\!\not\!\!\!\!\!\to}_{\mathscr{R}_m^d} t_1$).*

Proof. The proof proceeds by induction on $m + n$. The base case $m + n = 0$ holds vacuously. Suppose the lemma holds for all m' and n' with $m' + n' < \ell$. In the induction step, we have to prove that the lemma holds for all m and n with $m + n = \ell$. Observe that the inductive hypothesis implies the validity of the diagrams in Fig. 1, where $m' + n' < \ell$.

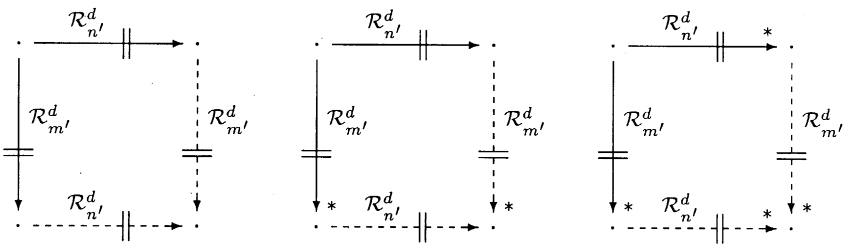


Fig. 1. The induction hypothesis in Lemma 3.7.

(1) Suppose $s = l_1\sigma_1 \to_{\mathscr{R}^d_m} r_1\sigma_1$ and $s = l_2\sigma_2 \to_{\mathscr{R}^d_n} r_2\sigma_2$. Since $\mathscr{R}$ is orthogonal, the rewrite rules coincide and will be denoted by $\rho : l \to r \Leftarrow s_1 == t_1, \ldots, s_k == t_k$ in the following. Obviously, $\sigma_1 = \sigma_2[\mathscr{V}ar(l)]$, i.e. the restrictions of σ_1 and σ_2 to $\mathscr{V}ar(l)$ coincide. So if $\mathscr{V}ar(r) \subseteq \mathscr{V}ar(l)$, then $r_1\sigma_1 = r_2\sigma_2$. Suppose otherwise that $\mathscr{V}ar(r) \not\subseteq \mathscr{V}ar(l)$. We show by induction on i that $\sigma_1 = \sigma_2[\mathscr{V}ar(l) \cup \bigcup_{j=1}^{i} \mathscr{V}ar(t_j)]$. It then follows $\sigma_1 = \sigma_2[\mathscr{E}\mathscr{V}ar(\rho)]$ and hence $r_1\sigma_1 = r_2\sigma_2$. If $i = 0$, then $\sigma_1 = \sigma_2[\mathscr{V}ar(l)]$. Let $i > 0$. According to the inductive hypothesis, $\sigma_1 = \sigma_2[\mathscr{V}ar(l) \cup \bigcup_{j=1}^{i-1} \mathscr{V}ar(t_j)]$. Since $\mathscr{V}ar(s_i) \subseteq \mathscr{V}ar(l) \cup \bigcup_{j=1}^{i-1} \mathscr{V}ar(t_j)$, it is sufficient to show $\sigma_1 = \sigma_2[\mathscr{V}ar(t_i)]$. By definition of $\nrightarrow\!\!\!\!\!\!\Vdash_{\mathscr{R}^d_m}$ and $\Vdash_{\mathscr{R}^d_n}$, there exist ground constructor terms u_1 and u_2 such that $s_i\sigma_1 \to^*_{\mathscr{R}^d_{m-1}} u_1 \,{}^*_{\mathscr{R}^d_{m-1}}\!\!\leftarrow t_i\sigma_1$ and $s_i\sigma_2 \to^*_{\mathscr{R}^d_{n-1}} u_2 \,{}^*_{\mathscr{R}^d_{n-1}}\!\!\leftarrow t_i\sigma_2$. It is an immediate consequence of $s_i\sigma_1 = s_i\sigma_2$, $s_i\sigma_1 \Vdash^*_{\mathscr{R}^d_{m-1}} u_1$, $s_i\sigma_2 \Vdash^*_{\mathscr{R}^d_{n-1}} u_2$, and the inductive hypothesis on ℓ that the two ground normal forms u_1 and u_2 coincide. Hence $t_i\sigma_1 \to^*_{\mathscr{R}^d_{m-1}} u_1 \,{}^*_{\mathscr{R}^d_{n-1}}\!\!\leftarrow t_i\sigma_2$. Thus, for all variables $x \in \mathscr{V}ar(t_i) \setminus (\mathscr{V}ar(l) \cup \bigcup_{j=1}^{i-1} \mathscr{V}ar(t_j))$, it follows from the definition of $\to^*_{\mathscr{R}^d_{m-1}}$ and $\to^*_{\mathscr{R}^d_{n-1}}$ that $x\sigma_1$ and $x\sigma_2$ are ground constructor terms because $x \in \mathscr{E}\mathscr{V}ar(\rho)$. Therefore, the fact that t_i is a constructor term implies $x\sigma_1 = x\sigma_2$.

(2) Since parallel reduction contracts pairwise disjoint redexes, it is sufficient to prove the lemma for the case where both $t \Vdash_{\mathscr{R}^d_m} t_1$ and $t \Vdash_{\mathscr{R}^d_n} t_2$ consist of a single $\to_{\mathscr{R}^d}$ step. In other words, we may assume $t \to_{\mathscr{R}^d_m} t_1$ and $t \to_{\mathscr{R}^d_n} t_2$. Furthermore, because $\to_{\mathscr{R}^d}$ is deterministic, the only interesting case is that where t is a redex, say $t = l\sigma \to_{\mathscr{R}^d_m} r\sigma = t_1$ for some rule $l \to r \Leftarrow c \in \mathscr{R}$, containing a proper subredex s which is contracted to s' in the step $t \to_{\mathscr{R}^d_n} t_2$.

Since $\mathscr{R}$ is orthogonal, there is a variable $x \in \mathscr{V}ar(l)$ such that s is a subterm of $x\sigma$. So $x\sigma = C[s]$ for some context $C[\,]$. Let q be the position in t such that $t|_q = s$. Consequently, for every descendant q' of q in t_1, we have $t_1|_{q'} = s$. Define $t_3 = t_1[q' \leftarrow s' \mid q' \in q\backslash t \to_{\mathscr{R}^d_m} t_1]$. Clearly, $t_1 \Vdash_{\mathscr{R}^d_n} t_3$.

It remains to be shown that $t_2 \Vdash_{\mathscr{R}^d_m} t_3$. To this end, let us consider $t = l\sigma \to_{\mathscr{R}^d_m} r\sigma = t_1$ again. By definition of $\Vdash_{\mathscr{R}^d_m}$, there exist ground constructor terms u_i such that $s_i\sigma \Vdash^*_{\mathscr{R}^d_{m-1}} u_i$ and $t_i\sigma \Vdash^*_{\mathscr{R}^d_{m-1}} u_i$ for all $s_i == t_i$ in c. Define σ' by $y\sigma = y\sigma'$ for all $y \neq x$ and $x\sigma' = C[s']$. We show $s_i\sigma' \Vdash^*_{\mathscr{R}^d_{m-1}} u_i$ and $t_i\sigma' \Vdash^*_{\mathscr{R}^d_{m-1}} u_i$. It then follows that $t_2 = t[q \leftarrow s'] = l\sigma' \to_{\mathscr{R}^d_m} r\sigma' = t_3$. Since $s_i\sigma \Vdash^*_{\mathscr{R}^d_n} s_i\sigma'$, $s_i\sigma \Vdash^*_{\mathscr{R}^d_{m-1}} u_i$, and u_i is a normal form, it follows from the inductive hypothesis that $s_i\sigma' \Downarrow^*_{\mathcal{R}^d_{m-1}} u_i$. Analogously, we obtain $t_i\sigma' \Downarrow^*_{\mathcal{R}^d_{m-1}} u_i$. □

Corollary 3.8. *$\rightarrow_{\mathcal{R}^d}$ is level-confluent (i.e., for every $n \in \mathbb{N}$, $\rightarrow_{\mathcal{R}^d_n}$ is confluent).*

Proof. Immediate consequence of Lemma 3.7. □

Theorem 3.9. *Every almost functional CTRS $\mathcal{R}$ is level-confluent.*

Proof. It follows from $\rightarrow_{\mathcal{R}^d_n} \subseteq \rightarrow_{\mathcal{R}_n}$ and Proposition 3.10 that $\rightarrow_{\mathcal{R}_n}$ is a compatible refinement of $\rightarrow_{\mathcal{R}^d_n}$. Hence, by the aforementioned result of Staples [17], $\rightarrow_{\mathcal{R}_n}$ is confluent if and only if $\rightarrow_{\mathcal{R}^d_n}$ is confluent. □

As a matter of fact, by carefully checking the proofs in [19], one finds that Theorem 3.9 can be proven in the same manner. The proof techniques, however, are totally different. Suzuki et al. [19] came to the result by using an extended parallel rewriting relation.

Proposition 3.10. *If $s \rightarrow^*_{\mathcal{R}_n} t$, then there is a term u such that $s \rightarrow^*_{\mathcal{R}^d_n} u$ and $t \rightarrow^*_{\mathcal{R}^d_n} u$.*

Proof. We proceed by induction on the depth n of $s \rightarrow^*_{\mathcal{R}_n} t$. The proposition holds vacuously for $n=0$. So let $n>0$. We further proceed by induction on the length ℓ of the reduction sequence $s \rightarrow^*_{\mathcal{R}_n} t$. Again, the case $\ell=0$ holds vacuously. Suppose the claim is true for ℓ. In order to show it for $\ell+1$, we consider $s = C[l\sigma] \rightarrow_{\mathcal{R}_n} C[r\sigma] = t' \rightarrow^{\ell}_{\mathcal{R}_n} t$, where $s \rightarrow_{\mathcal{R}_n} t'$ by the rule $\rho: l \rightarrow r \Leftarrow s_1 == t_1, \dots, s_k == t_k$. It follows from the inductive hypothesis on ℓ that there is a term u' such that $t' \rightarrow^*_{\mathcal{R}^d_n} u'$ and $t \rightarrow^*_{\mathcal{R}^d_n} u'$. Since $s \rightarrow_{\mathcal{R}_n} t'$, there are ground constructor terms u_i such that $s_i\sigma \rightarrow^*_{\mathcal{R}_{n-1}} u_i \; {}^*_{\mathcal{R}_{n-1}}\!\!\leftarrow t_i\sigma$. By the inductive hypothesis on n and the fact that u_i is a normal form, we conclude $s_i\sigma \rightarrow^*_{\mathcal{R}^d_{n-1}} u_i \; {}^*_{\mathcal{R}^d_{n-1}}\!\!\leftarrow t_i\sigma$. Now if $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$, then $s \rightarrow_{\mathcal{R}^d_n} t'$ and the claim follows. Suppose otherwise that $\mathcal{V}ar(r) \not\subseteq \mathcal{V}ar(l)$ and let $x \in \mathcal{EV}ar(\rho)$. Then $x \in \mathcal{V}ar(t_j)$ for some $s_j == t_j$. Since $t_j\sigma \rightarrow^*_{\mathcal{R}^d_{n-1}} u_j$, t_j is a constructor term, and u_j is a ground constructor term, it follows that $x\sigma \rightarrow^*_{\mathcal{R}^d_{n-1}} u_x$ for some ground constructor subterm u_x of u_j. Note that u_x is unique because $\rightarrow_{\mathcal{R}^d_{n-1}}$ is confluent ($\rightarrow_{\mathcal{R}^d}$ is level-confluent by Corollary 3.8). Define σ' by $x\sigma' = u_x$ for every $x \in \mathcal{EV}ar(\rho)$ and $y\sigma' = y\sigma$ otherwise. Observe that $z\sigma \rightarrow^*_{\mathcal{R}^d_{n-1}} z\sigma'$ for every variable $z \in \mathcal{D}(\sigma)$. Let $s' = C[r\sigma']$. According to the above, $t' \rightarrow^*_{\mathcal{R}^d_{n-1}} s'$. Observe that also $s \rightarrow_{\mathcal{R}^d_n} s'$ because $s_j\sigma' \rightarrow^*_{\mathcal{R}^d_{n-1}} u_j \; {}^*_{\mathcal{R}^d_{n-1}}\!\!\leftarrow t_j\sigma'$ for every $s_j == t_j$ in c (it is a consequence of $s_j\sigma \rightarrow^*_{\mathcal{R}^d_{n-1}} u_j$, $s_j\sigma \rightarrow^*_{\mathcal{R}^d_{n-1}} s_j\sigma'$ and confluence of $\rightarrow_{\mathcal{R}^d_{n-1}}$ that $s_j\sigma' \rightarrow^*_{\mathcal{R}^d_{n-1}} u_j$). It now follows from confluence of $\rightarrow_{\mathcal{R}^d_n}$ in conjunction with $t' \rightarrow^*_{\mathcal{R}^d_{n-1}} s'$ and $t' \rightarrow^*_{\mathcal{R}^d_n} u'$ that s' and u' have a common reduct u w.r.t. $\rightarrow_{\mathcal{R}^d_n}$. Clearly, u is a common reduct of s and t w.r.t. $\rightarrow_{\mathcal{R}^d_n}$ as well. □

Since term rewriting is mainly concerned with computing normal forms, the next lemma is of interest.

Lemma 3.11. *For every $n \in \mathbb{N}$, the sets of normal forms $NF(\rightarrow_{\mathscr{R}_n})$ and $NF(\rightarrow_{\mathscr{R}_n^d})$ coincide.*

Proof. Obviously, $NF(\rightarrow_{\mathscr{R}_n}) \subseteq NF(\rightarrow_{\mathscr{R}_n^d})$ because $\rightarrow_{\mathscr{R}_n^d} \subseteq \rightarrow_{\mathscr{R}_n}$. We prove $NF(\rightarrow_{\mathscr{R}_n^d}) \subseteq NF(\rightarrow_{\mathscr{R}_n})$ indirectly. To this end, suppose there is a term $s \in NF(\rightarrow_{\mathscr{R}_n^d})$ but $s \notin NF(\rightarrow_{\mathscr{R}_n})$. Since s is not a normal form w.r.t. $\rightarrow_{\mathscr{R}_n}$, there is a rule $\rho : l \rightarrow r \Leftarrow s_1 == t_1, \ldots, s_k == t_k \in \mathscr{R}$, a context $C[\,]$ and a substitution σ such that $s = C[l\sigma] \rightarrow_{\mathscr{R}_n} C[r\sigma]$. In particular, for every $s_j == t_j$ in c, there is a ground constructor term u_j such that $s_j\sigma \rightarrow^*_{\mathscr{R}_{n-1}} u_j \; {}^*_{\mathscr{R}_{n-1}}\!\!\leftarrow t_j\sigma$. It follows as in the proof of Proposition 3.10 that $s_j\sigma' \rightarrow^*_{\mathscr{R}^d_{n-1}} u_j \; {}^*_{\mathscr{R}^d_{n-1}}\!\!\leftarrow t_j\sigma'$. Hence $s = C[l\sigma] = C[l\sigma'] \rightarrow_{\mathscr{R}_n^d} C[r\sigma']$. This is a contradiction to $s \in NF(\rightarrow_{\mathscr{R}_n^d})$. $\square$

We would like to point out that all of the preceding results remain valid if we replace orthogonality with almost orthogonality.

4. Conditional term graph rewriting

In this section, we use the term-based approach of [12] to term graph rewriting rather than those of [4] or [16]. In so doing, it is possible to completely argue within the framework of term rewriting and to avoid concepts from different fields. We first recapitulate some basic notions. Most of them stem from [12].

Let $\mathscr{F}$ be a signature and $\mathscr{V}$ be a set of variables. Let M be a countably infinite set of objects called *marks* (we will use natural numbers as marks). Let $\mathscr{F}^* = \{f^\mu \mid f \in \mathscr{F},\ \mu \in M\}$ be the set of *marked function symbols*. For all $f^\mu \in \mathscr{F}^*$, the arity of f^μ coincides with that of f. Moreover, we define $symbol(f^\mu) = f$ and $mark(f^\mu) = \mu$. Analogously, let $\mathscr{V}^* = \{x^\mu \mid x \in \mathscr{V},\ \mu \in M\}$ be the set of *marked variables*, $symbol(x^\mu) = x$, and $mark(x^\mu) = \mu$. The set of *marked terms* over $\mathscr{F}^*$ and $\mathscr{V}^*$ is defined in the usual way and is denoted by $\mathscr{T}(\mathscr{F}^*, \mathscr{V}^*)$. The set of all marks appearing in a marked term $t^* \in \mathscr{T}(\mathscr{F}^*, \mathscr{V}^*)$ is denoted by $marks(t^*)$. The set $\mathscr{T}_w(\mathscr{F}^*, \mathscr{V}^*)$ of *well-marked terms* over $\mathscr{F}^*$ and $\mathscr{V}^*$ is the subset of $\mathscr{T}(\mathscr{F}^*, \mathscr{V}^*)$ such that $t^* \in \mathscr{T}_w(\mathscr{F}^*, \mathscr{V}^*)$ if and only if, for every pair (t_1^*, t_2^*) of subterms of t^*, $mark(root(t_1^*)) = mark(root(t_2^*))$ implies $t_1^* = t_2^*$. For example, the term $0^1 +^0 0^1$ is well-marked but $0^1 +^1 0^1$ is not. If a term t^* is well-marked and $\mu \in marks(t^*)$, then $t^* \backslash \mu$ denotes the unique subterm s^* of t^* for which $mark(root(s^*)) = \mu$ holds. Well-marked terms have an exact correspondance to directed acyclic graphs; the reader is referred to [12] for details. In contrast to [12], we are solely interested in well-marked terms. Thus, throughout the whole paper, marked stands for well-marked. Two subterms t_1^* and t_2^* of a marked term t^* are shared in t^* if $t_1^* = t_2^*$; e.g. 0^1 and 0^1 are shared in $0^1 +^0 0^1$.

If t^* is a marked term, then $e(t^*)$ denotes the unmarked term obtained from t^* by erasing all marks. Two marked terms s^* and t^* are *bisimilar* [1] (denoted by $s^* \sim t^*$) if and only if $e(s^*) = e(t^*)$. The marked terms s^* and t^* are *isomorphic* (denoted by $s^* \cong t^*$) if and only if t^* can be obtained from s^* by a *renaming* of marks, that is, there exists a bijective function $\Phi : marks(s^*) \to marks(t^*)$ such that $\Phi(s^*) = t^*$, where the extension of Φ to $\mathcal{T}_w(\mathcal{F}^*, \mathcal{V}^*)$ is defined by

$$\Phi(s^*) = \begin{cases} x^{\Phi(\mu)} & \text{if } s^* = x^\mu,\ x \in \mathcal{V}, \\ f^{\Phi(\mu)}(\Phi(t_1^*), \ldots, \Phi(t_n^*)) & \text{if } s^* = f^\mu(t_1^*, \ldots, t_n^*). \end{cases}$$

Note that $s^* \cong t^*$ implies $s^* \sim t^*$. The marks of a marked term s^* are called *fresh* w.r.t. another marked term t^* if $marks(s^*) \cap marks(t^*) = \emptyset$. A *marked substitution* $\sigma^* : \mathcal{V}^* \to \mathcal{T}(\mathcal{F}^*, \mathcal{V}^*)$ is a substitution which satisfies $x^\mu \sigma^* \sim x^\nu \sigma^*$ for all $x^\mu, x^\nu \in \mathcal{D}(\sigma^*)$ with $symbol(x^\mu) = symbol(x^\nu)$. This definition of marked substitution ensures that the unmarked substitution σ obtained from σ^* by erasing all marks is well-defined (i.e., σ really is a substitution). Let σ_1^* and σ_2^* be marked substitutions with $\mathcal{D}(\sigma_1^*) = \mathcal{D}(\sigma_2^*) = \{x_1^{\mu_1}, \ldots, x_n^{\mu_n}\}$. σ_1^* and σ_2^* are *isomorphic* ($\sigma_1^* \cong \sigma_2^*$) if $D^\mu(x_1^{\mu_1}\sigma_1^*, \ldots, x_n^{\mu_n}\sigma_1^*) \cong D^\mu(x_1^{\mu_1}\sigma_2^*, \ldots, x_n^{\mu_n}\sigma_2^*)$, where D is a fresh symbol of arity n and μ is a fresh mark w.r.t. every $x_i^{\mu_i}\sigma_j^*$, where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant 2$. The notion *marked context* is defined in the obvious way.

For instance, $0^1 +^0 0^1 \cong 0^2 +^0 0^2$ but $0^1 +^0 0^1 \not\cong 0^1 +^0 0^2$. On the other hand, $0^1 +^0 0^1 \sim 0^1 +^0 0^2$. Moreover, $\sigma_1^* = \{x_1 \mapsto 0^1 +^0 0^1, x_2 \mapsto 0^1 +^0 0^1\} \not\cong \{x_1 \mapsto 0^1 +^0 0^1, x_2 \mapsto 0^3 +^2 0^3\} = \sigma_2^*$.

Definition 4.1. A rule $l^* \to r^* \Leftarrow c^*$ is a *marked version* of a rule $l \to r \Leftarrow c$ in $\mathcal{R}$ if $e(l^*) = l$, $e(r^*) = r$, $e(c^*) = c$, and, for all $x^\mu, y^\nu \in \mathcal{V}ar(l^* \to r^* \Leftarrow c^*)$, $symbol(x^\mu) = symbol(y^\nu)$ if and only if $mark(x^\mu) = mark(y^\nu)$.

The last condition can be rephrased as: every marked occurrence of a variable $x \in \mathcal{V}ar(l \to r \Leftarrow c)$ must have the same mark in $l^* \to r^* \Leftarrow c^*$. For the sake of simplicity, marks on variables in marked rewrite rules will be omitted in the following because these marks are unique anyway. So on the one hand, variables in rewrite rules are maximally shared. On the other hand, by using fresh and mutually distinct marks for the right-hand side and the conditional part of a rewrite rule, we adopt a "minimal structure sharing scheme" (different structure sharing schemes are discussed in [12]).

Definition 4.2. Let $\mathcal{R}$ be a 3-CTRS with strict equality. Let s^* and t^* be marked terms. Let $\Rightarrow_{\mathcal{R}_0} = \emptyset$ and for $n > 0$, define $s^* \Rightarrow_{\mathcal{R}_n} t^*$ if there exists a marked version $l^* \to r^* \Leftarrow s_1^* == t_1^*, \ldots, s_k^* == t_k^*$ of a rewrite rule $\rho : l \to r \Leftarrow s_1 == t_1, \ldots, s_k == t_k$ from $\mathcal{R}$, a marked substitution σ^* and a marked context $C^*[, \ldots,]$ such that

- $s^* = C^*[l^*\sigma^*, \ldots, l^*\sigma^*]$ and $t^* = C^*[r^*\sigma^*, \ldots, r^*\sigma^*]$,
- $l^*\sigma^*$ is not a subterm of $C^*[, \ldots,]$,

[1] The origin of the notion "bisimilarity" is explained in [2].

- for every $1 \leqslant i \leqslant k$, there are marked ground constructor terms u_i^* and v_i^* such that $s_i^* \sigma^* \Rightarrow_{\mathscr{R}_{n-1}}^* u_i^*$, $t_i^* \sigma^* \Rightarrow_{\mathscr{R}_{n-1}}^* v_i^*$, and $u_i^* \sim v_i^*$,[2]
- all marks on function symbols in r^*, s_i^*, t_i^*, and $x\sigma^*$ (for every variable $x \in \mathscr{E}\mathscr{V}ar(\rho)$) are mutually distinct and fresh w.r.t. s^*.

We call $\Rightarrow_{\mathscr{R}} = \bigcup_{n \geq 0} \Rightarrow_{\mathscr{R}_n}$ *(term) graph rewrite relation* w.r.t. $\mathscr{R}$.

$l^*\sigma^*$ is called the *contracted marked redex* in s^*. We use the notation $s^* \Rightarrow_{\mathscr{R}_n}^{l^*\sigma^*} t^*$ in order to specify the contracted marked redex. Note that all shared subterms $l^*\sigma^*$ are replaced simultaneously by $r^*\sigma^*$.

Definition 4.3. The *deterministic* graph rewrite relation $\Rightarrow_{\mathscr{R}^d}$ is defined analogously to $\Rightarrow_{\mathscr{R}}$: in a $\Rightarrow_{\mathscr{R}_n^d}$ rewrite step, if $\mathscr{V}ar(r) \not\subseteq \mathscr{V}ar(l)$, then it is additionally required that $x\sigma^*$ is a marked ground constructor term for every extra variable x in $l \to r \Leftarrow c$.

In order to illustrate how graph rewriting works, let $\mathscr{R}$ be the CTRS $\mathscr{R}_{fib}$ from Section 3 augmented by the rewrite rules $double(x) \to x + x$ and $snd(\langle x, y \rangle) \to y$. There is the $\Rightarrow_{\mathscr{R}}$ (in fact, $\Rightarrow_{\mathscr{R}^d}$) reduction sequence:

$$\begin{aligned} double^0(snd^1(fib^2(s^3(0^4)))) &\Rightarrow_{\mathscr{R}} snd^1(fib^2(s^3(0^4))) +^5 snd^1(fib^2(s^3(0^4))) \\ &\Rightarrow_{\mathscr{R}} snd^1(t^*) +^5 snd^1(t^*) \\ &\Rightarrow_{\mathscr{R}} (0^8 +^{12} s^9(0^{10})) +^5 (0^8 +^{12} s^9(0^{10})) \\ &\Rightarrow_{\mathscr{R}} s^9(0^{10}) +^5 s^9(0^{10}) \end{aligned}$$

because $fib^6(0^4) \Rightarrow_{\mathscr{R}} \langle 0^8, s^9(0^{10}) \rangle^7$. In the derivation, t^* denotes the marked term $\langle s^9(0^{10}), 0^8 +^{12} s^9(0^{10}) \rangle^{11}$.

4.1. Adequacy

Next we will show that the mapping e which erases all marks from a well-marked term is an adequate mapping in the sense of Kennaway et al. [10], that is to say, it is surjective, preserves normal forms, preserves reductions, and is cofinal. Surjectivity ensures that every term can be represented as a directed acyclic graph (well-marked term). The normal form condition ensures that a graph is a final result of a computation if the term which it represents also is, and vice versa. Preservation of reduction ensures that every graph reduction sequence represents some term reduction sequence. Cofinality ensures that for every term rewriting computation, there is a graph rewriting computation which can be mapped, not necessarily to the term rewriting computation, but to some extension of it. Recall that $\mathscr{R}$ denotes an almost functional CTRS unless stated otherwise.

[2] Note that $u_i^* \cong v_i^*$ is not required.

Theorem 4.4. *For all $n \in \mathbb{N}$, $\Rightarrow_{\mathscr{R}_n^d}$ is an adequate implementation of $\rightarrow_{\mathscr{R}_n^d}$, that is,*

1. *e is surjective*,
2. *$\forall t^* \in \mathscr{T}_w(\mathscr{F}^*, \mathscr{V}^*)$: $t^* \in NF(\Rightarrow_{\mathscr{R}_n^d})$ if and only if $e(t^*) \in NF(\rightarrow_{\mathscr{R}_n^d})$*,
3. *$\forall s^* \in \mathscr{T}_w(\mathscr{F}^*, \mathscr{V}^*)$: if $s^* \Rightarrow^*_{\mathscr{R}_n^d} t^*$, then $e(s^*) \rightarrow^*_{\mathscr{R}_n^d} e(t^*)$*,
4. *$\forall s^* \in \mathscr{T}_w(\mathscr{F}^*, \mathscr{V}^*)$: if $e(s^*) \rightarrow^*_{\mathscr{R}_n^d} u$, then there is a $t^* \in \mathscr{T}_w(\mathscr{F}^*, \mathscr{V}^*)$ such that $s^* \Rightarrow^*_{\mathscr{R}_n^d} t^*$ and $u \rightarrow^*_{\mathscr{R}_n^d} e(t^*)$.*

Proof. We use induction on n.

(1) Surjectivity is obvious.

(2) The *if* direction is easily shown. For an indirect proof of the *only if* direction, suppose $e(t^*) \notin NF(\rightarrow_{\mathscr{R}_n^d})$, i.e., $e(t^*) = C[l\sigma] \rightarrow_{\mathscr{R}_n^d} C[r\sigma]$ by using the rule $\rho: l \rightarrow r \Leftarrow s_1 == t_1, \ldots, s_k == t_k$ at position p. So, for every $s_i == t_i$, there is a ground constructor term u_i such that $s_i\sigma \rightarrow^*_{\mathscr{R}_{n-1}^d} u_i$ and $t_i\sigma \rightarrow^*_{\mathscr{R}_{n-1}^d} u_i$. Let l^* and σ^* be marked version of l and σ such that $t^*|_p = l^*\sigma^*$. Let $l^* \rightarrow r^* \Leftarrow s_1^* == t_1^*, \ldots, s_k^* == t_k^*$ be a marked version of ρ such that all marks on r^*, s_i^*, and t_i^* are fresh w.r.t. t^* and mutually distinct. Furthermore, σ^* is extended to $\mathscr{E}\mathscr{V}ar(\rho)$ in the usual way: for all $z \in \mathscr{E}\mathscr{V}ar(\rho)$ let $z\sigma^*$ be a marked version of $z\sigma$ such that all marks are mutually distinct and fresh w.r.t. t^*, r^*, s_i^*, and t_i^*. Let $C^*[,\ldots,]$ be the marked context such that $t^* = C^*[l^*\sigma^*, \ldots, l^*\sigma^*]$ and $l^*\sigma^*$ is not a subterm of $C^*[,\ldots,]$. Since $e(s_i^*\sigma^*) = s_i\sigma \rightarrow^*_{\mathscr{R}_{n-1}^d} u_i$, $e(t_i^*\sigma^*) = t_i\sigma \rightarrow^*_{\mathscr{R}_{n-1}^d} u_i$, and u_i is a ground constructor term, it follows from the inductive hypothesis that there exist marked terms v_i^* and w_i^* such that $s_i^*\sigma^* \Rightarrow^*_{\mathscr{R}_{n-1}^d} v_i^*$, $t_i^*\sigma^* \Rightarrow^*_{\mathscr{R}_{n-1}^d} w_i^*$, and $e(v_i^*) = u_i = e(w_i^*)$. The latter particularly implies that v_i^* and w_i^* are ground constructor terms and $v_i^* \sim w_i^*$. Therefore, $t^* \Rightarrow_{\mathscr{R}_n^d} C^*[r^*\sigma^*, \ldots, r^*\sigma^*]$ which contradicts $t^* \in NF(\Rightarrow_{\mathscr{R}_n^d})$.

(3) We proceed by induction on the length ℓ of $s^* \Rightarrow^*_{\mathscr{R}_n^d} t^*$. The base case $\ell = 0$ clearly holds. Thus consider $s^* \Rightarrow^{l^*\sigma^*}_{\mathscr{R}_n^d} u^* \Rightarrow^{\ell}_{\mathscr{R}_n^d} t^*$. According to the inductive hypothesis on ℓ, $e(u^*) \rightarrow^*_{\mathscr{R}_n^d} e(t^*)$. Since $s^* \Rightarrow^{l^*\sigma^*}_{\mathscr{R}_n^d} u^*$, we have $s^* = C^*[l^*\sigma^*, \ldots, l^*\sigma^*]$, $l^*\sigma^*$ is not a subterm of $C^*[,\ldots,]$, $u^* = C^*[r^*\sigma^*, \ldots, r^*\sigma^*]$, and, for every $s_i^* == t_i^*$, there are marked ground constructor terms u_i^* and v_i^* such that $s_i^*\sigma^* \Rightarrow^*_{\mathscr{R}_{n-1}^d} u_i^*$, $t_i^*\sigma^* \Rightarrow^*_{\mathscr{R}_{n-1}^d} v_i^*$, and $u_i^* \sim v_i^*$. Let $\sigma = e(\sigma^*)$, i.e., $x\sigma = e(x\sigma^*)$ for all $x \in \mathscr{D}(\sigma^*)$. By the inductive hypothesis on n, $e(s_i^*)\sigma \rightarrow^*_{\mathscr{R}_{n-1}^d} e(u_i^*) = e(v_i^*) \;{}_{\mathscr{R}_{n-1}^d}^{*}\!\!\leftarrow e(t_i^*)\sigma$. Hence $l\sigma \rightarrow_{\mathscr{R}_n^d} r\sigma$ and $e(s^*) \rightarrow^+_{\mathscr{R}_n^d} e(t^*)$.

(4) We use induction on the length ℓ of $e(s^*) \rightarrow^{\ell}_{\mathscr{R}_n^d} u$. The proof is illustrated in Fig. 2. The case $\ell = 0$ holds vacuously. So we consider $e(s^*) \rightarrow^{\ell}_{\mathscr{R}_n^d} \bar{u} \rightarrow_{\mathscr{R}_n^d} u$. By the inductive hypothesis on ℓ, there exists a $\bar{t}^* \in \mathscr{T}_w(\mathscr{F}^*, \mathscr{V}^*)$ such that $s^* \Rightarrow^*_{\mathscr{R}_n^d} \bar{t}^*$ and $\bar{u} \rightarrow^*_{\mathscr{R}_n^d} e(\bar{t}^*)$. Let $\bar{t} = e(\bar{t}^*)$. Suppose $\bar{u} = C[l\sigma] \rightarrow_{\mathscr{R}_n^d} C[r\sigma] = u$ by using the rule $\rho: l \rightarrow r \Leftarrow s_1 == t_1, \ldots, s_k == t_k$ at the position p, i.e., $C[l\sigma]|_p = l\sigma$. By the parallel moves lemma for $\rightarrow_{\mathscr{R}_n^d}$, there is a $v \in \mathscr{T}(\mathscr{F}, \mathscr{V})$ such that $u \mathrel{+\!\!\!\!\!\!\rightarrow}^*_{\mathscr{R}_n^d} v$ and $\bar{t} \mathrel{+\!\!\!\!\!\!\rightarrow}_{\mathscr{R}_n^d} v$. In particular, the redexes contracted in the step $\bar{t} \mathrel{+\!\!\!\!\!\!\rightarrow}_{\mathscr{R}_n^d} v$ are the descendants $p \backslash \bar{u} \rightarrow^*_{\mathscr{R}_n^d} \bar{t}$ of p in $\bar{t}$. Let $Q = p \backslash \bar{u} \rightarrow^*_{\mathscr{R}_n^d} \bar{t}$. Note that $Q \subseteq \mathscr{P}os(\bar{t})$ consists of pairwise independent positions. For every $q \in Q$, $\bar{t}^*|_q$ can be written as $\bar{t}^*|_q = l_q^*\tau_q^*$, where l_q^* is a marked version of l and τ_q^* is a marked substitution. As in the proof of (2), one can show

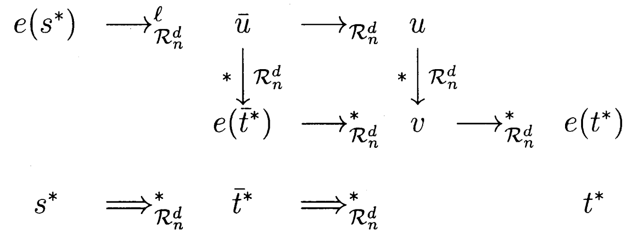


Fig. 2. Proof of Theorem 4.4.

that $l_q^* \tau_q^* \Rightarrow_{\mathscr{R}_n^d} r^* \tau_q^*$. Let

$$Q' = \{q' \in \mathscr{P}os(\bar{t}^*) \mid \bar{t}^*|_{q'} = l_q^* \tau_q^* \text{ for some } q \in Q\}.$$

Note that $Q \subseteq Q'$. It is not difficult to prove that Q' consists of pairwise independent positions. Let t^* be the marked term obtained from $\bar{t}^*$ by contracting all the redexes $l_q^* \tau_q^*$. Let $\tau_q = e(\tau_q^*)$. Since $\bar{t} \mathrel{+\!\!\!\!\!\!\!\;+\!\!\!\!\rightarrow}_{\mathscr{R}_n^d} v$ by contracting the redexes in Q and $\bar{t} \mathrel{+\!\!\!\!\!\!\!\;+\!\!\!\!\rightarrow}_{\mathscr{R}_n^d} e(t^*)$ by contracting the redexes in Q', it follows that $v \mathrel{+\!\!\!\!\!\!\!\;+\!\!\!\!\rightarrow}_{\mathscr{R}_n^d} e(t^*)$ by contracting the redexes in $Q' \backslash Q$. All in all, $s^* \Rightarrow_{\mathscr{R}_n^d}^* t^*$ and $u \rightarrow_{\mathscr{R}_n^d}^* e(t^*)$. □

Corollary 4.5. *For all $n \in \mathbb{N}$, $\Rightarrow_{\mathscr{R}_n}$ is an adequate implementation of $\rightarrow_{\mathscr{R}_n}$.*

Proof. One can prove statements (1)–(3) as in Theorem 4.4. Statement (4) remains to be shown: for $s^* \in \mathscr{T}_w(\mathscr{F}^*, \mathscr{V}^*)$ and $u \in \mathscr{T}(\mathscr{F}, \mathscr{V})$ with $e(s^*) \rightarrow_{\mathscr{R}_n}^* u$, there must be a $t^* \in \mathscr{T}_w(\mathscr{F}^*, \mathscr{V}^*)$ such that $s^* \Rightarrow_{\mathscr{R}_n}^* t^*$ and $u \rightarrow_{\mathscr{R}_n}^* e(t^*)$.

By Proposition 3.10, there is a term v such that $e(s^*) \rightarrow_{\mathscr{R}_n^d}^* v \;{}_{\mathscr{R}_n^d}^*\!\!\leftarrow u$. By Theorem 4.4, there is a marked term t^* such that $s^* \Rightarrow_{\mathscr{R}_n^d}^* t^*$ and $v \rightarrow_{\mathscr{R}_n^d}^* e(t^*)$. Now the claim is a consequence of $\Rightarrow_{\mathscr{R}_n^d} \subseteq \Rightarrow_{\mathscr{R}_n}$, $u \rightarrow_{\mathscr{R}_n^d}^* v \rightarrow_{\mathscr{R}_n^d}^* e(t^*)$, and $\rightarrow_{\mathscr{R}_n^d} \subseteq \rightarrow_{\mathscr{R}_n}$. □

Corollary 4.6. *$\Rightarrow_{\mathscr{R}}$ is an adequate implementation of $\rightarrow_{\mathscr{R}}$.*

Proof. Immediate consequence of Corollary 4.5. □

It is a direct consequence of the preceding results that $\Rightarrow_{\mathscr{R}}$ is a sound and complete implementation of $\rightarrow_{\mathscr{R}}$ in the sense of Barendregt et al. [4]. Recall that soundness ensures that the graph implementation of a CTRS cannot give incorrect results, while completeness ensures that graph rewriting gives all results.

Corollary 4.7. *$\Rightarrow_{\mathscr{R}}$ is a sound and complete implementation of $\rightarrow_{\mathscr{R}}$, i.e.,*
1. *$s^* \Rightarrow_{\mathscr{R}}^* t^* \in NF(\Rightarrow_{\mathscr{R}})$ implies $e(s^*) \rightarrow_{\mathscr{R}}^+ e(t^*) \in NF(\rightarrow_{\mathscr{R}})$ (soundness),*
2. *$\forall s^* \in \mathscr{T}_w(\mathscr{F}^*, \mathscr{V}^*)$: if $e(s^*) \rightarrow_{\mathscr{R}_n}^* u \in NF(\rightarrow_{\mathscr{R}})$, then there is a marked term t^* such that $s^* \Rightarrow_{\mathscr{R}}^* t^* \in NF(\Rightarrow_{\mathscr{R}})$ and $e(t^*) = u$ (completeness).*

Proof. Follows directly from Corollary 4.6. □

Note that in the entire subsection, there is only one place at which we made use of the fact that $\mathscr{R}$ is orthogonal: Theorem 4.4(4) crucially depends on the fact that the parallel moves lemma holds for $\rightarrow_{\mathscr{R}_n^d}$. Since the parallel moves lemma remains valid if $\mathscr{R}$ is almost orthogonal, so do all of the preceding results if we replace orthogonality with almost orthogonality.

We conclude this subsection by showing why we need $u_i^* \sim v_i^*$ but not $u_i^* \cong v_i^*$ in Definition 4.2.

Example 4.8. In the CTRS

$$\mathscr{R} = \begin{cases} g(x) \rightarrow c(x,x) & \\ f(x) \rightarrow x & \Leftarrow g(x) \rightarrow c(d,d) \end{cases}$$

$f^0(d^1)$ rewrites to d^1 because $g^2(d^1) \Rightarrow_{\mathscr{R}} c^3(d^1,d^1)$ and $e(c^3(d^1,d^1)) = c(d,d) = e(c^4(d^5,d^6))$. If $c^3(d^1,d^1) \cong c^4(d^5,d^6)$ were required, then $f^0(d^1)$ would be a normal form w.r.t. $\Rightarrow_{\mathscr{R}}$. Since $f(d)$ is not a normal form w.r.t. $\rightarrow_{\mathscr{R}}$, graph rewriting would not be an adequate implementation of conditional term rewriting.

4.2. Confluence

In this subsection, it will be shown that orthogonal conditional rules give rise to a subcommutative deterministic graph rewrite relation (up to isomorphism). This implies that the graph rewrite relation is level-confluent modulo $\cong$. Similar results for unconditional systems were achieved by Staples [18] and Barendregt et al. [4]. In order to prove the above-mentioned statement, the following auxiliary result is useful.

Lemma 4.9. *Let $\Rightarrow_n$ denote $\Rightarrow_{\mathscr{R}_n^d}$ or $\Rightarrow_{\mathscr{R}_n}$. If $s^* \cong t^* \Rightarrow_n^{\ell} u^*$ (so the reduction of t^* to u^* consists of ℓ graph rewrite steps), then there is a marked term v^* such that $s^* \Rightarrow_n^{\ell} v^* \cong u^*$.*

Proof. Since $s^* \cong t^*$, there is a renaming of marks $\Phi : marks(t^*) \rightarrow marks(s^*)$ with $\Phi(t^*) = s^*$. We show that Φ can be extended to a renaming on $marks(t^*) \cup marks(u^*)$ such that $\Phi(t^*) = s^* \Rightarrow_n^{\ell} \Phi(u^*) \cong u^*$.

The claim obviously holds for $\ell = 0$. We show the lemma for $\ell = 1$, the whole claim then follows by induction on the length of the reduction sequence. We proceed further by induction on the depth n. Suppose $t^* \Rightarrow_n^{l^*\sigma^*} u^*$, where a marked version of the rule $\rho : l \rightarrow r \Leftarrow s_1 == t_1, \ldots, s_k == t_k$ is used. Then for every $1 \leqslant i \leqslant k$, there are marked ground constructor terms u_i^* and v_i^* such that $s_i^*\sigma^* \Rightarrow_{n-1}^* u_i^*$, $t_i^*\sigma^* \Rightarrow_{n-1}^* v_i^*$, and $u_i^* \sim v_i^*$. Let M_1 be the set of all fresh marks used in r_1^* and in the sequences $s_i^*\sigma^* \Rightarrow_{n-1}^* u_i^*$ and $t_i^*\sigma^* \Rightarrow_{n-1}^* v_i^*$, $1 \leqslant i \leqslant k$. Moreover, let M_2 be a set of (fresh) marks with $M_2 \cap (M_1 \cup marks(s^*) \cup marks(t^*)) = \emptyset$ and $card(M_2) = card(M_1)$, where $card(M_i)$ denotes the cardinality of M_i. Let $\Phi' : M_1 \rightarrow M_2$ be an arbitrary bijective function. Now we extend Φ from $marks(t^*)$ to $M_1 \cup marks(t^*)$ by

$$\Phi(\mu) = \Phi'(\mu) \quad \text{if } \mu \in M_1.$$

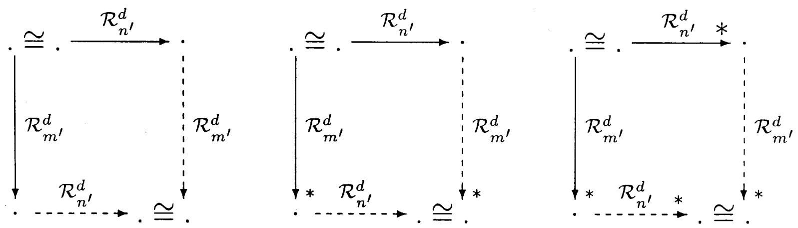


Fig. 3. The induction hypothesis of Lemma 4.10.

Note that $\Phi : M_1 \cup marks(t^*) \to M_2 \cup marks(s^*)$ is bijective. By the inductive hypothesis on n, we have $\Phi(s_i^*\sigma^*) \Rightarrow^*_{n-1} \Phi(u_i^*) \cong u_i^*$ and $\Phi(t_i^*\sigma^*) \Rightarrow^*_{n-1} \Phi(v_i^*) \cong v_i^*$ for all $1 \leqslant i \leqslant k$. Since $e(\Phi(u_i^*)) = e(u_i^*) = e(v_i^*) = e(\Phi(v_i^*))$, we infer $s^* = \Phi(t^*) \Rightarrow_n \Phi(u^*)$. Finally, $\Phi(u^*) \cong u^*$ because the restriction of Φ to $marks(u^*)$ is a renaming of marks. □

The next lemma shows that the deterministic graph rewrite relation is subcommutative modulo $\cong$.

Lemma 4.10. *For all $m, n \in \mathbb{N}$, the following statements hold*:

1. *If $s^* = l_1^*\sigma_1^* \Rightarrow_{\mathscr{R}^d_m} r_1^*\sigma_1^*$ and $s^* = l_2^*\sigma_2^* \Rightarrow_{\mathscr{R}^d_n} r_2^*\sigma_2^*$, then $r_1^*\sigma_1^* \cong r_2^*\sigma_2^*$.*
2. *If $s^* \Rightarrow^{l_1^*\sigma_1^*}_{\mathscr{R}^d_m} \bar{s}^*$, $t^* \Rightarrow^{l_2^*\sigma_2^*}_{\mathscr{R}^d_n} \bar{t}^*$, and $s^* \cong t^*$, then there are marked terms $\tilde{s}^*$ and $\tilde{t}^*$ such that* (i) *$\bar{s}^* \Rightarrow^{l_2^*\sigma_2^*}_{\mathscr{R}^d_n} \tilde{s}^*$ or $\bar{s}^* = \tilde{s}^*$*, (ii) *$\bar{t}^* \Rightarrow^{l_1^*\sigma_1^*}_{\mathscr{R}^d_m} \tilde{t}^*$ or $\bar{t}^* = \tilde{t}^*$, and* (iii) *$\tilde{s}^* \cong \tilde{t}^*$.*

Proof. The proof is similar to that of Lemma 3.7. Again, we proceed by induction on $m + n$. The base case $m + n = 0$ holds vacuously. Suppose the lemma holds for all m' and n' with $m' + n' < \ell$. In the induction step, we have to prove that the lemma holds for all m and n with $m + n = \ell$. By using Lemma 4.9, it is not difficult to prove that the inductive hypothesis implies the validity of the diagrams in Fig. 3, where $m' + n' < \ell$ and $\to$ stands for $\Rightarrow$.

(1) Let $s^* = l_1^*\sigma_1^* \Rightarrow_{\mathscr{R}^d_m} r_1^*\sigma_1^*$ and $s^* = l_2^*\sigma_2^* \Rightarrow_{\mathscr{R}^d_n} r_2^*\sigma_2^*$. Clearly, $l_1^* \to r_1^* \Leftarrow c_1^*$ and $l_2^* \to r_2^* \Leftarrow c_2^*$ are marked versions of the same rewrite rule $\rho : l \to r \Leftarrow c \in \mathscr{R}$ because $\mathscr{R}$ is orthogonal. Apparently, the restrictions of σ_1^* and σ_2^* to $\mathscr{V}ar(l)$ Coincide. So if $\mathscr{V}ar(r) \subseteq \mathscr{V}ar(l)$, then $r_1^*\sigma_1^* \cong r_2^*\sigma_2^*$ since r_1^* and r_2^* are freshly marked. Suppose otherwise that $\mathscr{V}ar(r) \not\subseteq \mathscr{V}ar(l)$. We show that in this case $\sigma_1^* \cong \sigma_2^*$ holds. Since $\sigma_1^* = \sigma_2^*[\mathscr{V}ar(l)]$, it remains to be shown that $\sigma_1^* \cong \sigma_2^*[\mathscr{E}\mathscr{V}ar(\rho)]$. We show by induction on i that $\sigma_1^* \cong \sigma_2^*[\mathscr{V}ar(l) \cup \bigcup_{j=1}^{i} \mathscr{V}ar(t_j)]$. If $i = 0$, then $\sigma_1^* = \sigma_2^*[\mathscr{V}ar(l)]$. So let $i > 0$. According to the inductive hypothesis, $\sigma_1^* \cong \sigma_2^*[\mathscr{V}ar(l) \cup \bigcup_{j=1}^{i-1} \mathscr{V}ar(t_j)]$. Since $\mathscr{V}ar(s_i) \subseteq \mathscr{V}ar(l) \cup \bigcup_{j=1}^{i-1} \mathscr{V}ar(t_j)$, it is sufficient to show $\sigma_1^* \cong \sigma_2^*[\mathscr{V}ar(t_i)]$. There are marked ground constructor terms $u_1^*, u_2^*, v_1^*, v_2^*$ such that $s_i^*\sigma_1^* \Rightarrow^*_{\mathscr{R}^d_{m-1}} u_1^*$, $t_i^*\sigma_1^* \Rightarrow^*_{\mathscr{R}^d_{m-1}} u_2^*$, where $u_1^* \sim u_2^*$, and $s_i^*\sigma_2^* \Rightarrow^*_{\mathscr{R}^d_{n-1}} v_1^*$, $t_i^*\sigma_2^* \Rightarrow^*_{\mathscr{R}^d_{n-1}} v_2^*$, where $v_1^* \sim v_2^*$. It now follows from the inductive hypothesis on ℓ in combination with $s_i^*\sigma_1^* \cong s_i^*\sigma_2^*$ that $u_1^* \cong v_1^*$. Thus $u_2^* \sim u_1^* \sim v_1^* \sim v_2^*$. As in the proof of Lemma 3.7, for every extra variable x, there are

$$\begin{array}{ccccccc}
t^* & \Longrightarrow_{\mathcal{R}_n^d} & & & \bar{u}^* & \cong & \bar{t}^* \\
\Big\Downarrow \mathcal{R}_m^d & & & & \Big\Downarrow \mathcal{R}_m^d & & \Big\Downarrow \mathcal{R}_m^d \\
\bar{s}^* & \Longrightarrow_{\mathcal{R}_n^d} & \tilde{s}^* & \cong & \tilde{u}^* & \cong & \tilde{t}^*
\end{array}$$

Fig. 4.

marked ground constructor terms u_x^* and v_x^* such that $x\sigma_1^* = u_x^*$, $x\sigma_2^* = v_x^*$, and $u_x^* \sim v_x^*$. Since the marks on instantiated extra variables are fresh and mutually distinct, we finally derive $\sigma_1^* \cong \sigma_2^*[\mathscr{V}ar(t_i)]$.

(2) By Lemma 4.9, it is sufficient to prove that if $t^* \Rightarrow_{\mathscr{R}_m^d}^{l_1^*\sigma_1^*} \bar{s}^*$ and $t^* \Rightarrow_{\mathscr{R}_n^d}^{l_2^*\sigma_2^*} \bar{t}^*$, then (i) $\bar{s}^* \Rightarrow_{\mathscr{R}_n^d}^{l_2^*\sigma_2^*} \tilde{s}^*$ or $\bar{s}^* = \tilde{s}^*$, (ii) $\bar{t}^* \Rightarrow_{\mathscr{R}_m^d}^{l_1^*\sigma_1^*} \tilde{t}^*$ or $\bar{t}^* = \tilde{t}^*$, and (iii) $\tilde{s}^* \cong \tilde{t}^*$ for some marked terms $\tilde{s}^*$ and $\tilde{t}^*$. We distinguish three cases:

(a) $l_1^*\sigma_1^* = l_2^*\sigma_2^*$,

(b) $l_1^*\sigma_1^*$ is neither a subterm of $l_2^*\sigma_2^*$ nor conversely,

(c) $l_1^*\sigma_1^*$ is a proper subterm of $l_2^*\sigma_2^*$.

(a) With the aid of (1), this follows easily.

(b) The proof is analogous to Proposition 3.19, (1), case 1 in [12]. We may write $t^* = C^*[l_{i_1}^*\sigma_{i_1}^*, \ldots, l_{i_p}^*\sigma_{i_p}^*]$, where $i_j \in \{1,2\}$ for every index $j \in \{1,\ldots,p\}$. Without loss of generality, we may assume that $1 = i_1 \leqslant i_2 \leqslant \cdots \leqslant i_p = 2$. That is to say, $t^* = C^*[l_1^*\sigma_1^*, \ldots, l_2^*\sigma_2^*]$. Then $\bar{s}^* = C^*[r_1^*\sigma_1^*, \ldots, l_2^*\sigma_2^*]$ and $\bar{t}^* = C^*[l_1^*\sigma_1^*, \ldots, r_2^*\sigma_2^*]$. According to Lemma 4.9, we may assume that all marks on function symbols in r_1^*, r_2^*, $x\sigma_1^*$ and $y\sigma_2^*$ (for every variable $x \in \mathscr{E}\mathscr{V}ar(l_1 \to r_1 \Leftarrow c_1)$ and $y \in \mathscr{E}\mathscr{V}ar(l_2 \to r_2) \Leftarrow c_2)$) are pairwise distinct and fresh w.r.t. t^*. (If $(marks(r_1^*) \cup marks(\sigma_1^*)) \cap (marks(r_2^*) \cup marks(\sigma_2^*)) \neq \emptyset$, then we take marked versions $\bar{r}_2^*$ and $\bar{\sigma}_2^*$ with $(marks(r_1^*) \cup marks(\sigma_1^*)) \cap (marks(\bar{r}_2^*) \cup marks(\bar{\sigma}_2^*)) = \emptyset$, observe that $t^* \Rightarrow_{\mathscr{R}_n^d}^{l_2^*\bar{\sigma}_2^*} C^*[l_1^*\sigma_1^*, \ldots, \bar{r}_2^*\bar{\sigma}_2^*] = \bar{u}^*$, and apply Lemma 4.9; see Fig. 4.) Now if we contract the redex $l_2^*\sigma_2^*$ in $\bar{s}^*$ to $r_2^*\sigma_2^*$, then we obtain $\tilde{s}^* = C^*[r_1^*\sigma_1^*, \ldots, r_2^*\sigma_2^*]$. Analogously, contracting $l_1^*\sigma_1^*$ in $\bar{t}^*$ yields $\tilde{t}^* = C^*[r_1^*\sigma_1^*, \ldots, r_2^*\sigma_2^*]$.

(c) We proceed in analogy to Proposition 3.19, (1), case 2 in [12]. As in (b), we may write t^* as

$$t^* = C^*[l_1^*\sigma_1^*, \ldots, l_2^*\sigma_2^*] = C^*[l_1^*\sigma_1^*, \ldots, \bar{C}^*[l_1^*\sigma_1^*, \ldots, l_1^*\sigma_1^*]]$$

where $l_2^*\sigma_2^* = \bar{C}^*[l_1^*\sigma_1^*, \ldots, l_1^*\sigma_1^*]$ and $l_1^*\sigma_1^*$ is neither a subterm of $C^*[,\ldots,]$ nor of $\bar{C}^*[,\ldots,]$. Hence

$$\bar{s}^* = C^*[r_1^*\sigma_1^*, \ldots, \bar{C}^*[r_1^*\sigma_1^*, \ldots, r_1^*\sigma_1^*]]$$

$$\bar{t}^* = C^*[l_1^*\sigma_1^*, \ldots, r_2^*\sigma_2^*] = C^*[l_1^*\sigma_1^*, \ldots, \tilde{C}^*[l_1^*\sigma_1^*, \ldots, l_1^*\sigma_1^*]]$$

for some context $\tilde{C}^*[,\ldots,]$ which does not contain $l_1^*\sigma_1^*$. Again, by Lemma 4.9 we may assume that the marks on function symbols in r_1^*, r_2^*, $x\sigma_1^*$ and $y\sigma_2^*$ (for all

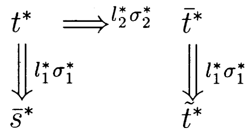


Fig. 5.

extra variables x and y) are pairwise distinct and fresh w.r.t. t^*. Observe that no occurrence of $\bar{C}^*[r_1^*\sigma_1^*,\ldots,r_1^*\sigma_1^*]$ can be found in $\bar{s}^*$ aside from those obtained by contracting the marked redex $l_1^*\sigma_1^*$ because we use fresh marks. For the same reason, $C^*[,\ldots,\tilde{C}^*[,\ldots,]]$ does not contain $l_1^*\sigma_1^*$. Now if $l_1^*\sigma_1^*$ is not a subterm of $\bar{t}^*$, then let $\tilde{t}^* = \bar{t}^*$. Otherwise define

$$\tilde{t}^* = C^*[r_1^*\sigma_1^*,\ldots,\tilde{C}^*[r_1^*\sigma_1^*,\ldots,r_1^*\sigma_1^*]]$$

and observe that $\bar{t}^* \Rightarrow_{\mathscr{R}}^{l_1^*\sigma_1^*} \tilde{t}^*$. The situation is depicted in Fig. 5, where $l_2^*\sigma_2^* = \bar{C}^*[l_1^*\sigma_1^*,\ldots,l_1^*\sigma_1^*] \Rightarrow_{\mathscr{R}} \tilde{C}^*[l_1^*\sigma_1^*,\ldots,l_1^*\sigma_1^*] = r_2^*\sigma_2^*$.

Next, we will show that $\bar{C}^*[r_1^*\sigma_1^*,\ldots,r_1^*\sigma_1^*] \Rightarrow_{\mathscr{R}_n^d} \tilde{C}^*[r_1^*\sigma_1^*,\ldots,r_1^*\sigma_1^*]$. To this end, recall that $l_2^*\sigma_2^* = \bar{C}^*[l_1^*\sigma_1^*,\ldots,l_1^*\sigma_1^*] \Rightarrow_{\mathscr{R}_n^d} r_2^*\sigma_2^*$. Thus, for every $s_i^* == t_i^*$ in c_2^*, there exist marked ground constructor terms u_i^* and v_i^* such that $s_i^*\sigma_2^* \Rightarrow_{\mathscr{R}_{n-1}^d}^* u_i^*$, $t_i^*\sigma_2^* \Rightarrow_{\mathscr{R}_{n-1}^d}^* v_i^*$, and $u_i^* \sim v_i^*$. Since $\mathscr{R}$ is orthogonal, for every occurrence of $l_1^*\sigma_1^*$, there is a variable $x \in \mathscr{V}ar(l_2)$ such that $x\sigma_2^* = C_x^*[l_1^*\sigma_1^*,\ldots,l_1^*\sigma_1^*]$ contains this particular occurrence. Define $\bar{\sigma}_2^*$ by $x\bar{\sigma}_2^* = C_x^*[r_1^*\sigma_1^*,\ldots,r_1^*\sigma_1^*]$ for all those variables x and $y\bar{\sigma}_2^* = y\sigma_2^*$ otherwise. Now $l_2^*\bar{\sigma}_2^* \Rightarrow_{\mathscr{R}_n^d} r_2^*\bar{\sigma}_2^* = \tilde{C}^*[r_1^*\sigma_1^*,\ldots,r_1^*\sigma_1^*]$. In order to see this, infer from the inductive hypothesis on ℓ in conjunction with $s_i^*\sigma_2^* \Rightarrow_{\mathscr{R}_{n-1}^d}^* u_i^*$ and $s_i^*\sigma_2^* \Rightarrow_{\mathscr{R}_m^d} s_i^*\bar{\sigma}_2^*$ that there is a marked ground constructor term $\bar{u}_i^*$ such that $s_i^*\bar{\sigma}_2^* \Rightarrow_{\mathscr{R}_{n-1}^d}^* \bar{u}_i^*$ and $u_i^* \cong \bar{u}_i^*$. Analogously, there is a marked ground constructor term $\bar{v}_i^*$ such that $t_i^*\bar{\sigma}_2^* \Rightarrow_{\mathscr{R}_{n-1}^d}^* \bar{v}_i^*$, and $v_i^* \cong \bar{v}_i^*$. Hence the claim follows from $\bar{u}_i^* \sim u_i^* \sim v_i^* \sim \bar{v}_i^*$. □

Corollary 4.11. *For every $n \in \mathbb{N}$, $\Rightarrow_{\mathscr{R}_n^d}$ is confluent modulo $\cong$.*

Proof. Immediate consequence of Lemma 4.10.

Theorem 4.12. *For every $n \in \mathbb{N}$, $\Rightarrow_{\mathscr{R}_n}$ is confluent modulo $\cong$.*

Proof. Because of $\Rightarrow_{\mathscr{R}_n^d} \subseteq \Rightarrow_{\mathscr{R}_n}$, Proposition 4.13, and Lemma 4.9, we conclude by the generalization of Staples' result (see Section 2) that $\Rightarrow_{\mathscr{R}_n}$ is confluent modulo $\cong$ if and only if $\Rightarrow_{\mathscr{R}_n^d}$ is confluent modulo $\cong$. □

Proposition 4.13. *If $s^* \Rightarrow_{\mathscr{R}_n}^* t^*$, then there are marked terms u^* and v^* such that $s^* \Rightarrow_{\mathscr{R}_n^d}^* u^*$, $t^* \Rightarrow_{\mathscr{R}_n^d}^* v^*$, and $u^* \cong v^*$.*

Proof. We proceed by induction on the depth n of $s^* \Rightarrow_{\mathscr{R}_n}^* t^*$. The proposition holds vacuously for $n = 0$. So let $n > 0$. We proceed further by induction on the length ℓ of the reduction sequence $s^* \Rightarrow^*_{\mathscr{R}_n} t^*$. Again, the case $\ell = 0$ holds vacuously. Suppose the claim is true for ℓ. In order to show it for $\ell + 1$, we consider $s^* = C^*[l^*\sigma^*, \dots, l^*\sigma^*] \Rightarrow^{l^*\sigma^*}_{\mathscr{R}_n} C^*[r^*\sigma^*, \dots, r^*\sigma^*] = \bar{t}^* \Rightarrow^{\ell}_{\mathscr{R}_n} t^*$, where $s^* \Rightarrow_{\mathscr{R}_n} \bar{t}^*$ by a marked version of the rule $\rho : l \rightarrow r \Leftarrow s_1 == t_1, \dots, s_k == t_k$. We show that there are marked terms $\bar{u}^*$ and $\bar{v}^*$ such that $s^* \Rightarrow_{\mathscr{R}^d_n} \bar{u}^*$, $\bar{t}^* \Rightarrow_{\mathscr{R}^d_n} \bar{v}^*$, and $\bar{u}^* \cong \bar{v}^*$. The whole claim then follows from the inductive hypothesis on ℓ in combination with Corollary 4.11 and Lemma 4.9. Since $s^* \Rightarrow_{\mathscr{R}_n} \bar{t}^*$, there are marked ground constructor terms u^*_i and v^*_i such that $s^*_i\sigma^* \Rightarrow^*_{\mathscr{R}_{n-1}} u^*_i$, $t^*_i\sigma^* \Rightarrow^*_{\mathscr{R}_{n-1}} v^*_i$, and $u^*_i \sim v^*_i$. By the inductive hypothesis on n and the fact that u^*_i is a normal form, we conclude that there are marked terms $\bar{u}^*_i$ and $\bar{v}^*_i$ such that $s^*_i\sigma^* \Rightarrow^*_{\mathscr{R}^d_{n-1}} \bar{u}^*_i \cong u^*_i$ and $t^*_i\sigma^* \Rightarrow^*_{\mathscr{R}^d_{n-1}} \bar{v}^*_i \cong v^*_i$. Note that $\bar{u}^*_i \sim u^*_i \sim v^*_i \sim \bar{v}^*_i$. So if $\mathscr{V}ar(r) \subseteq \mathscr{V}ar(l)$, then $s^* \Rightarrow_{\mathscr{R}^d_n} \bar{t}^*$ and the claim follows. Suppose otherwise that $\mathscr{V}ar(r) \not\subseteq \mathscr{V}ar(l)$ and let $x \in \mathscr{E}\mathscr{V}ar(\rho)$. Then $x \in \mathscr{V}ar(t_j)$ for some $s^*_j == t^*_j$. Since $t^*_j\sigma^* \Rightarrow^*_{\mathscr{R}^d_{n-1}} \bar{v}^*_j \cong v^*_j$, t^*_j is a marked constructor term, and $\bar{v}^*_j$ is a marked ground constructor term, it follows that $x\sigma^* \Rightarrow^*_{\mathscr{R}^d_{n-1}} \bar{v}^*_x$ for some ground constructor subterm $\bar{v}^*_x$ of $\bar{v}^*_j$. Note that even if x occurs more than once in t^*_j, every occurrence of $x\sigma^*$ in $t^*_j\sigma^*$ is reduced to $\bar{v}^*_x$ because redexes with identical marks are shared and are thus reduced simultaneously by $\Rightarrow_{\mathscr{R}^d_{n-1}}$. Define $\bar{\sigma}^*$ by $x\bar{\sigma}^* = \bar{v}^*_x$ for every $x \in \mathscr{E}\mathscr{V}ar(\rho)$ and $y\bar{\sigma}^* = y\sigma^*$ for every $y \in \mathscr{V}ar(l)$. Let $\bar{v}^* = C^*[r^*\bar{\sigma}^*, \dots, r^*\bar{\sigma}^*]$. Since $z\sigma^* = z\bar{\sigma}^*$ for every $z \in \mathscr{V}ar(l)$ and $z\sigma^* \Rightarrow^*_{\mathscr{R}^d_{n-1}} z\bar{\sigma}^*$ for every $z \in \mathscr{E}\mathscr{V}ar(\rho)$ we derive $\bar{t}^* \Rightarrow^*_{\mathscr{R}^d_{n-1}} \bar{v}^*$ (taking into account that $z\sigma^*$ gets fresh marks for every $z \in \mathscr{E}\mathscr{V}ar(\rho)$). We claim that also $s^* \Rightarrow_{\mathscr{R}^d_n} \bar{v}^*$. This is because for every i, $1 \leqslant i \leqslant k$, we can conclude from $s^*_i\sigma^* \Rightarrow^*_{\mathscr{R}^d_{n-1}} \bar{u}^*_i$, $s^*_i\sigma^* \Rightarrow^*_{\mathscr{R}^d_{n-1}} s^*_i\bar{\sigma}^*$, and confluence of $\Rightarrow_{\mathscr{R}^d_{n-1}}$ that $s^*_i\bar{\sigma}^* \Rightarrow_{\mathscr{R}^d_{n-1}} \tilde{u}^*_i \cong \bar{u}^*_i$. for some $\tilde{u}^*_i$. Analogously, $t^*_i\bar{\sigma}^* \Rightarrow^*_{\mathscr{R}^d_{n-1}} \tilde{v}^*_i \cong \bar{v}^*_i$. It then follows from $\tilde{u}^*_i \sim \bar{u}^*_i \sim \bar{v}^*_i \sim \tilde{v}^*_i$ that $l^*\sigma^* = l^*\bar{\sigma}^* \Rightarrow_{\mathscr{R}^d_n} r^*\bar{\sigma}^*$. Hence $s^* = C^*[l^*\bar{\sigma}^*, \dots, l^*\bar{\sigma}^*] \Rightarrow^{l^*\sigma^*}_{\mathscr{R}^d_n} C^*[r^*\bar{\sigma}^*, \dots, r^*\bar{\sigma}^*] = \bar{v}^*$. □

In contrast to the preceding sections, the results in this subsection do *not* extend to almost orthogonal systems. This can be seen in the following example taken from [16].

Example 4.14. In the almost orthogonal TRS

$$\mathscr{R} = \begin{cases} f(x) \rightarrow g(x,x) \\ f(a) \rightarrow g(a,a) \end{cases}$$

we have $f^0(a^1) \Rightarrow_{\mathscr{R}} g^2(a^1, a^1)$ and $f^0(a^1) \Rightarrow_{\mathscr{R}} g^2(a^3, a^4)$ but $g^2(a^1, a^1) \not\cong g^2(a^3, a^4)$.

We conclude this section with a simple corollary.

Corollary 4.15. *For every $n \in \mathbb{N}$, the sets $NF(\Rightarrow_{\mathscr{R}_n})$ and $NF(\Rightarrow_{\mathscr{R}^d_n})$ coincide.*

Proof. Follows from Theorem 4.4(2), Corollary 4.5(2), and Lemma 3.11. □

5. Concluding remarks

In this paper we have considered the class of *almost functional* CTRSs. It has been shown that graph rewriting is adequate for simulating term rewriting in these systems. In particular, conditional graph rewriting proved to be a sound and complete implementation (w.r.t. the computation of normal forms) of almost functional CTRSs. Furthermore, every almost functional CTRS is level-confluent and the same holds for its graph implementation. All these results can be extended to *almost orthogonal* systems except for the last one.

We stress that the results remain valid if we continue to allow infeasible conditional critical pairs. A conditional critical pair

$$\langle C[r_2]\sigma, r_1\sigma\rangle \Leftarrow c_1\sigma, c_2\sigma$$

induced by an overlap of the two conditional rewrite rules $l_1 \to r_1 \Leftarrow c_1$ and $l_2 \to r_2 \Leftarrow c_2$ is called *infeasible* if the condition $c_1\sigma, c_2\sigma$ is unsolvable. For example, in the CTRS

$$\begin{aligned}
split(x,[\,]) &\to ([\,],[\,])\\
split(x,y:ys) &\to (xs,y:zs) \Leftarrow x \leqslant y == True, split(x,ys) == (xs,zs)\\
split(x,y:ys) &\to (y:xs,zs) \Leftarrow x \leqslant y == False, split(x,ys) == (xs,zs)\\
qsort([\,]) &\to [\,]\\
qsort(x:xs) &\to qsort(ys) +\!\!+ (x:qsort(zs)) \Leftarrow split(x,xs) == (ys,zs)
\end{aligned}$$

the conditional critical pair

$$\begin{aligned}
\langle (xs,y:zs),(y:xs',zs')\rangle \Leftarrow\ & x \leqslant y == True, split(x,ys) == (xs,zs),\\
& x \leqslant y == False, split(x,ys) == (xs',zs')
\end{aligned}$$

is infeasible because the condition $x \leqslant y == True$, $x \leqslant y == False$ has no solution.

It has been shown by Suzuki et al. [19] that every orthogonal-oriented 3-CTRS is level-confluent provided it is *properly oriented* and *right-stable*. It is quite natural to ask oneself whether graph rewriting is an adequate implementation of these systems too. Our proof for almost functional CTRSs heavily depends on the fact that there exists a closely related deterministic reduction relation which satisfies the parallel moves lemma. In case of oriented 3-CTRSs, however, this approach does not work. Consider for example the orthogonal properly oriented right-stable 3-CTRS

$$\begin{aligned}
&a \to b\\
&b \to a\\
&c \to y \Leftarrow a \to y.
\end{aligned}$$

Then $c \rightarrow_{\mathscr{R}} a$ as well as $c \rightarrow_{\mathscr{R}} b$ but c is a normal form w.r.t. $\rightarrow_{\mathscr{R}^d}$ because neither a nor b are (ground constructor) normal forms. So the question of whether our results also hold for orthogonal properly oriented right-stable 3-CTRSs remains open.

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