Spectral Shift Function in the Large Coupling Constant Limit

O. Safronov

Department of Mathematics, Royal Institute of Technology, 10044 Stockholm, Sweden
E-mail: safronov@math.kth.se

Communicated by L. Carleson

Received July 10, 2000; accepted November 14, 2000

Given two selfadjoint operators $H_0$ and $V = V_+ - V_-$, we study the spectrum of the operator $H(\alpha) = H_0 + \alpha V$, $\alpha > 0$. We consider the quantity $\zeta(\alpha; H(\alpha), H_0)$ which coincides with Krein's spectral shift function of the pair $(H(\alpha), H_0)$ if $V$ is of trace class and study its asymptotic behavior as $\alpha \to \infty$. Applications to differential operators are given.

\section*{0. INTRODUCTION}

Let $H_0 = H_0^*$ be a selfadjoint operator in a Hilbert space $\mathcal{H}$ and let $V = V^*$ be of the form
\begin{equation}
V = V_+ - V_-, \quad V_\pm \geq 0.
\end{equation}
We study the motion of the spectrum of the operator
\begin{equation}
H(\alpha) = H_0 + \alpha V, \quad \alpha > 0
\end{equation}
with growth of $\alpha$. In the case of trace class perturbations $V$, Krein \cite{Krein} and Lifshits \cite{Lifshits} have introduced the spectral shift function $\zeta(\alpha; H(x), H_0)$ which belongs to $L_1(\mathbb{R})$ and satisfies the relation
\begin{equation}
\text{Tr} \left[ \phi(H(\alpha)) - \phi(H_0) \right] = \int_{-\infty}^{+\infty} \zeta(\alpha, \lambda) \phi'(\lambda) \, d\lambda,
\end{equation}
with an arbitrary function $\phi$ of a suitable class. Let $E_M(\delta)$ denote the spectral projection of an operator $M = M^*$ corresponding to an interval $\delta$. Then due to the spectral theorem the trace formula (0.2) can be formally rewritten as
\begin{equation}
\zeta(\alpha, \lambda) = \text{Tr} \left[ E_{H(\alpha)}(-\infty, \lambda) - E_{H_0}(-\infty, \lambda) \right].
\end{equation}
Later Birman and Krein [5] (see also [9, 22]) found an application of the spectral shift function to the scattering theory. Namely they proved the formula

$$\det S(\lambda) = \exp(-2\pi i \zeta(\lambda, \alpha)),$$

where $S(\lambda)$ is the scattering matrix for the pair $H(\sigma), H_0$. Recently Pushnitski found a new representation of this function (see [16]) for sign-definite perturbations $V \geq 0$. In [10] this representation was obtained by Gesztesy and Makarov for the case of no-sign-definite perturbations (0.1).

Then the Gesztesy–Makarov formula (see (1.7)) was extended by Pushnitski [18] to the case when $V$ is not necessarily the trace class. However, one of the main conditions on $H_0$ and $H(\alpha)$ in [18] was that there exists a function $f$, such that $f(H(\alpha)) - f(H_0)$ is a nuclear operator. In our paper the r.h.s. of (1.7) is treated as a definition of $\zeta(\lambda; H(\alpha), H_0)$. It is still unclear whether this definition leads to new classes of perturbations or not.

We study the leading term (in the power expansion) of the asymptotics of $\zeta(\lambda, \sigma)$ as $\sigma \to \infty$ and typically it does not depend on $\lambda$. A similar "independancy" theorem for negative $V = -V_-$ was obtained in [17]. In this paper we treat the more difficult case of no-sign-definite perturbations of the form (0.1). It should be mentioned also that the idea of the independancy theorem is presented in [3], where only the discrete spectrum of $H(\alpha)$ has been studied.

In Section 1 we formulate the problem and describe the main result in detail. It should be noted that if $\lambda \in \rho(H_0)$ is a "regular" point for $H_0$, then $\zeta(\lambda, \alpha)$ coincides with the difference between the number of eigenvalues of $H(\alpha)$ moving via $\lambda$ to the left and to the right. This remark allows us to study the asymptotics of $\zeta(\lambda, \sigma)$ for $\lambda \in \rho(H_0)$ with help of a special perturbation theory developed in [19, 20]. The stability theorem obtained in the present paper reduces the case of an arbitrary $\lambda$ to that of $\lambda \in \rho(H_0)$. In applications to differential operators this approach allows one to use known results.

If $V < 0$, then the function $\zeta(\lambda, \sigma)$ is monotone decreasing in $\sigma$ (for a fixed $\lambda = \lambda$) and coincides with a certain integral of the counting function of the spectrum of an auxiliary compact selfadjoint operator. A suitable version of the corresponding representation formula can be found in [16, 17]. These arguments have allowed Pushnitski [17] to prove for $V < 0$ that the leading term in the asymptotics is independent of $\lambda$.

The class of the perturbations (0.1) requires, fist of all, a new version of the representation formula for $\zeta(\lambda, \sigma)$. Such a formula is obtained in [10]. Moreover, the technique used in [19, 20] can be also simplified with the help of this representation.
Different types of asymptotic formulae with respect to the large coupling constant in the gaps of the spectrum $\sigma(H_0)$ were studied in the papers \cite{1, 3, 11}. In the present paper we investigate the "spectral flow" of $H(x)$ not only in the gaps of $\sigma(H_0)$ but even on the continuous spectrum of $H_0$.

1. MAIN RESULT (THEOREM 1.1)

1. Notations. Throughout the paper formulae and statements with double indices ($\pm$ or $\mp$) are understood independently, as pairs of formulae. Below $\mathcal{H}$ is a separable Hilbert space. By $\mathcal{D}(T)$, $\rho(T)$ and $\sigma(T)$ we denote the domain, the resolvent set and the spectrum of a linear operator $T$ respectively. For a selfadjoint operator $T$ let $E_T(\delta)$ be the spectral measure of a Borel set $\delta \subset \mathbb{R}$ and

$$2T_\pm := |T| \pm T.$$ 

By $S_{\infty}$ we denote the space of compact operators. For $T = T^* \in S_{\infty}$ and $s > 0$ let $n_\pm(s, T) = \text{rank} \ E_T(s, +\infty)$, and for $T \in S_{\infty}$ and $s > 0$ let $n(s, T) = n_+(s^2, T^*T)$. Recall \cite{1} that for a pair of compact operators the following two inequalities hold:

$$n(s_1 + s_2, T_1 + T_2) \leq n(s_1, T_1) + n(s_2, T_2), \quad s_1, s_2 > 0,$$

and (the Horn inequality)

$$n(s_1, T_1) + n(s_2, T_2), \quad s_1, s_2 > 0. \quad (1.1)$$

These inequalities are applicable not only to compact operators but at least to all bounded normal operators for which the right hand side is finite. Moreover one can write the estimates which are equivalent to the Weyl inequalities for selfadjoint operators $T_1, T_2$:

$$n_\pm(s_1 + s_2, T_1 + T_2) \leq n_\pm(s_1, T_1) + n_\pm(s_2, T_2), \quad s_1, s_2 > 0.$$

For $0 < p < \infty$ the class $S_p$ is defined as the set of all compact operators $T$ such that the following functional is finite:

$$\|T\|_{S_p}^p := \int_0^{\infty} s^p - 1 n(s, T) \ ds < \infty.$$
The functional \( \| \cdot \|_p \) is a norm for \( p \geq 1 \) and a quasinorm for \( p < 1 \). For \( 0 < p < \infty \) the class \( \mathcal{L}_p \) is defined as the set of all compact operators \( T \) such that the following functional (which is a quasinorm) is finite:
\[
\| T \|_p := \sup_{s > 0} s^n s^p(n, T) < \infty.
\]

2. Here we introduce some notions from the theory of index of a pair of projections. This material can also be found in \([2, 10]\).

Let \( R = R^* \) be a bounded operator and let the spectrum of \( R \) in the interval \( \delta = (a, b) \) be discrete or empty. Then for every selfadjoint compact operator \( K \) and \( \lambda \in \delta \) we introduce the number
\[
\eta(\lambda; R + K, R) := \text{index}(E_{R+K}(-\infty, \lambda), E_R(-\infty, \lambda)),
\]
where
\[
\text{index}(P, Q) = \dim \ker(P - Q - I) - \dim \ker(P - Q + I).
\]

Some properties of the function \( \eta \) should be mentioned here. For example, we have the “chain rule,”
\[
\eta(\lambda; R + K_1 + K_2, R) = \eta(\lambda; R + K_1 + K_2, R + K_1) + \eta(\lambda; R + K_1, R),
\]
which holds for \( K_j = K_j^* \in \mathbb{S}_\omega, j = 1, 2 \). For signdefinite perturbations \( K \) the following Birma–Schwinger principle holds true.

**Proposition 1.1.** Let \( K = \pm G^* G \) where \( G \in \mathbb{S}_\omega \) and let \( \lambda \in \rho(R) \). Then
\[
\eta(\lambda; R + K, R) = \pm \lim_{s \to 1 \mp 0} n_{\pm}(s, T), \tag{1.2}
\]
where
\[
T = G(R - \lambda I)^{-1} G^*.
\]

**Proof.** In fact Proposition 1.1 is a slight modification of Corollary 4.8 of \([10]\). In particular it was shown that under the conditions \( \pm K \geq 0, K \in \mathbb{S}_1, \lambda \in \rho(R + K) \) the value of \( \pm \eta(\lambda; R + K, R) \) coincides with the number of eigenvalues of \( R + tK \) which cross \( \lambda \) as \( t \) grows from 0 to 1. Thus, according to the classical Birma–Schwinger principle (see, for example, \([3]\)) the relation (1.2) holds true for \( K \in \mathbb{S}_1 \). Moreover it was proved in \([10]\) that if
\[ \lambda \in \rho(R+K), \text{ then the left hand side of (1.2) is continuous with respect to small perturbations of } K \text{ in the operator norm. Therefore for general } K \in S_\infty, \lambda \in \rho(R+K), \text{ the equality (1.2) is obtained by } S_\infty \text{-approximation of } K. \] Now note that both sides of (1.2) are left continuous with respect to \( K \).

Therefore the assumption \( \lambda \in \rho(R+K) \) becomes unnecessary.

As a consequence we obtain

**Proposition 1.2.** Let \( \lambda \in \rho(R) \) and \( 2K_\pm = |K| \pm K \). Then

\[
\lim_{s \to 0^-} n_-(s, T^+) \leq \eta(\lambda; R+K, R) \leq n_+(1, T^-),
\]

(1.3)

where

\[
T^\pm = \sqrt{K_\pm}(R-\lambda I)^{-1} \sqrt{K_\pm}.
\]

**Proof.** By the chain rule

\[
\eta(\lambda; R+K, R) = \eta(\lambda; R+K, R \pm K_\pm) + \eta(\lambda; R \pm K_\pm, R).
\]

Thus, according to Proposition 1.1, it remains to prove that

\[
\pm \eta(\lambda; R+K, R \pm K_\pm) \geq 0.
\]

But this also follows from the Proposition 1.1 if we assume that \( \lambda \in \rho(R \pm K_\pm) \).

It is enough to note that the last condition is irrelevant since the function \( \eta \) is continuous from the left with respect to \( \lambda \) and \( \mu \in \rho(R \pm K_\pm) \) for sufficiently small \( \lambda - \mu > 0 \).

3. Let now \( H_0, V \) be selfadjoint operators in the Hilbert space \( \mathcal{H} \) and \( J = J^* \) be the sign of \( V \). Suppose that \( |H_0 + iI|^{1/2} \in \mathcal{D}(|V|^{1/2}) \) and

\[
|V|^{1/2} |H_0 + iI|^{-1/2} \in S_\infty.
\]

(1.4)

The family of selfadjoint operators

\[
H(\alpha) = H_0 + \alpha V, \quad \alpha > 0,
\]

is introduced via the second resolvent identity (see Section 6 or [22, Sects. 1.9, 1.10]). For a real valued function \( f(\cdot) \) (defined on \( \mathbb{R} \)) such that

\[
\pm f(\pm 1) > 0,
\]
we introduce the operator

\[ V_f = f(J) |V| \]

and define the family of perturbed operators

\[ H_f(\alpha) = H_0 + \alpha V_f, \quad \alpha > 0. \]

The main purpose of our paper is to describe the properties of the spectrum of \( H(\alpha) \) as \( \alpha \) grows. The family of operators \( H_f(\alpha) \) plays only an auxiliary role. We shall always assume that the limit

\[ X_\lambda := \lim_{\varepsilon \to \pm 0} \sqrt{|V|} (H_0 - \varepsilon I)^{-1} \sqrt{|V|} \tag{1.5} \]

exists in the operator norm for a.e. \( \lambda \) and for these \( \lambda \)

\[ B_\lambda := \text{Im} X_\lambda \in S_1. \tag{1.6} \]

Below we also use the notation

\[ A_\lambda := \text{Re} X_\lambda, \quad \lambda \in \mathbb{R}. \]

For a.e. \( \lambda \in \mathbb{R} \), we define the function

\[ \zeta(\lambda; H(\alpha), H_0) = \int_{-\infty}^{+\infty} \eta(0; J + \alpha (A_\lambda + t B_\lambda), J) \, dw(t), \tag{1.7} \]

\[ dw(t) = \pi^{-1}(1 + t^2)^{-1} \, dt, \]

whose relation to the spectral shift function (s.s.f.) was established in [10, 18]. Namely, it was shown that \( \zeta \) coincides with the spectral shift function under quite wide conditions on \( V, H_0 \).

However, we prefer to call \( \zeta \) the “generalized spectral shift function” since the quantity (1.7) could be well defined even if the usual definition of the spectral shift function can not be applied. For convergence of the integral (1.7) see Subsection 6.2.

If the limit (1.5) exists and the relation (1.6) is fulfilled only for a.e. \( \lambda \in A \), where \( A \) is a measurable subset of \( \mathbb{R} \) then we say that the function (1.7) is defined on \( A \).

We are interested in the asymptotics of the function \( \zeta(\lambda; H(\alpha), H_0) \) as \( \alpha \to \infty \).
Here we present our main result which requires some additional conditions. In most of applications to differential operators there exists a point \( \mu = \bar{\mu} \in \rho(H_0) \), for which the asymptotics of \( \zeta(\mu; H_0, H_0) \) is already known (see \([19, 20]\)). Very often we can impose the following condition: there exists a point \( \mu = \bar{\mu} \in \rho(H_0) \) and a number \( p > 0 \) such that

\[
\zeta(\mu; H_0(x), H_0) \sim (C_+ |f(1)|^p + C_- |f(-1)|^p) x^p, \\
\alpha \to \infty, \quad \pm f(\pm 1) > 0, 
\]

where the constants \( C_\pm \) do not depend on \( f \) (but may depend on \( V_\pm \)).

It is natural to say that \( C_+ |f(1)|^p x^p \) and \( -C_- |f(-1)|^p x^p \) are contributions of \( V_+ \) and \( V_- \) to the asymptotics of \( \zeta \). The condition (1.8) means that these contributions are homogeneous of order \( p \) with respect to the multiplication of \( V_+ \) or \( V_- \) by positive constants. Roughly speaking, (1.8) relates the spectral asymptotics for \( V \) to the difference of the separate asymptotics for \( V_+ \) and \( V_- \). It was already mentioned that (1.8) is fulfilled in most of applications.

Our second condition will be imposed on the operator

\[
Q = Q(t) := \alpha(A_\alpha + tB_\alpha - X_\alpha),
\]

which is the variation of the second argument of the function \( \eta \) when we pass in (1.7) from \( \lambda \) to the point \( \mu \). This operator should be “small” in the sense that

\[
\int_{-\infty}^{+\infty} m(\epsilon, Q(t))(1 + t^2)^{-1} dt = o(x^p), \quad \alpha \to \infty, \quad \forall \epsilon > 0. \quad (1.9)
\]

**Theorem 1.1.** Let \( \mu = \bar{\mu} \in \rho(H_0) \) and let the conditions (1.8), (1.9) be fulfilled. Then for the function

\[
\psi(\alpha) := \zeta(\lambda; H(\alpha), H_0) - \zeta(\mu; H(\alpha), H_0)
\]

the following relation holds for a.e. \( \lambda \in \mathbb{R} \)

\[
\psi(\alpha) = o(x^p), \quad \alpha \to \infty.
\]

Theorem 1.1 says that under conditions (1.8), (1.9) the asymptotics of \( \zeta \) as \( \alpha \to \infty \) does not depend on \( \lambda \).
2. PROOF OF THEOREM 1.1

1. We start with the following auxiliary

**Proposition 2.1.** Assume that \( f(-1) < 0 \) and \( f(1) > 0 \). Then

\[
\zeta(\lambda_1 H_f, H_0) = \int_{-\infty}^{+\infty} \eta(0; f(J)^{-1} + \pi(A_J + tB_J), f(J)^{-1}) \, dw(t),
\]

\[
dw(t) = \pi^{-1}(1 + t^2)^{-1} \, dt.
\]

**Proof.** For every pair of selfadjoint bounded operators \( R, K \), such that \( 0 \not\in \rho(R) \) and \( K \in S_\infty \) we introduce

\[
R_s = SRS, \quad K_s = SKS,
\]

where \( S = S^* \) is a bounded invertible operator. Then in order to establish (2.10) by using (1.7) it is sufficient to prove that

\[
\eta(0; R_s + K_s, R_s) = \eta(0; R + K, R).
\]

Moreover the proof of (2.11) can be reduced to the cases \( K \geq 0 \) and \( K \leq 0 \). But if \( K \) is of definite sign, the quantity \( \eta(0; R + K, R) \) coincides with the number of eigenvalues of the operator \( R + tK \) which pass point 0 as \( t \) grows from 0 to 1. Therefore (2.11) follows from equivalence of the two statements:

\[
0 \not\in \sigma(R + tK) \iff 0 \not\in \sigma(R_s + tK_s).
\]

Now to complete the proof we take \( S = |f(J)|^{-1/2} \), \( R = \text{sign } f(J) \) and \( K_s = \pi(A_J + tB_J) \) in (2.11).

2. The rest of the section is devoted to the proof of Theorem 1.1. Consider the function

\[
\eta(0; J + \pi(A_J + tB_J), J).
\]

To compare this function with \( \eta(0; J + \pi X_\mu, J), \mu \in \rho(H_0) \), we use its additivity,

\[
\zeta(\lambda) := \eta(0; J + \pi(A_J + tB_J), J) - \eta(0; J + \pi X_\mu, J)
\]

\[
= \eta(0; J + \pi(A_J + tB_J), J + \pi X_\mu).
\]
Denoting \( \mathcal{Y} := \{ x \in \mathbb{R}_+ : 0 \in \rho(J + \alpha X_\mu) \} \) and using Proposition 1.2 we obtain

\[
- \lim_{\varepsilon \to 0} n_-(s, T^{(s)}) \leq \zeta(s) \leq n_+(1, T^{(-1)}), \quad \alpha \in \mathcal{Y},
\]

where

\[
T^{(\pm)} := \sqrt{Q_\pm} (J + \alpha X_\mu)^{-1} \sqrt{Q_\pm}
\]

and

\[
Q = Q(t) = \pi(A_\mu + tB_\mu - X_\mu).
\]

By the Horn inequality (1.1),

\[
n_\pm(1, T^{(\mp)}) \leq 2n(\varepsilon, Q) + n(\varepsilon^{-1}, (J + \alpha X_\mu)^{-1}).
\]

Since the first term in the right side obeys the estimate

\[
\int_{-\infty}^{+\infty} n(\varepsilon, Q(t))(1 + t^2)^{-1} dt = o(\varepsilon^n), \quad \alpha \to \infty,
\]

it is sufficient to establish that

\[
\limsup_{\varepsilon \to 0} \varepsilon n(\varepsilon^{-1}, (J + \alpha X_\mu)^{-1}) \leq C(\varepsilon)
\]

and

\[
\lim_{\varepsilon \to 0} C(\varepsilon) = 0. \tag{2.12}
\]

In order to prove the latter, we use the equality

\[
n(\varepsilon^{-1}, (J + \alpha X_\mu)^{-1}) = \text{rank } E_{(J + \alpha X_\mu)(-\varepsilon, \varepsilon)},
\]

which gives the estimate

\[
n(\varepsilon^{-1}, (J + \alpha X_\mu)^{-1}) \leq \eta(\varepsilon; J + \alpha X_\mu, J) - \eta(-\varepsilon; J + \alpha X_\mu, J) =: m(\varepsilon).
\]

From the definition of the function \( \eta \) we obtain

\[
m(\varepsilon) = \eta(0, (f_+(J) + \alpha X_\mu), f_+(J)) - \eta(0, (f_-(J) + \alpha X_\mu), f_-(J)),
\]

where

\[
f_\pm(J) = J^{\mp} \varepsilon I.
\]
Now everything is prepared for a representation of $m(\alpha)$ in terms of spectral shift functions. Namely, denoting

$$H^\pm(\alpha) = H_0 + \alpha f_\pm(J)^{-1} |V|$$

and using the representation for $\zeta(\mu, H^\pm(\alpha), H_0)$ (see (2.10)), we obtain

$$m(\alpha) = \zeta(\mu, H^+(\alpha), H_0) - \zeta(\mu, H^-(\alpha), H_0).$$

Therefore

$$m(\alpha) \sim (C_+ + C_-)[(1 - \epsilon)^{-p} - (1 + \epsilon)^{-p}] \propto, \quad \alpha \to \infty,$$

and

$$0 \leq \lim_{\epsilon \to 0} C(\epsilon) \leq \lim_{\epsilon \to 0} \limsup_{\alpha \to \infty} \alpha^{-p} m(\alpha) = 0.$$ 

The proof is complete.

3. LIMITS OF COMPACT OPERATORS

Here we present some conditions on $V$ and $H_0$ which ensure existence of (1.5) and guarantee (1.6). We begin with formulating a statement which immediately follows from the results of [15] (see also [4]). For $0 < q \leq 1$ we define $q^*$

$$q^* = \begin{cases} q, & \text{if } q < 1; \\ q^* > 1, & \text{is arbitrary, if } q = 1. \end{cases}$$

**Proposition 3.1.** Assume that for every bounded open interval $\delta \subset \mathbb{R}$ the following inclusion holds:

$$|V|^{1/2} E_{H_0}(\delta) \in S_{2q^*}, \quad 0 < q \leq 1. \quad (3.13)$$

Then for a.e. $\lambda \in \mathbb{R}$ the limit

$$X(\lambda, \delta) := \lim_{\epsilon \to 0} \sqrt{|V|} E_{H_0}(\delta)(H_0 - \lambda I)^{-1} \sqrt{|V|}$$

exists in the $S_{q^*}$-norm and

$$\text{Im} X(\lambda, \delta) \in S_{q^*}.$$
The following statement is a direct consequence of Proposition 3.1; moreover it can be found in [17]. (A slightly different version is given in [22].)

**Proposition 3.2.** Assume that for every bounded interval \( \delta \subset \mathbb{R} \)
\[
|V|^{1/2} E_{H_0}(\delta) \in S_2. \tag{3.14}
\]
Then for a.e. \( \lambda \in \mathbb{R} \) the limit (1.5) exists and the condition (1.6) is fulfilled.

The next statement was proved in [17] for the case of semibounded operators \( H_0 \). The general case (when \( H_0 \) is not semibounded) does not require any changes in the proof.

**Proposition 3.3.** Let the condition (3.13) be satisfied and
\[
p \geq q, \quad \text{if } q < 1;
p > 1, \quad \text{if } q = 1.
\]
Assume that
\[
|V|^{1/2} |H_0 + iI|^{-1/2} \in \mathcal{S}_{2p}. \tag{3.15}
\]
Then for a.e. \( \lambda \in \mathbb{R} \) the condition (1.9) holds.

4. REGULAR POINTS OF \( H_0 \)

For applications of our abstract result we need to establish some conditions which guarantee (1.8). For a regular point \( \mu \in \rho(H_0) \) we introduce the operators
\[
X^\pm_\mu := \sqrt{V}_\pm (H_0 - \mu I)^{-1} \sqrt{V}_\pm
\]
and
\[
X^0_\mu := \sqrt{V}_+ (H_0 - \mu I)^{-1} \sqrt{V}_-.
\]
Note that if \( V \) commutes with \( H_0 \), then \( X^0_\mu = 0 \) and therefore \( \xi(\mu; H_0(x), H_0) = n_-(x^{-1}, f(1) X^+_{\mu}) - n_+(x^{-1}, f(-1) X^-_{\mu}) \). In most of applications \( X^0_\mu \neq 0 \) but is small in the asymptotic sense (see (4.17) below).
Theorem 4.1. Let \( \mu = \bar{\mu} \in \rho(H_0) \). Assume that there exist constants \( C_0^+ \), \( C_0^- \) such that

\[
\lim_{s \to 0} n_+(s, X_\mu^+) \sim C_0^+ s^{-p}, \quad s \to 0, \quad p > 0,
\]

and

\[
m(s, X_\mu^0) = o(s^{-p}), \quad s \to 0.
\] (4.17)

Then (1.8) is fulfilled with \( C_\pm = C_\pm^0 \), in particular

\[
\zeta(\mu; H(x), H_0) \sim (C_+^0 - C_-^0) x^p, \quad x \to \infty.
\] (4.18)

Proof. It is sufficient to establish (4.18). In our case

\[
\zeta(\mu; H(x), H_0) = \eta(0; J + \alpha X_\mu, J).
\]

We are going to compare this function with

\[
\eta(0; J + \alpha L, J), \quad L := X_\mu^+ + X_\mu^-.
\]

The chain rule implies

\[
\zeta_0(x) := \eta(0; J + \alpha X_\mu, J) - \eta(0; J + \alpha L, J)
\]

\[= \eta(0; J + \alpha X_\mu, J + \alpha L). \]

Denoting \( \mathcal{H}_0 := \{x \in \mathbb{R}_+ : 0 \in \rho(J + \alpha L)\} \) and using Proposition 1.2 we obtain

\[
- \lim_{s \to 1-0} n_-(s, T_0^{(+)}) \leq \zeta_0(x) \leq n_+(1, T_0^{(-)}), \quad x \in \mathcal{H}_0,
\]

where

\[
T_0^{(\pm)} := \sqrt{Q_\pm} (J + \alpha L)^{-1} \sqrt{Q_\pm}
\]

and

\[
Q^0 := \alpha(X_\mu - L).
\]

By the Horn inequality (1.1),

\[
n_\pm(1, T_0^{(\pm)}) \leq 2n(e, Q^0) + m(e^{-1}, (J + \alpha L)^{-1}).
\]

By (4.17) for the first term in the right hand side we obtain

\[
m(e, Q^0) = o(e^p), \quad x \to \infty.
\]
It remains to prove that
\[ \lim \sup_{n \to \infty} n(\alpha_n^{-1}, (J + \alpha L)^{-1}) \leq C(\varepsilon) \]
and
\[ \lim_{\varepsilon \to 0} C(\varepsilon) = 0, \]
which can be done in the similar way as in the proof of (2.12). Theorem 4.1 is proved.

5. APPLICATIONS

In this section we give some applications of Theorem 1.1 to the spectral theory of partial differential operators.

1. Below we write \( \mathcal{H} := L^2(\mathbb{R}^d) \) and denote \( D_j = -i\partial/\partial x_j \), \( D = -iV = (D_1, \ldots, D_d) \). We also set \( \omega_d = \text{vol} \{ x \in \mathbb{R}^d : |x| < 1 \} \).

The first example deals with the Dirac operator perturbed by a decreasing electric potential. Let \( g = (g_1, g_2, g_3) \) and \( g_0 \) be \((4 \times 4)\)-Dirac matrices; \( I \) denotes the unit matrix. The Dirac matrices satisfy the relations
\[ g_j g_k + g_k g_j = \delta_{jk} I, \quad j, k = 0, 1, 2, 3. \] (5.19)

Let us consider the unperturbed Dirac operator in \( \mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^4) \)
\[ H_0 = g \cdot D + g_0, \]
and perturb the operator \( H_0 \) by a real potential
\[ H(\alpha) = H_0 + \alpha V, \quad \alpha > 0, \] (5.20)
\[ V \in L_3(\mathbb{R}^3), \quad \bar{V} = V. \] (5.21)

The operator (5.20) needs to be correctly defined. Under the condition (5.21) it is, in general, impossible to introduce the operator as the difference of two operators, but it can be defined via the second resolvent identity. The corresponding scheme is given in [22, Sects. 1.9 and 1.10] (see also Section 6 of the present paper).
The spectrum of the operator $H_0$ is absolutely continuous and covers the complement of the interval $A = (-1, 1)$. The essential spectrum of the operator $A(x)$ coincides with the spectrum of $H_0$. Besides, the operator $H(x)$ has a discrete spectrum in the gap $A$. It is clear that the function (1.7) of the pair $H(x)$ and $H_0$ exists on the interval $A$.

**Theorem 5.1.** Let $H_0$ be the Dirac operator and $\mu \in A$. Under the condition (5.21) the following asymptotics holds

$$
\zeta(\mu; H(x), H_0) \sim \frac{1}{3\pi^2} \int V^3_+ dx - \int V^3_- dx, \quad x \to \infty.
$$

**Proof.** It is sufficient to note that the condition (4.16), (4.17) are fulfilled with $p = 3$ and

$$
C_+^0 = \frac{1}{3\pi^2} \int V^3_+ dx.
$$

For the reference concerning (4.16) see [6, 12]. The relation (4.17) is obtained in [20]. In fact the proof of (5.22) can also be found in [20].

Now we are going to apply our abstract theorem to the Dirac operator.

Note that the condition (3.14) is fulfilled if and only if

$$
V \in L_1(\mathbb{R}^3).
$$

The inclusion (3.15) follows from the results of [6], but it can be also found in [20]. Combining Theorem 1.1 with Proposition 3.3 and Theorem 5.1, we obtain

**Theorem 5.2.** Let $H_0$ be the Dirac operator. Under the condition

$$
V \in L_1(\mathbb{R}^3) \cap L_1(\mathbb{R}^3)
$$

the following asymptotics holds for a.e. $\lambda \in \mathbb{R}$

$$
\zeta(\lambda; H(x), H_0) \sim \frac{1}{3\pi^2} \int V^3_+ dx - \int V^3_- dx, \quad x \to \infty.
$$

Now keeping $H_0$ as above, assume that $V$ is the operator of multiplication by a matrix

$$
V(x) = g \cdot A(x) = \sum_{j=1}^{3} g_j A_j,
$$
where
\[ \mathbf{A}(x) = (A_1, A_2, A_3), \quad A_j(x) \in \mathbb{R}, \quad j = 1, ..., 3, \]
is the magnetic vector potential
\[ \mathbf{A} \in L_3(\mathbb{R}^3) \cap L_1(\mathbb{R}^3). \] (5.25)

Then obviously
\[ \left( \prod_{j=0}^{3} g_j \right) H(x) = -H(x) \left( \prod_{j=0}^{3} g_j \right) \]
and therefore the point spectrum of \( H(x) \) is symmetric with respect to zero, i.e.,
\[ \zeta(\lambda; H(x), H_0) = -\zeta(-\lambda; H(x), H_0), \quad \forall \lambda > 0, \quad \lambda \in \mathbb{R}. \]

Consequently
\[ \zeta(0; H(x), H_0) = 0, \quad \forall \lambda > 0. \]

Applying Theorem 1.1 we obtain

**Theorem 5.3.** Let \( H(x) \) be the magnetic Dirac operator and the vector potential satisfies (5.25). Then for a.e. \( \lambda \in \mathbb{R} \)
\[ \zeta(\lambda; H(x), H_0) = o(\lambda^3), \quad \lambda \to \infty. \]

2. Considering the next example we set \( \mathcal{H} = L_2(\mathbb{R}^d) \) and as \( H_0 \) we take the polyharmonic operator, i.e.,
\[ H_0 = (-\Delta)^s, \quad s > 0. \] (5.26)

For simplicity we assume that
\[ s = d/2s > 1. \]

Introduce the perturbed operator \( H(x) = H_0 + \alpha V \), where \( V \) is a measurable real-valued function, such that
\[ V \in L_{\infty}(\mathbb{R}^d). \] (5.27)

In this case (see [3, Sect. 2]) the condition (3.15) is fulfilled with
\[ p = \infty. \]
The relation (3.14) holds if and only if

\[ V \in L_1(\mathbb{R}^d). \]  \hspace{1cm} (5.28)

Now we may apply Theorem 1.1, putting \( \mu = -1 \).

**Theorem 5.4.** Let \( H_0 \) be defined as in (5.26) and let (5.27), (5.28) be fulfilled. If \( \varepsilon = d/2 \gamma > 1 \), then for a.e. \( \lambda = \tilde{\lambda} \in \mathbb{R} \) we have

\[ -\zeta(\tilde{\lambda}; H(x), H_0) = (2\pi)^{-d} \omega_d x^\tilde{\lambda} \int V^\infty \, dx + o(x^\tilde{\lambda}), \quad x \to \infty. \]  \hspace{1cm} (5.29)

**Proof.** If \( \mu = -1 \), then \( \zeta(\mu; H(x), H_0) = \eta(0; J + X_\mu, J) \). Now according to Proposition 1.1 we obtain the equality

\[ \zeta(\mu; H(x), H_0) = -n_-(1 - 0, xP), \]

where

\[ P = (H_0 + 1)^{-1/2} \left( H_0 + 1 \right)^{-1/2}. \]

By the minimax principle, the quantity \( n_-(1 - 0, xP) \) coincides with the maximal dimension of subspaces \( \mathcal{F} \subset \mathcal{H} \) such that

\[ -\varepsilon(Pu, u) \geq (u, u), \quad \forall u \in \mathcal{F}. \]

The substitution \( u = (H_0 + 1)^{1/2} v \) leads us to the inequality

\[ -\varepsilon(Vu, v) \geq \| (H_0 + 1)^{1/2} v \|^2. \]

Therefore, again by the minimax principle, we conclude that

\[ n_-(1 - 0, xP) = \text{rank} \, E_{H_0(\infty, -\infty, -1)}. \]  \hspace{1cm} (5.30)

The asymptotics of the right hand side of (5.30) is well known (see [8] and references therein) and coincides with the r.h.s. of (5.29). Thus the full-scale assertion of the theorem is a consequence of Theorem 1.1 and Proposition 3.3.

Obviously instead of the polyharmonic operator we can take more general operators. For example, if \( s = 1 \) then instead of (5.26) we can consider the Schrödinger operator

\[ H_0 = -A + f(x), \quad x \in \mathbb{R}^d, \quad d \geq 3. \]
where $f \in L_\infty$ is a real valued function. For the case when the perturbation is the multiplication by a function

$$V \in C^\infty_0(\mathbb{R}^d),$$

the condition (3.14) has been established in [17] (under much weaker assumptions) and therefore

$$-\zeta(\lambda; H(\lambda), H_0) \sim (2\pi)^{-d} \omega_2 \lambda^{d/2} \int V d^2 x, \quad \lambda \to \infty.$$ 

For no-signdefinite $V$ this asymptotics has been known only for the case when $\lambda$ lies in a gap of $\sigma(H_0)$. Now we are able to control the “spectral flow” on the continuous spectrum. Note once again that (5.31) is not the most general condition on $V$. Less restrictive conditions for the case $V \leq 0$ (in terms of $L_p$-classes) are given in [17].

6. APPENDIX

1. Let us give the construction of the operator $H(\lambda)$, which corresponds to the formal sum $H_0 + \lambda V$. For $z \in \rho(H_0)$ define the following operator of the class $S_\infty$:

$$T(z, \lambda) = \lambda(\sqrt{|V|} (|H_0| + I)^{-1/2}) \frac{|H_0| + I}{H_0 - z I} (\sqrt{|V|} (|H_0| + I)^{-1/2})^*.$$ 

It is easy to check (see, e.g., [22, Lemma 1.10.5]) that

$$0 \in \rho(I + JT(z, \lambda)) \quad \text{for all} \quad z \in \mathbb{C} \setminus \mathbb{R}.$$ 

Under the assumption (1.4), there exists a unique self-adjoint operator $H(\lambda)$ (see [22, Sects. 1.9, 1.10]), such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ its resolvent satisfies the equation

$$(H(z) - z I)^{-1} - (H_0 - z I)^{-1} = -\lambda(\sqrt{|V|} (H_0 - z I)^{-1} (I + JT(z, \lambda))^{-1} (J \sqrt{|V|} (H_0 - z I)^{-1})).$$

2. Finally we establish that the integral (1.7) converges. By the chain rule,

$$\forall t \geq 0 \quad \psi(t) := \eta(0; J + \lambda(A_2 + tB_2), J) - \eta(0; J + \lambda A_2, J) = \eta(0; J + \lambda(A_2 + tB_2), J) + \lambda A_2).$$
Assume that $0 \in \rho(J + \varkappa A)$. (6.32)

Then using Proposition 1.2 we obtain

$$-n_-(1 - 0, T^{(1)}) \leq \psi(t) \leq n_+(1, T^{(-)}),$$

where

$$T^{(\pm)} := \sqrt{\Pi_{\pm}} (J + \varkappa A)^{-1}_\pm \sqrt{\Pi_{\pm}}$$

and

$$\Pi = \alpha B_\perp.$$

By the Horn inequality (1.1),

$$n_{\pm}(1, T^{(\mp)}) \leq 2n(\varepsilon, \Pi) + n(\varepsilon^{-1}, (J + \varkappa A)^{-1}).$$

Taking into account that $n(\varepsilon^{-1}, (J + \varkappa A)^{-1}) = \text{rank} E_{J + \varkappa A}[-\varepsilon, \varepsilon]$ we obtain

$$|\psi(t)| \leq 2n(\varepsilon, \alpha B_\perp) + \text{rank} E_{J + \varkappa A}[-\varepsilon, \varepsilon]. \quad (6.33)$$

Note that (6.33) holds true without the assumption (6.32). Since $B_\perp \in S_1$, the following integral is finite

$$\int_{-\infty}^{+\infty} n(\varepsilon, \alpha B_\perp)(1 + t^2)^{-1} dt < \infty.$$

Consequently the integral (1.7) converges.

REFERENCES


13. M. G. Krein, On the trace formula in perturbation theory, *Mat. Sb.* **33** (75), No. 3 (1953), 597–626. [In Russian]


