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Estimates for Hilbertian Koszul homology

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Abstract

The objective of this paper is to give new kind of estimates for Hilbertian Koszul homology, inspired by commutative algebra, in multivariable Fredholm theory.

Keywords: Multivariable Fredholm theory; Koszul complex; Homology; Lech's formula

0. Introduction

The Fredholm index of a single operator admits a generalization to several variables via Koszul complexes over Hilbert spaces, which is, in general, difficult to calculate. In particular, in sharp contrast with rich results on Noetherian algebraic modules, over Hilbert modules currently there are essentially no systematic estimates for higher Koszul homology groups.

In [13–15], we initiated a study of Fredholm theory through the asymptotic behavior of higher powers of a tuple $\bar{T}$. See also Eschmeier's [12]. In this paper, the asymptotic methods lead to estimates for all powers of $\bar{T}$.

Let $\bar{T} = (T_1, \dots, T_n)$ $(n \in \mathbb{N})$ be a Fredholm tuple of commuting operators on a Hilbert space H. This means that the homology groups $H_i(K(T_1, \dots, T_n))$ $(i = 0, 1, \dots, n)$ of the associated Koszul complex $K(T_1, \dots, T_n)$ of T_i over the Hilbert space H are all finite-dimensional. Let $\mathrm{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ be a multi-index, and $\bar{T}^{\mathrm{k}} = (T_1^{k_1}, \dots, T_n^{k_n})$. If $\bar{T}$ is Fredholm, then so

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is $\bar{T}^{\mathrm{k}}$. For convenience, let $h_i(k_1, \dots, k_n) = \dim(H_i(K(T_1^{k_1}, \dots, T_n^{k_n})))$. The main result of this paper is

Theorem 1. *For any Fredholm tuple $(T_1, \dots, T_n)$, there exist $e_0, e_1, \dots, e_n \in \mathbb{Z}$, and a constant $C > 0$ such that for all $i = 0, 1, \dots, n$, and $k_1, k_2, \dots, k_n \in \mathbb{N}$,*

$$k_1 k_2 \cdots k_n \cdot e_i \leqslant h_i(k_1, \dots, k_n) \leqslant k_1 k_2 \cdots k_n \left(e_i + \frac{C}{\min k_i} \right).$$

A few remarks follow:

- Considering the multi-index k is indeed useful, say, in [14], where $n = 2$, and $\sup_k h_i(1, k) < \infty$ implies $e_i = 0$.
- Clearly, our result implies that $e_i = \lim_{k \to \infty} \frac{h_i(T_1^k, \dots, T_n^k)}{k^n}$ (see Corollary 2.3 in [12]) and that index $\mathrm{index}(\bar{T}) = \sum_{i=0}^{n} (-1)^i e_i$ by the multiplicity formula $\mathrm{index}(T_1^{k_1}, \dots, T_n^{k_n}) = k_1 \cdots k_n \,\mathrm{index}(T_1, \dots, T_n)$.
- When H is replaced by a finitely generated module over a Noetherian ring, the corresponding function h_i is dominated by a polynomial of k_i with degree $n - i$, hence $e_i = 0$ except for possibly e_0 [26]. It is not clear whether $\lim_{k \to \infty} \frac{h_i(k, \dots, k)}{k^{n-i}}$ exists.

Two main ingredients in the proof of Theorem 1. Many arguments in this paper refine those of Eschmeier's [12] in order to obtain quantitative results. *The first set of techniques is sheaf theoretic*. First touched upon by Markoe [22], sheaf theory for operators was systematically investigated later [25], and the primary reference is the monograph [11]. *The second set is commutative algebra in nature,* and is more closely related to our previous work. In particular, we own a deep intellectual debt to C. Lech [14,18,19], from which we borrow many ideas.

Both sets of techniques are well known, and in fact easy, to experts in algebra and analysis, respectively. What we do here is to bring them together to yield estimates which appear of value in operator theory.

1. Background

Definitions. In order to study the spectral theory of a tuple of commuting operators, instead of a single operator, J.L. Taylor, in 1970, introduced a seminal approach via Koszul complexes over Banach spaces [28,29]. For a commuting tuple $\bar{T} = (T_1, \dots, T_n)$ on a Banach space H, its Koszul complex $K(T_1, \dots, T_n)$ is

$$K(\bar{T}): \quad 0 \to H \otimes \bigwedge^n \mathbb{C}^n \to H \otimes \bigwedge^{n-1} \mathbb{C}^n \to \cdots \to H \otimes \bigwedge^0 \mathbb{C}^n \to 0.$$

Here $\bigwedge^n \mathbb{C}^n$ is the kth exterior power of $\mathbb{C}^n$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for $\mathbb{C}^n$, and let c_i be the creation operator associated with e_i, that is, $c_i(\xi) = e_i \wedge \xi$ for $\xi \in \bigwedge \mathbb{C}^n$. Then the boundary operator is $B = T_1 \otimes c_1^* + \cdots + T_n \otimes c_n^*$. The tuple $(T_1, \dots, T_n)$ is called *invertible* if the complex $K(\bar{T})$ is exact.

Subsequently, a multivariable Fredholm theory is formulated: a tuple $\bar{T}$ of commuting operators is *Fredholm* if $K(\bar{T})$ has a finite-dimensional homology group at each stage, that is,

$$\dim_{\mathbb{C}} H_i\big(K(\bar{T})\big) < \infty$$

for all $i = 0, 1, \dots, n$ [1,2,7,8,11,30]. We also write $H_i(\bar{T})$ instead of $H_i(K(\bar{T}))$ for convenience. The $n+1$ homology groups of $K(\bar{T})$ are the multivariable analogs of the kernel $\ker(T)$ and cokernel H/TH of a single operator $T \in B(H)$. When $(T_1, \dots, T_n)$ is Fredholm, define the multivariable *Fredholm index* by

$$\operatorname{index}(T_1, \dots, T_n) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H_i\big(K(\bar{T})\big),$$

the Euler characteristic of $K(\bar{T})$.

The multivariable index $\operatorname{index}(\bar{T})$ is connected with a variety of problems in both classical analysis and algebraic topology [3,11,20,21]. Currently, however, there is essentially no effective computational tools, especially for higher homology groups $H_i(\cdot)$, that is, for those groups with $i > 0$. Most known examples are, or are reduced to, acyclic tuples: $H_i(\cdot) = 0$ except for $i = 0$, hence $\operatorname{index}(\cdot) = \dim(H_0(\cdot))$. Consequently, there is a current need to get a better grasp on those higher homology groups.

Motivation. Our approach to $H_i(\cdot)$ originates from an effort to generalize the following simple arguments from [14] to several variables: for a single Fredholm operator T acting on a separable Hilbert space H, by the definition of Fredholm index, and the multiplicity formula,

$$\begin{aligned}
\operatorname{index}(T) &= \frac{\operatorname{index}(T^k)}{k} \\
&= \frac{\dim(\ker(T^k))}{k} - \frac{\dim(H/T^kH)}{k} \\
&= \lim_{k\to\infty} \frac{\dim(\ker(T^k))}{k} - \lim_{k\to\infty} \frac{\dim(H/T^kH)}{k}.
\end{aligned}$$

Here both limits exist, and are in fact integers. This leads to links to commutative algebra through the Hilbert function $k \to \dim(H/T^kH)$, [10], and a celebrated result of J.-P. Serre, relating the Euler characteristics of Koszul complexes to Samuel multiplicities [27].

2. Correction modules $C(M, L; J)$

This section is purely commutative algebra. We introduce a notion of correction modules, which, simple as it is, seems not discussed explicitly in literature. For operator theorists wanting more algebraic references, see standard texts [4,10] for Samuel multiplicity, and see [14,18,19], and [26] for Lech's formulas.

Definition 2. Let R be a ring, $J \subset R$ be an ideal, and $M \subset L$ be a submodule of an R-module L. Define the *correction module* of M in L with respect to an ideal J to be

$$C(M, L; J) = \frac{M \cap JL}{JM}.$$

Remark. When R and L are Noetherian, the Artin–Rees lemma is useful for the study of the asymptotic behavior of $C(M, L; J^k)$ when $k \to \infty$.

Lemma 3. *Let R be a local Noetherian ring, $I = (x_1, \ldots, x_n) \subset R$ be its maximal ideal, and $I_{\mathrm{k}} = (x_1^{k_1}, \ldots, x_n^{k_n})$ for any $\mathrm{k} \in \mathbb{N}^n$.*

If L is a finitely generated R-module, and $M \subset L$ is a submodule, then there exists a constant C such that for all $\mathrm{k} \in \mathbb{N}^n$,

$$\operatorname{length}\big(C(M, L; I_k)\big) \leqslant k_1 k_2 \cdots k_n \cdot \frac{C}{\min k_j}.$$

Proof. Let $N = L/M$ be the quotient module. For any ideal $J \subset R$, applying the functor $(\cdot) \otimes_R R/J$, which is only right-exact, to a short exact sequence of R-modules

$$0 \to M \to L \to N \to 0, \tag{2.1}$$

we get a right-exact sequence

$$\to M/JM \to L/JL \to N/JN \to 0. \tag{2.2}$$

By the definition of correction module, it follows

$$0 \to C(M, L; J) \to M/JM \to L/JL \to N/JN \to 0. \tag{2.3}$$

Consider $J = I_{\mathrm{k}}$, and by the algebraic Lech's formula (see Lemma 4), there exists a constant C_E for the modules $E = M$, L, or N, such that

$$k_1 \cdots k_n \cdot e(E) \leqslant \operatorname{length}(E/I_{\mathrm{k}}E) \leqslant k_1 \cdots k_n \left(e(E) + \frac{C_E}{\min k_j} \right). \tag{2.4}$$

Here

$$e(E) = n! \lim_{t \to \infty} \frac{\operatorname{length}(E/I^t E)}{t^n}$$

is the Samuel multiplicity of E with respect to I. By the additivity of Samuel multiplicity over short exact sequence (2.1), we have

$$e(L) = e(M) + e(N).$$

Now the proof is completed by observing

$$\operatorname{length}\bigl(C(M,L;I_{\mathrm{k}})\bigr)=\operatorname{length}(M/I_{\mathrm{k}}M)+\operatorname{length}(N/I_{\mathrm{k}}N)-\operatorname{length}(L/I_{\mathrm{k}}L)$$
$$\leqslant k_1k_2\cdots k_n\cdot\frac{C_M+C_N}{\min k_j}.\qquad\square$$

Remarks. (1) We derive the name of $C(M,L;J)$ from (2.3).

(2) For the proof of Theorem 1, the only case we need is $R=\mathcal{O}_0$, the local ring of germs of holomorphic functions around the origin in $\mathbb{C}^n$.

For readers' convenience we record the following.

Lemma 4 *(Lech's inequality). Let $J=(x_1,\dots,x_n)$ be an ideal of a local ring R, generated by x_i, and let M be a Noetherian R-module such that* $\operatorname{length}(M/JM)<\infty$*, then there exists a constant C such that*

$$k_1\cdots k_n e(M,J)\leqslant\operatorname{length}\bigl(M/\bigl(x_1^{k_1},\dots,x_n^{k_n}\bigr)M\bigr)\leqslant k_1\cdots k_n\left(e(M,J)+\frac{C}{\min_j k_j}\right),$$

here $e(M,J)$ is the Samuel multiplicity of M with respect to J.

The original proof of Lech is contained in the proof of Theorem 2 in [18], which in fact only covers the case $M=R$. The (Hilbert) module case is treated in [14]. Both proofs can be easily adopted to prove Lemma 4.

3. Difference between $H_p(L_\bullet/JL_\bullet)$ and $H_p(L_\bullet)/JH_p(L_\bullet)$ as correction modules

This section is still purely algebraic. Let R be any commutative ring, and

$$L_\bullet:\quad\cdots\to L_p\to L_{p-1}\to\cdots$$

be a complex of R-modules, with $H_p(L_\bullet)$ denoting the homology group at the pth stage, $p\in\mathbb{Z}$. For any ideal $J\subset R$, we represent the difference between $H_p(L_\bullet/JL_\bullet)$ and $H_p(L_\bullet)/JH_p(L_\bullet)$ as correction modules in this section. This is also considered in [12]. Here we refine the arguments in [12] and obtain more quantitative results.

Since the difference between $H_p(L_\bullet/JL_\bullet)$ and $H_p(L_\bullet)/JH_p(L_\bullet)$ is often encountered, say in base change theorems, in algebraic geometry, our results here may be of interests to algebraists.

Recall that for any R-module M, there exists a natural morphism (say, by [5])

$$H_p(L_\bullet)\otimes_R M\to H_p(L_\bullet\otimes_R M).$$

Here we will consider $M=J$, and R/J.

Let $Z_p\subset L_p$ be the set of closed elements, that is, $Z_p=\ker(L_p\to L_{p-1})$, and let $B_p\subset L_p$ be the set of boundary elements, that is, $B_p=\operatorname{Image}(L_{p+1}\to L_p)$. Note that $H_p(L_\bullet)=Z_p/B_p$.

Lemma 5. *For the natural morphism $j: H_p(L_\bullet)/JH_p(L_\bullet)\to H_p(L_\bullet/JL_\bullet)$, the cokernel is isomorphic to*

$$\operatorname{coker}(j)\cong C(B_{p-1},L_{p-1};J).$$

The kernel $\ker(j)$ *is resolved by an exact sequence of correction modules*

$$0 \to C(B_p, Z_p; J) \to C(B_p, L_p; J) \to C(Z_p, L_p; J) \to \ker(j) \to 0.$$

Proof. The standard strategy in algebra is to analyze the natural morphism

$$j : H_p(L_\bullet)/JH_p(L_\bullet) \to H_p(L_\bullet/JL_\bullet)$$

by embedding it into a commutative diagram. To resolve $H_p(L_\bullet/JL_\bullet)$, we consider the long exact sequence associated with

$$0 \to JL_\bullet \to L_\bullet \to L_\bullet/JL_\bullet \to 0,$$

and get the first row of the diagram (3.1). To resolve $H_p(L_\bullet)/JH_p(L_\bullet)$ we consider the straightforward short exact sequence which leads to the second row of the diagram (3.1). Together with the natural morphisms j_1, $j_2 = \mathrm{id}$, and j, we obtain a commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & H_p(JL_\bullet) \arrow[r, "d_1"] & H_p(L_\bullet) \arrow[r, "d_2"] & H_p(L_\bullet/JL_\bullet) \arrow[r, "\delta"] & H_{p-1}(JL_\bullet) \arrow[r, "d_3"] & H_{p-1}(L_\bullet) \\
0 \arrow[r] & JH_p(L_\bullet) \arrow[r] \arrow[u, "j_1"] & H_p(L_\bullet) \arrow[r] \arrow[u, "j_2"] & H_p(L_\bullet)/JH_p(L_\bullet) \arrow[r] \arrow[u, "j"] & 0. &
\end{tikzcd} \tag{3.1}$$

The cokernel part is easier. By the second commutative square in the diagram (3.1), and the exactness of the first row in (3.1),

$$\operatorname{coker}(j) = \operatorname{coker}(d_2) \cong \operatorname{Image}(\delta) = \ker(d_3).$$

Let $Z_*(JL_\bullet)$ (respectively $B_*(JL_\bullet)$) denote the closed (respectively boundary) elements of the complex $JL_\bullet$. Then

$$\ker(d_3) = \frac{Z_{p-1}(JL_\bullet) \cap B_{p-1}}{B_{p-1}(JL_\bullet)} = \frac{JL_{p-1} \cap Z_{p-1} \cap B_{p-1}}{JB_{p-1}} = \frac{JL_{p-1} \cap B_{p-1}}{JB_{p-1}}.$$

Now consider $\ker(j)$. By the second commutative square, and exactness of both rows in (3.1),

$$\ker(j) = \frac{\ker(d_2)}{JH_p(L_\bullet)} = \frac{\operatorname{Image}(d_1)}{JH_p(L_\bullet)}.$$

Note that

$$\operatorname{Image}(d_1) = \frac{JL_p \cap Z_p + B_p}{B_p} \quad \text{and} \quad JH_p(L_\bullet) = \frac{JZ_p + B_p}{B_p}.$$

Hence we can resolve $\ker(j)$ by $C(Z_p, L_p; J)$

$$0 \to \frac{(JL_p \cap Z_p) \cap (JZ_p + B_p)}{JZ_p} \to \frac{JL_p \cap Z_p}{JZ_p} \to \frac{JL_p \cap Z_p + B_p}{JZ_p + B_p} \to 0. \tag{3.2}$$

Observe that

$$(JL_p \cap Z_p) \cap (JZ_p + B_p) = (JL_p \cap Z_p) \cap B_p + JZ_p = JL_p \cap B_p + JZ_p.$$

Here the first equality is because if $x \in JZ_p$, $y \in B_p$ such that $x + y \in JL_p \cap Z_p$, then $y \in JL_p \cap Z_p$ since $x \in JZ_p \subset JL_p \cap Z_p$. Hence the left-hand side of (3.2) is isomorphic to

$$\frac{JL_p \cap B_p + JZ_p}{JZ_p} \cong \frac{JL_p \cap B_p}{(JL_p \cap B_p) \cap JZ_p} = \frac{JL_p \cap B_p}{JZ_p \cap B_p}.$$

But the last one is resolved by correction modules $C(B_p, Z_p; J)$ and $C(B_p, L_p; J)$

$$0 \to \frac{JZ_p \cap B_p}{JB_p} \to \frac{JL_p \cap B_p}{JB_p} \to \frac{JL_p \cap B_p}{JZ_p \cap B_p} \to 0. \tag{3.3}$$

Now patching (3.2) and (3.3) completes the proof of the lemma. □

4. Parametrized Koszul complexes

In this section sheaf theory comes into the play. For more background interested readers should see [11], especially those arguments related to Lemma 2.1.5, Proposition 9.4.5, and Theorem 10.3.13. Here our approach is slightly more algebraic. It allows conceptual proofs and leads to conjectures for further development.

We start with a connection between a Hilbert module H over a ring R, associated with an operator tuple $\bar{T} = (T_1, \dots, T_n)$, and its sheaf model [11,25],

$$\mathtt{h} = \mathcal{O}(H)/(z - \bar{T})\mathcal{O}(H),$$

as well as its stalk at the origin $\mathtt{h}_0 = \mathcal{O}_0(H)/(z - \bar{T})\mathcal{O}_0(H)$. Here R is any of the following three rings

$$\mathbb{C}[z_1, \dots, z_n], \quad \mathcal{O}(\mathbb{C}^n), \quad \text{and} \quad \mathcal{O}(U),$$

with U being a Stein neighborhood of the Taylor spectrum $\sigma(\bar{T})$. In any case, and even for $\mathcal{O}_0$, let $I = (z_1, \dots, z_n)$ be the maximal ideal at the origin. We usually assume that $\dim(H/IH) < \infty$ and $0 \in \sigma(\bar{T})$.

In an effort to relate H to $\mathtt{h}_0$, Douglas and Yan showed in [9] that the Hilbert function of H, with respect to I, is greater than or equal to the Hilbert function of $\mathtt{h}_0$. In [15] we showed that the inequality between the two Hilbert functions is in fact an equality. This plays a key role in the proof of the semi-continuity of Samuel multiplicity over Hilbert modules. The result from [15] can be reformulated as that the completions of H and $\mathtt{h}_0$ in the so-called *I-adic topology* [27] are isomorphic,

$$\hat{H} \cong \hat{\mathtt{h}}_0, \tag{4.1}$$

which is better suited for generalization. In particular, an easy consequence of (4.1) is

$$H/IH \cong \mathtt{h}_0/I\mathtt{h}_0. \tag{4.2}$$

Next we aim at the homological generalizations of (4.1) and (4.2). First, rewrite the completion $\hat{H}$ as an inverse limit

$$\hat{H} = \varprojlim_{k\to\infty} H/I^k H.$$

Let $I_k = (z_1^k, \dots, z_n^k) \subset R$. Then, by basic facts on inverse limits

$$\hat{H} = \varprojlim_{k\to\infty} H/I_k H.$$

Observe that $H/I_k H$ can be written as the 0th homology group of the Koszul complex $K(T_1^k, \dots, T_n^k; H)$ of $(T_1^k, \dots, T_n^k)$ on H. On the other hand, the sheaf model $\mathtt{h}$ can be written as the 0th homology group $H_0(z - \bar{T}, \mathcal{O}(H))$ of the Koszul complex of $z - \bar{T} = (z_1 - T_1, \dots, z_n - T_n)$ on $\mathcal{O}(H)$.

To generalize Eq. (4.1), observe that, for each $i = 0, 1, \dots, n$, we can form an inverse system of the Koszul homology groups, [6,16],

$$\bigl\{H_i\bigl(T_1^k, \dots, T_n^k; H\bigr),\ k = 1, 2, \dots\bigr\}.$$

Definition 6. For each $i = 0, 1, \dots, n$, we define

$$\hat{H}_i = \varprojlim_{k\to\infty} H_i\bigl(T_1^k, \dots, T_n^k; H\bigr).$$

For the sheaf side, as generalization of the sheaf model $\mathtt{h} = \mathtt{h}_{(0)}$, we call

$$\mathtt{h}_{(i)} = H_i\bigl(z - \bar{T}, \mathcal{O}(H)\bigr), \quad i = 0, 1, \dots, n,$$

the homological sheaf models of H.

The modules $\hat{H}_i$ are reminiscent of Grothendieck's local cohomology modules in algebraic geometry [16]. According to Markoe [22], $\mathtt{h}_{(i)}$ is in fact a coherent analytic sheaf around the origin for each i when $\bar{T}$ is Fredholm. The significance of $\hat{H}_i$ and $\mathtt{h}_{(i)}$ is yet to be understood. As a first step, and as a generalization of (4.1), we offer the following conjecture. Let $\mathtt{h}_{(i),0}$ denote the stalk at the origin, and $\hat{\mathtt{h}}_{(i),0}$ its I-adic completion.

Conjecture. *For any Fredholm tuple $\bar{T}$ and each $i = 0, 1, \dots, n$, we have a natural isomorphism $\hat{H}_i \cong \hat{\mathtt{h}}_{(i),0}$ of modules over the ring of power series $\mathbb{C}\langle z_1, \dots, z_n\rangle$.*

As for Eq. (4.2), observe that $\mathtt{h}_0/I\mathtt{h}_0$ can be written as the 0th homology group of the Koszul complex $K(z - \bar{T}; R/I \otimes H)$ of $z - \bar{T} = (z_1 - T_1, \dots, z_n - T_n)$ on $\mathcal{O}_0(H)/I\mathcal{O}_0(H) = R/I \otimes_{\mathbb{C}} H$. Note that $\mathcal{O}_0/I\mathcal{O}_0$ are isomorphic to R/I, as Artinian rings, for any of $R = \mathbb{C}[z_1, \dots, z_n]$, $\mathcal{O}(\mathbb{C}^n)$, and $\mathcal{O}(U)$. For each of these three rings, we generalize (4.2) to

Lemma 7. *Let $f = (f_1, \dots, f_n)$ be a regular sequence in R, and (f) be the ideal generated by f_j. Then*

$$H_i\bigl(f_1(\bar{T}), \dots, f_n(\bar{T}); H\bigr) \cong H_i\bigl(z - \bar{T}, R/(f) \otimes_{\mathbb{C}} H\bigr).$$

Remarks. (1) Here f being regular means that the Koszul complex of f on R yields a free resolution of $R/(f)$ [10]. In particular, length$(R/(f)) < \infty$.

(2) Under the condition $f^{-1}(0) = 0$, Lemma 7 is already covered in [12] which, in turn, is modeled after the proof of Theorem 10.3.13 in [11]. Modulo technical matters, what is new here is just the way it is presented.

(3) Our proof is essentially only a series observations in homological algebra, which can establish the result for a larger category, and motivates a further conjecture—see the remark at the end of the paper.

Proof. When $i = 0$, both sides are directly verified to be $H/\sum f_j(\bar{T})H$. In fact one has

$$H_0\big(z - \bar{T}, R/(f) \otimes_{\mathbb{C}} H\big) \cong \big(R/(f)\big) \otimes_R H.$$

The natural map from the left to the right is the class of $x \otimes y \in R/(f) \otimes_{\mathbb{C}} H$ being sent to the class of $x \otimes y \in (R/(f)) \otimes H$. It is clearly surjective with kernel being the submodule generated by $rx \otimes_{\mathbb{C}} y - x \otimes_{\mathbb{C}} ry$, which is the same as that generated by $(z_i - T_i)(x \otimes_{\mathbb{C}} y) = z_i x \otimes_{\mathbb{C}} y - x \otimes_{\mathbb{C}} T_i y$ [10].

For general i, since f is regular, the Koszul homology is also given by the derived functors $\mathrm{Tor}_i^R(\cdot,\cdot)$,

$$H_i\big(f_1(\bar{T}), \ldots, f_n(\bar{T}); H\big) = \mathrm{Tor}_i^R\big(R/(f), H\big).$$

Let R_w denote the ring R with variables written in $w = (w_1, \ldots, w_n)$, and consider H as a module over R_w. Then $H_i(z - \bar{T}, R/(f) \otimes_{\mathbb{C}} H)$, viewed as a module over $R_z \otimes R_w$, is naturally a module over R_w. Hence, the functors $F_i : M \to H_i(z - \bar{T}, R/(f) \otimes_{\mathbb{C}} M)$ can be regarded as over the category of $R_w (= R)$-modules.

To show the sequence of functors $F_i : M \to H_i(z - \bar{T}, R/(f) \otimes_{\mathbb{C}} M)$ coincide with the derived functors $M \to \mathrm{Tor}_i^R(R/(f), M)$ for any R-module M, we only need to show that $F = (F_i)$ is a universal δ-functor—here we use the machinery in homological algebra as explained in Section 3.1, [17]. Being a δ-functor is clear by definition. To show it is universal, it suffices to show that F_i is coeffaceable for $i > 0$. This can be verified by (1) the category of all R-modules have enough projectives, and (2) $F_i(R) = 0$ when $i > 0$. The first is algebraic folklore, and the second follows easily from the definition of regular sequence.

Next we give more details for

$$H_i\big(z - w, \mathcal{O}(U)/(f) \otimes_{\mathbb{C}} \mathcal{O}(U)\big) = 0 \quad (i > 0)$$

for readers' convenience [10]. Because $(z - w)$ forms a regular sequence in $\mathcal{O}(U) \otimes_{\mathbb{C}} \mathcal{O}(U)$, the Koszul homology can be calculated via

$$\mathrm{Tor}_i^{\mathcal{O}(U)\otimes_{\mathbb{C}}\mathcal{O}(U)}\big(\mathcal{O}(U) \otimes_{\mathbb{C}} \mathcal{O}(U)/(z - w), \mathcal{O}(U)/(f) \otimes_{\mathbb{C}} \mathcal{O}(U)\big).$$

Since (f) is regular by assumption, the Koszul complex $K(f, \mathcal{O}(U))$ provides a free resolution of $\mathcal{O}(U)/(f)$, hence $K(f, \mathcal{O}(U)) \otimes_{\mathbb{C}} \mathcal{O}(U)$ a free resolution of $\mathcal{O}(U)/(f) \otimes_{\mathbb{C}} \mathcal{O}(U)$. Hence the above Tor_i can be calculated through the complex

$$K\big(f, \mathcal{O}(U)\big) \otimes_{\mathbb{C}} \mathcal{O}(U) \otimes_{\mathcal{O}(U)\otimes_{\mathbb{C}}\mathcal{O}(U)} \mathcal{O}(U) \otimes_{\mathbb{C}} \mathcal{O}(U)/(z-w)$$
$$\cong K\big(f, \mathcal{O}(U)\big) \otimes_{\mathbb{C}} \mathcal{O}(U)/(z-w).$$

The last term, regarded as a complex of $\mathcal{O}(U)$-modules in the variable z, is isomorphic to $K(f, \mathcal{O}(U))$, which is acyclic, hence $H_i(\cdots) = 0$. $\square$

Let $J = (f) = (z_1^{k_1}, \dots, z_n^{k_n})$, then $R/(f) \otimes_{\mathbb{C}} H \cong \mathcal{O}_0(H)/J\mathcal{O}_0(H)$. If $\mathcal{L}_\bullet = K(z - \bar{T}, \mathcal{O}_0(H))$ denotes the Koszul complex of $z - \bar{T}$ on $\mathcal{O}_0(H)$, then, by Lemma 7,

$$H_i\big(T_1^{k_1}, \dots, T_n^{k_n}\big) \cong H_i(\mathcal{L}_\bullet / J\mathcal{L}_\bullet).$$

Since $\mathcal{O}_0(H)$ in $\mathcal{L}_\bullet$ is an infinitely generated $\mathcal{O}_0$-module when $\dim(H) = \infty$, a standard strategy for parametrized complexes is to find a complex of finitely generated $\mathcal{O}_0$-modules with isomorphic homology groups, which will allow us to apply results from Section 3.

Lemma 8. *If $\bar{T}$ is Fredholm, then there exists a complex $\mathcal{E}_\bullet$ of finitely generated $\mathcal{O}_0$-modules: $\cdots \to E_i \to E_{i-1} \to \cdots$, such that for $J = 0$, or any $\mathrm{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ and $J = (z_1^{k_1}, \dots, z_n^{k_n})$,*

$$H_i(\mathcal{L}_\bullet / J\mathcal{L}_\bullet) \cong H_i(\mathcal{E}_\bullet / J\mathcal{E}_\bullet), \quad i \in \mathbb{Z}.$$

Proof. This is essentially due to [11] and [12]. The construction of $\mathcal{E}_\bullet$ is detailed in [11]. The isomorphism between homology groups is verified in [12]. $\square$

Proof of Theorem 1. Since the components E_i in $\mathcal{E}_\bullet$ are finitely generated $\mathcal{O}_0$-modules, so are the homology groups $H_i(\mathcal{E}_\bullet)$. By Lemma 4, the function

$$\phi(\mathrm{k}) = \dim\big[H_i(\mathcal{E}_\bullet)/J H_i(\mathcal{E}_\bullet)\big]$$

satisfies, for some constant C,

$$\phi(\mathrm{k}) \leqslant k_1 \cdots k_n \left(e_i + \frac{C}{\min k_j}\right),$$

here $e_i = e_i(\bar{T})$ is the Samuel multiplicity of $H_i(\mathcal{E}_\bullet) = H_i(\mathcal{L}_\bullet)$ with respect to I. Now, the estimates on correction modules, together with the representation of the difference between $H_i(\mathcal{E}_\bullet)/J H_i(\mathcal{E}_\bullet)$ and $H_i(\mathcal{E}_\bullet / J\mathcal{E}_\bullet)$ as correction modules, completes the proof of the upper bound in Theorem 1.

The lower bound is much easier, and is in fact part of Theorem 2.4 in [12]. Our treatment here is just slightly different. For fixed k, we claim that $h_i(k_1, \dots, k_n) = e_i \cdot k_1 \cdots k_n$ when the tuple is $\bar{T} - \lambda$, where λ is in a small neighborhood of the origin except for a possibly thin subvariety. Because $h_i(k_1, \dots, k_n)$ is upper semi-continuous around the origin as a function of λ in $\bar{T} - \lambda$, we get the lower bound.

Since the singularity set of the coherent sheaf $H_i(\mathcal{L}_\bullet)$ is thin, we can choose small λ such that, with respect to the tuple $\bar{T} - \lambda$, $H_i(\mathcal{L}_\bullet)$ is free, and, for any primary ideal J, the following are naturally isomorphic: $H_i(\mathcal{L}_\bullet / J\mathcal{L}_\bullet) \cong H_i(\mathcal{L}_\bullet)/J H_i(\mathcal{L}_\bullet)$ (see Grauert's comparison theorem [5,17]).

Here the latter has dimension $e_i \cdot \dim(\mathcal{O}_0/J)$ since $H_i(\mathcal{L}_\bullet)$ is free. Choosing $J = (z_1^{k_1}, \ldots, z_n^{k_n})$ gives us the claim. □

Because $H_i(z - \bar{T}, \mathcal{O}(H))$ is coherent around the origin when $\bar{T}$ is Fredholm [22], and e_i is the Samuel multiplicity of the stalk of $H_i(z - \bar{T}, \mathcal{O}(H))$ at the origin, a straightforward consequence is, by the invariance of Samuel multiplicity of stalks of a coherent analytic sheaf, that is $H_i(z - \bar{T}, \mathcal{O}(H))$ in our case, the local constancy of $e_i(\bar{T} - \lambda)$.

Corollary 9. *If $\bar{T}$ is Fredholm, then the function $\lambda \in \mathbb{C}^n \mapsto e_i(\bar{T} - \lambda)$ is locally constant in a neighborhood of the origin.*

In other words, let Ω be a connected component of the Fredholm domain $\mathbb{C}^n \setminus \sigma_e(\bar{T})$, then $e_i(\bar{T} - \lambda)$ is a constant for $\lambda \in \Omega$.

Motivated by the base change formula for Fredholm index $\mathrm{index}(f_1(\bar{T}), \ldots, f_n(\bar{T}))$ [24], it is natural to ask whether similar formulas hold for $H_i(f_1(\bar{T}), \ldots, f_n(\bar{T}))$ and $e_i(f_1(\bar{T}), \ldots, f_n(\bar{T}))$. In general, $H_i(\cdot)$ is too unstable to enjoy a nice base change formula.

For e_i, however, we observe that the proof of the base change formula in Theorem 10.3.16 in [11] goes, roughly, as follows. The key in reduction is that $\mathrm{index}(\bar{T} - \lambda)$ is locally constant in λ. For a neighborhood $U \supset \sigma(\bar{T})$ of the Taylor spectrum $\sigma(T)$, and a map $F = (f_1, \ldots, f_n) : U \to \mathbb{C}^n$ with $F(0) = 0$, we can consider $\mathrm{index}(F(\bar{T}) - \lambda)$ such that the fibre $(F - \lambda)^{-1}(0)$ is simple, that is, a collection of k distinct points $\{p_1, \ldots, p_k\}$, here k being the mapping degree of f at 0. Then over each simple point p_i, the contribution to index can be counted directly, hence leading to the base change formula. Now, based on Corollary 9, the whole proof in [11] carries over for $e_i(\cdot)$.

Corollary 10. *Let $\bar{T}$ be a Fredholm tuple, and $F \in \mathcal{O}(U)^n$ be an n-tuple of analytic functions defined on an open neighborhood U of the Taylor spectrum $\sigma(\bar{T})$. Assume that $F(0) = 0$ and $0 \notin \sigma_e(F(\bar{T}))$, and let $m_z(F)$ denote the multiplicity of F at z. Then, for each $i = 0, 1, \ldots, n$,*

$$e_i\big(F(\bar{T})\big) = \sum_{z \in F^{-1}(0) \cap \sigma(\bar{T})} m_z(F) e_i(\bar{T} - z).$$

We end the paper with a remark when $f = (f_1, \ldots, f_n)$ in $R = \mathbb{C}[z_1, \ldots, z_n]$, $\mathcal{O}(\mathbb{C}^n)$, or $\mathcal{O}(U)$, is not necessarily a regular sequence.

If we rewrite $R/(f)$ as the 0th Koszul homology of f on R, then Lemma 7 becomes

$$H_i\big(f_1(\bar{T}), \ldots, f_n(\bar{T}); H\big) \cong H_i\big(z - \bar{T}, H_0(f, R) \otimes_{\mathbb{C}} H\big).$$

Hence it motivates

Conjecture. *For general f, there exists a spectral sequence, with E^2 page*

$$E^2_{pq} \cong H_p\big(z - \bar{T}, H_q(f, R) \otimes_{\mathbb{C}} H\big),$$

convergent to $H_{p+q}(f_1(\bar{T}), \ldots, f_n(\bar{T}); H)$.

Adopting this viewpoint, Lemma 7, that is when f is regular, actually follows immediately from Grothendieck's spectral sequence of composition functors [23,31]. For the general case, we will address the conjecture by constructing spectral sequences directly from double complexes in a coming work.

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