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Relative Hopf Modules for (Dual) Quasi-Hopf Algebras

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INTRODUCTION

Quasi-bialgebras and quasi-Hopf algebras were introduced by Drinfel'd [6] and used in his treatment of the Knizhnik–Zamolodchikov equations [7]. From a purely algebraic point of view, they appear naturally as follows: Let (H, Δ, ε) be a bialgebra and $F \in H \otimes H$ an invertible element such that $(\varepsilon \otimes \mathrm{Id})(F) = (\mathrm{Id} \otimes \varepsilon)(F) = 1$ (such an element was called by Drinfel'd a "gauge transformation") and define a new comultiplication on H by $\Delta_F(h) = F\Delta(h)F^{-1}$, then $(H, \Delta_F, \varepsilon)$ is no longer a bialgebra, but it is a quasi-bialgebra. In this way, we can get a lot of quasi-bialgebras (and quasi-Hopf algebras). If H is a finite-dimensional cocommutative Hopf algebra and $\omega \in (H \otimes H \otimes H)^*$ is a normalized 3-cocycle in the Sweedler cohomology, then a structure of a quasitriangular quasi-Hopf algebra can be constructed on the k-linear space $H^* \otimes H$, denoted by $D^{\omega}(H)$ [2]. This generalizes Dijkgraaf–Pasquier–Roche's quasi-Hopf algebra $D^{\omega}(G)$ [1]. There exists a construction more general than the above $D^{\omega}(H)$, namely the quantum double of any finite-dimensional quasi-Hopf algebra H, denoted by $D(H)$ [8].

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It is well known [15] that a left integral in a finite-dimensional Hopf algebra always exists and it is unique up to a scalar multiple. The proof in [15] uses the fundamental structure theorem on Hopf modules by Larson and Sweedler [11]. Now, quasi-Hopf H-bimodules for a quasi-Hopf algebra H have been introduced by F. Hausser and F. Nill in [9]. If H is a quasi-Hopf algebra and M is an H-bimodule, roughly speaking, M is called a right quasi-Hopf H-bimodule if there exists a bimodule map ρ: $M \to M \otimes H$ which turns M into an "almost" right H-comodule (in a sense similar to the fact that the comultiplication of a quasi-Hopf algebra is "almost" coassociative). Also, they proved a structure theorem for quasi-Hopf bimodules and used it as the essential tool to show that a finite-dimensional quasi-Hopf algebra is a Frobenius algebra, and therefore the space of left (right) integrals in H is one-dimensional.

In this paper we shall study relative Hopf modules for (dual) quasi-bialgebras. If H is a dual quasi-bialgebra, a right H-comodule algebra A is defined as being an algebra in the tensor category of right H-comodules $\mathscr{M}^H$ (so that A is not necessarily associative, because the associativity constraints of $\mathscr{M}^H$ are not the trivial ones, but the ones given by the reassociator φ). An example of an H-comodule algebra is the following: if H is finite-dimensional then, in general, H^* is not necessarily an algebra in $\mathscr{M}^H$, but if we define on H^* a new multiplication and we denote this new structure on H^* by H_0^*, then H_0^* becomes a right H-comodule algebra. This construction is a dual case of [3, Proposition 2.2] (see Remark 2.2 below). For usual Hopf algebras that is just the right coadjoint coaction of H on H^*.

A general procedure for constructing comodule algebras over dual quasi-bialgebras is the following: if A is a right H-comodule algebra (in the Hopf sense or dual quasi-Hopf sense) and $\tau \in (H \otimes H)^*$ a gauge transformation, and if we denote by H_τ the dual quasi-bialgebra obtained by twisting the multiplication of H via τ, then we can change the multiplication of A via τ^{-1} (we denote by $A_{\tau^{-1}}$ the resulting structure on A) such that $A_{\tau^{-1}}$ is also a right H_τ-comodule algebra. Moreover, if H is finite-dimensional, then the smash product $A \# H^*$ (in the Hopf sense or quasi-Hopf sense) is isomorphic to the smash product (in the quasi-Hopf sense) $A_{\tau^{-1}} \# H_\tau^*$ as algebras (Proposition 2.3). For the definition of smash products in the quasi-Hopf sense see Section 1, Preliminaries.

In Section 2, we shall describe the category of (H, A)-Hopf modules when H is a finite-dimensional dual quasi-Hopf algebra. In fact, as in the Hopf case, we have that the above category is isomorphic to the category of modules over the smash product $A \# H^*$. Next, we shall prove that [5, Theorem 1] is still true for our Hopf modules, and the existence of a right H-comodule map λ: $H \to A$ such that $\lambda(1) = 1$ is equivalent to the fact that A is an injective H-comodule (Proposition 2.9, Corollary 2.10). The main result of this section is a structure theorem for our Hopf modules (Theorem 2.11) which generalizes [5, Theorem 3]. Whether this structure theorem is functorially related to existence and uniqueness of integrals in a finite-dimensional quasi-Hopf algebra remains unclear at the moment.

Finally, in Section 3 we will define the dual concept, $[C, H]$-Hopf modules for a quasi-Hopf algebra H and a coalgebra C in the tensor category $\mathscr{M}_H$. We shall prove that if C is finite-dimensional then the above category is isomorphic to a category of modules over the smash product $C^* \# H$. Since the proof of Theorem 2.11 is not so easily dualized, for the sake of the reader we will just sketch the proof of Theorem 3.5. Further, we dualize (without proofs) the results given in Section 2.

1. PRELIMINARIES

In this section we recall some definitions and results and fix notation. Throughout, k will be a fixed field and all algebras, linear spaces, etc., will be over k; unadorned $\otimes$ means $\otimes_k$. For coalgebras and dual quasi-Hopf algebras we shall use Σ-notation: $\Delta(h) = \sum h_1 \otimes h_2$, etc.

Definition 1.1. Let H be a k-algebra, and $\Delta: H \to H \otimes H$, $\varepsilon: H \to k$ two algebra homomorphisms. H is called a quasi-bialgebra if there exists an invertible element $\Phi \in H \otimes H \otimes H$ such that, for all elements $h \in H$, we have:

$$(\mathrm{Id} \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes \mathrm{Id})(\Delta(h))\Phi^{-1}, \tag{1.1}$$

$$(\mathrm{Id} \otimes \mathrm{Id} \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{Id} \otimes \mathrm{Id})(\Phi) = (1 \otimes \Phi)(\mathrm{Id} \otimes \Delta \otimes \mathrm{Id})(\Phi)(\Phi \otimes 1), \tag{1.2}$$

$$(\varepsilon \otimes \mathrm{Id})(\Delta(h)) = 1 \otimes h \quad \text{and} \quad (\mathrm{Id} \otimes \varepsilon)(\Delta(h)) = h \otimes 1, \tag{1.3}$$

$$(\mathrm{Id} \otimes \varepsilon \otimes \mathrm{Id})(\Phi) = 1 \otimes 1 \otimes 1, \tag{1.4}$$

where $\mathrm{Id} = \mathrm{id}_H$. The map Δ is called the coproduct or the comultiplication and ε the counit. H is called a quasi-Hopf algebra if, moreover, there exist an anti-automorphism S of the algebra H and elements α and β of H such that, for all $h \in H$, we have:

$$\sum S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad \sum h_1 \beta S(h_2) = \varepsilon(h)\beta, \tag{1.5}$$

$$\sum X^1 \beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad \sum S(x^1)\alpha x^2 \beta S(x^3) = 1, \tag{1.6}$$

where $\Phi = \sum X^1 \otimes X^2 \otimes X^3$, $\Phi^{-1} = \sum x^1 \otimes x^2 \otimes x^3$ (formal notation), and we used the Σ-notation: $\Delta(h) = \sum h_1 \otimes h_2$. In this case, S is called the antipode of H.

Note that every Hopf algebra with bijective antipode is a quasi-Hopf algebra with $\Phi = 1 \otimes 1 \otimes 1$ and $\alpha = \beta = 1$.

We note the following two consequences of the definitions of S, α, β: $\varepsilon(\alpha)\varepsilon(\beta) = 1$, $\varepsilon \circ S = \varepsilon$. Moreover, (1.2) and (1.4) imply $(\varepsilon \otimes I \otimes I)(\Phi) = (I \otimes I \otimes \varepsilon)(\Phi) = 1$.

If we denote:

$$\sum A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (\Phi \otimes 1)(\Delta \otimes \mathrm{Id} \otimes \mathrm{Id})(\Phi^{-1}), \tag{1.7}$$

$$\sum B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes \mathrm{Id} \otimes \mathrm{Id})(\Phi)(\Phi^{-1} \otimes 1) \tag{1.8}$$

and we define

$$\rho = \sum S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4, \tag{1.9}$$

$$\delta = \sum B^1\beta S(B^4) \otimes B^2\beta S(B^3), \tag{1.10}$$

$$f = \sum (S \otimes S)(\Delta^{\mathrm{cop}}(x^1))\rho\Delta(x^2\beta S(x^3)), \tag{1.11}$$

where $\Delta^{\mathrm{cop}}(h) = \sum h_2 \otimes h_1$, then f is invertible with inverse given by

$$f^{-1} = \sum \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\mathrm{cop}}(x^3)), \tag{1.12}$$

and the relations (see [1, 8])

$$f\Delta(h)f^{-1} = (S \otimes S)(\Delta^{\mathrm{cop}} \circ S^{-1}(h)) \qquad \text{for all } h \in H, \tag{1.13}$$

$$\rho = f\Delta(\alpha), \qquad \delta = \Delta(\beta)f^{-1}, \tag{1.14}$$

$$(1 \otimes f)(\mathrm{Id} \otimes \Delta)(f)\Phi(\Delta \otimes \mathrm{Id})(f^{-1})(f^{-1} \otimes 1) = (S \otimes S \otimes S)(\Phi^{321}), \tag{1.15}$$

hold, where if $\Phi = \sum X^1 \otimes X^2 \otimes X^3$ then $\Phi^{321} = \sum X^3 \otimes X^2 \otimes X^1$.

If H is a quasi-bialgebra with Drinfel'd associator Φ, then we shall denote the tensor components of Φ with big letters, for instance

$$\Phi = \sum X^1 \otimes X^2 \otimes X^3 = \sum T^1 \otimes T^2 \otimes T^3 = \sum V^1 \otimes V^2 \otimes V^3,$$

etc., and the tensor components of Φ^{-1} with small letters, for instance

$$\Phi^{-1} = \sum x^1 \otimes x^2 \otimes x^3 = \sum t^1 \otimes t^2 \otimes t^3 = \sum v^1 \otimes v^2 \otimes v^3,$$

etc.

Together with a quasi-Hopf algebra $H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ we also have H^{op}, H^{cop}, and $H^{\mathrm{op,cop}}$ as quasi-Hopf algebras, where "op" means opposite multiplication and "cop" means opposite comultiplication. The quasi-Hopf structures are obtained by putting $\Phi_{op} = \Phi^{-1}$, $\Phi_{cop} = (\Phi^{-1})^{321}$, $\Phi_{op,cop} = \Phi^{321}$, $S_{op} = S_{cop} = (S_{op,cop})^{-1} = S^{-1}$, $\alpha_{op} = S^{-1}(\beta)$, $\beta_{op} = S^{-1}(\alpha)$, $\alpha_{cop} = S^{-1}(\alpha)$, $\beta_{cop} = S^{-1}(\beta)$, $\alpha_{op,cop} = \beta$, and $\beta_{op,cop} = \alpha$.

Next we recall that the definition of a quasi-bialgebra is "twist covariant" in the following sense. An invertible element $F \in H \otimes H$ is called a gauge transformation if it satisfies the relation $(\varepsilon \otimes \mathrm{Id})(F) = (\mathrm{Id} \otimes \varepsilon)(F) = 1$. If $(H, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra and $F \in H \otimes H$ is a gauge transformation, then one can define a new quasi-bialgebra $H_F = (H, \Delta_F, \varepsilon, \Phi_F)$ (see [10]) by taking the algebra structure of H and

$$\Delta_F\colon H \to H \otimes H, \qquad \Delta_F(h) = F\Delta(h)F^{-1},$$

$$\Phi_F = F_{23}(\mathrm{Id} \otimes \Delta)(F)\Phi(\Delta \otimes \mathrm{Id})(F^{-1})F_{12}^{-1},$$

where F_{ij} means F acting non-trivially in the ith and jth positions of $H \otimes H \otimes H$ and F^{-1} is the inverse of F.

Suppose that $(H, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra. If U, V, W are left H-modules, define $a_{U,V,W}\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ by

$$a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)).$$

Then the category ${}_H\mathscr{M}$ of left H-modules becomes a tensor category (see [10] for the terminology) with tensor product $\otimes$ given via Δ, associativity constraints $a_{U,V,W}$, unit k as a trivial H-module, and the usual left and right unit constraints.

We now recall the concept of a module algebra over a quasi-bialgebra introduced in [3].

DEFINITION 1.2. Let H be a quasi-bialgebra and A a k-linear space. We say that A is a (left) H-module algebra if A is an algebra in the tensor category ${}_H\mathscr{M}$, i.e., A has a multiplication and a usual unit 1_A satisfying the following conditions:

$$(ab)c = \sum (X^1 \cdot a)\big[(X^2 \cdot b)(X^3 \cdot c)\big], \tag{1.16}$$

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b), \tag{1.17}$$

$$h \cdot 1_A = \varepsilon(h)1_A, \tag{1.18}$$

for all $a, b, c \in A$ and $h \in H$, where $\Phi = \sum X^1 \otimes X^2 \otimes X^3$ and $h \otimes a \mapsto h \cdot a$ is the H-module structure of A.

For a left H-module algebra A as above we define the smash product $A \# H$ as follows: as vector space $A \# H$ is $A \otimes H$ (elements $a \otimes h$ will be written $a \# h$) with multiplication given by

$$(a \# h)(b \# g) = \sum (x^1 \cdot a)\big(x^2 h_1 \cdot b\big) \# x^3 h_2 g, \tag{1.19}$$

for all $a, b \in A$, $h, g \in H$. This $A \# H$ is an associative algebra and it is defined by a universal property (as Heyneman and Sweedler did for Hopf algebras, see [3]). It is easy to see that H is a subalgebra of $A \# H$ via $h \mapsto 1 \# h$, A is a k-subspace of $A \# H$ via $a \mapsto a \# 1$, and $(1 \# h)(a \# 1) = \sum h_1 \cdot a \# h_2$. In the Hopf case, if the antipode S is bijective then $A \# H = (1 \# H)(A \# 1) \cong H \otimes A$ as vector space, where the correspondence is given by

$$a \# h = \sum (1 \# h_2)\big(S^{-1}(h_1) \cdot a \# 1\big).$$

In the quasi-Hopf case we have the following:

LEMMA 1.3. *Let H be a quasi-Hopf algebra, A an H-module algebra, and $a \in A$, $h \in H$. Then in $A \# H$,*

$$a \# h = \sum \big(1 \# x^3 h_2\big)\big(S^{-1}\big(S(x^1)\alpha x^2 h_1 X^1 \beta S(X^2)\big) \cdot a \# X^3\big).$$

Proof. We calculate:

$$\begin{aligned}
&\sum \big(1 \# x^3 h_2\big)\big(S^{-1}\big(S(x^1)\alpha x^2 h_1 X^1 \beta S(X^2)\big) \cdot a \# X^3\big)\\
&\quad = \sum x^3_1 (h_2)_1 S^{-1}\big(S(x^1)\alpha x^2 h_1 X^1 \beta S(X^2)\big) \cdot a \# x^3_2 (h_2)_2 X^3\\
&\quad = \sum x^3_1 S^{-1}\big(S(x^1)\alpha x^2 h_1 X^1 \beta S\big((h_2)_1 X^2\big)\big) \cdot a \# x^3_2 (h_2)_2 X^3\\
&\quad = \sum x^3_1 S^{-1}\big(S(x^1)\alpha x^2 X^1 (h_1)_1 \beta S\big(X^2 (h_1)_2\big)\big) \cdot a \# x^3_2 X^3 h_2 \qquad \text{by (1.1)}\\
&\quad = \sum S^{-1}\big(S(x^1)\alpha x^2 X^1 \beta S\big(x^3_1 X^2\big)\big) \cdot a \# x^3_2 X^3 h \qquad \text{by (1.5)}\\
&\quad = \sum S^{-1}\big(S\big(X^1_1 x^1 y^1\big)\alpha X^1_2 x^2 y^2_1 \beta S\big(X^2 x^3 y^2_2\big)\big) \cdot a \# X^3 y^3 h \qquad \text{by (1.2)}\\
&\quad = \sum S^{-1}\big(S(x^1)\alpha x^2 \beta S(x^3)\big) \cdot a \# h \qquad \text{by (1.5)}\\
&\quad = a \# h \qquad \text{by (1.6)}.
\end{aligned}$$

∎

Now recall from [13] the following:

DEFINITION 1.4. A dual quasi-bialgebra (H, M, u, φ) is a coassociative coalgebra H with counit ε together with coalgebra morphisms $M\colon H \otimes H \to H$ (the multiplication—we denote $M(h, g) = hg$ for all $h, g \in H$) and $u\colon k \to H$ (the unit—we denote $u(1) = 1$), and an invertible element

$\varphi \in (H \otimes H \otimes H)^*$ (the reassociator), such that:

$$\sum h_1(g_1k_1)\varphi(h_2, g_2, k_2) = \sum \varphi(h_1, g_1, k_1)(h_2g_2)k_2, \quad (1.20)$$

$$1h = h1 = h, \quad (1.21)$$

$$\sum \varphi(h_1, g_1, k_1l_1)\varphi(h_2g_2, k_2, l_2)$$
$$= \sum \varphi(g_1, k_1, l_1)\varphi(h_1, g_2k_2, l_2)\varphi(h_2, g_3, k_3), \quad (1.22)$$

$$\varphi(h, 1, g) = \varepsilon(h)\varepsilon(g), \quad (1.23)$$

for all $h, g, k, l \in H$.

A dual quasi-bialgebra is called a dual quasi-Hopf algebra, if there exist a coalgebra antihomomorphism S: $H \to H$ and elements $\alpha, \beta \in H^*$ satisfying for all $h \in H$:

$$\sum S(h_1)\alpha(h_2)h_3 = \alpha(h)1, \qquad \sum h_1\,\beta(h_2)S(h_3) = \beta(h)1, \quad (1.24)$$

$$\sum \varphi(h_1\,\beta(h_2), S(h_3), \alpha(h_4)h_5)$$
$$= \sum \varphi^{-1}(S(h_1), \alpha(h_2)h_3\,\beta(h_4), S(h_5)) = \varepsilon_H(h) \quad (1.25)$$

We note the following two consequences of α, β, and S: $\alpha(1)\beta(1) = 1$, $S(1) = 1$. Moreover, it is easy to see that (1.22)–(1.23) also imply the identities:

$$\varphi(1, g, h) = \varphi(g, h, 1) = \varepsilon(g)\varepsilon(h) \qquad \text{for all } g, h \in H. \quad (1.26)$$

Also, the definition of a dual quasi-bialgebra is "twist coinvariant." A convolution invertible element $\tau \in (H \otimes H)^*$ is called a gauge transformation if it satisfies the relation $\tau(h, 1) = \tau(1, h) = \varepsilon(h)$ for all $h \in H$. If (H, M, u, φ) is a dual quasi-bialgebra and $\tau \in (H \otimes H)^*$ is a gauge transformation one can define a new dual quasi-bialgebra $H_\tau = (H, M_\tau, u, \varphi_\tau)$ by taking the coalgebra structure of H and

$$g \cdot_\tau h = \sum \tau(g_1, h_1)g_2h_2\tau^{-1}(g_3, h_3),$$

$$\varphi_\tau(g, h, l)$$
$$= \sum \tau(h_1, l_1)\tau(g_1, h_2l_2)\varphi(g_2, h_3, l_3)\tau^{-1}(g_3h_4, l_4)\tau^{-1}(g_4, h_5),$$

for all $g, h, l \in H$, where τ^{-1} is the convolution inverse of τ.

Suppose that (H, M, u, φ) is a dual quasi-bialgebra. If U, V, W are right H-comodules, define $a'_{U,V,W}$: $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$ by

$$a'_{U,V,W}((u \otimes v) \otimes w) = \sum \varphi(u_{(1)}, v_{(1)}, w_{(1)})u_{(0)} \otimes (v_0 \otimes w_{(0)}),$$

where $u \mapsto \sum u_{(0)} \otimes u_{(1)}$ is the H-comodule structure map of U, etc.

Then the category $\mathscr{M}^H$ of right H-comodules becomes a tensor category with tensor product $\otimes$ given via M, associativity constraints $a'_{U,V,W}$, unit k as a trivial right H-comodule, and the usual left and right unit constraints.

Finally, if $\mathscr{C}$ and $\mathscr{D}$ are tensor categories then, roughly speaking, we say that $T: \mathscr{C} \to \mathscr{D}$ is a monoidal functor if it respects the tensor products (in the sense that for any two objects V, W in $\mathscr{C}$ there exists a functorial isomorphism $c_{V,W}: T(V) \otimes T(W) \to T(V \otimes W)$ such that c respects the associativity constraints), the unit objects, and the left and right unit constraints (for the complete definition see [13, p. 421]).

2. (H, A)-HOPF MODULES

Let H be a dual quasi-bialgebra over a field k. In this section we study the notion of (H, A)-Hopf bimodules for an H-comodule algebra A in the tensor category $\mathscr{M}^H$. We shall prove a structure theorem for these Hopf modules, generalizing [5, Theorem 3]. From [12], we know that the Structure Theorem of Hopf Modules also holds for Hopf algebras in braided, monoidal categories, if there exist (co)equalizers.

We now introduce the concept of a comodule algebra over a dual quasi-bialgebra.

Definition 2.1. Let H be a dual quasi-bialgebra and A a k-linear space. We say that A is a right H-comodule algebra if A is an algebra in the tensor category $\mathscr{M}^H$, i.e., A has a multiplication and a usual unit 1_A satisfying the conditions

$$(ab)c = \sum \varphi(a_{(1)}, b_{(1)}, c_{(1)}) a_{(0)}(b_{(0)}c_{(0)}), \tag{2.1}$$

$$\sum (ab)_{(0)} \otimes (ab)_{(1)} = \sum a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)}, \tag{2.2}$$

$$\rho_A(1_A) = 1_A \otimes 1_H, \tag{2.3}$$

for all $a, b, c \in A$, where $\rho_A: A \to A \otimes H$, $\rho_A(a) = \sum a_{(0)} \otimes a_{(1)}$ is the H-comodule structure of A.

Similarly, we say that a k-linear space A is a left H-comodule algebra if A is an algebra in the tensor category of left H-comodules ${}^H\mathscr{M}$ (note that in ${}^H\mathscr{M}$ the associativity constraint $\underline{a}'$ is given via φ^{-1}).

Remark 2.2. Let H be a finite-dimensional quasi-bialgebra and A a k-vector space. Then A is an H-module algebra if and only if A is an H^*-comodule algebra.

Indeed, if A is a left H-module algebra then, as in the Hopf case, A becomes a right H^*-comodule via $\rho_A(a) = \sum_{i=1}^n e_i \cdot a \otimes e^i$, where $\{e_i\}_{i=\overline{1,n}}$

and $\{e^i\}_{i=\overline{1,n}}$ are dual bases in H and H^*. Moreover, (1.16), (1.17), and (1.18) imply (2.1), (2.2), and (2.3) respectively, so A is a right H^*-comodule algebra.

Conversely, if A is a right H^*-comodule algebra then A is a left H-module algebra via the action $h \cdot a = \Sigma a_{(1)}(h)a_{(0)}$.

If H is a quasi-Hopf algebra then H is not necessarily an algebra in ${}_H\mathscr{M}$. But, following [3], if we define on H a new multiplication given by

$$g \bullet h = \sum X^1 g S(x^1 X^2)\alpha x^2 X_1^3 h S(x^3 X_2^3)$$

and we denote this new structure on H by H_0, then H_0 becomes a left H-module algebra with unit β and with the left adjoint action, that is, $g \cdot h = \Sigma g_1 h S(h_2)$ for all $g, h \in H$. Now, if H is finite-dimensional and $\{e_i\}_{i=\overline{1,n}}$ and $\{e^i\}_{i=\overline{1,n}}$ are dual bases in H and H^*, then by the above Remark 2.2 we obtain that H_0 is a right H^*-comodule algebra, where the structure H^*-comodule map is for any $h \in H$ given by

$$\rho_{H_0}(h) = \sum_{i,j=1}^{n} e_i h S(e_j) \otimes e^i e^j.$$

Dual, if we start with a finite-dimensional dual quasi-Hopf algebra H then H_0^* is a right H-comodule algebra, where $H_0^* = H^*$ as linear spaces, but for all $p, q \in H^*$, $h \in H$, the multiplication is given by

$$\begin{aligned}\langle p \diamond q, h\rangle = \sum \varphi(h_1, S(h_3), h_7 S(h_9))\varphi^{-1}(S(h_4), h_6, S(h_{10}))\\ \times \alpha(h_5)p(h_2)q(h_8).\end{aligned}$$

The unit for H_0^* is $\beta \in H^*$ and the structure H-comodule map is for any $p \in H^*$ given by

$$\rho_{H_0^*}(p) = \sum_{i,j=1}^{n} e^i p S(e^j) \otimes e_i e_j.$$

Note that, if H is a Hopf algebra, then $g \bullet h = gh$ $(p \diamond q = pq)$; therefore $H_0 = H$ $(H_0^* = H^*)$ as algebras. In this case we just obtain the right coadjoint coaction of H^* on H (respectively, of H on H^*). Also, other examples of module algebras over a finite-dimensional quasi-Hopf algebra have been given in [3, Remark 2.4, Example 2.11]. So using the above remark we can obtain new examples of comodule algebras over a dual quasi-Hopf algebra.

Next we shall prove that a twist preserves the class of right comodule algebras over a dual quasi-bialgebra.

PROPOSITION 2.3. *Let H be a dual quasi-bialgebra, $\tau \in (H \otimes H)^*$ a gauge transformation, and A a right H-comodule algebra via $\rho_A(a) = \sum a_{(0)} \otimes a_{(1)}$. If we introduce on A a new multiplication by*

$$a \odot b = \sum \tau^{-1}(a_{(1)}, b_{(1)}) a_{(0)} b_{(0)}, \tag{2.4}$$

where τ^{-1} is the convolution inverse of τ, and we denote by $A_{\tau^{-1}}$ the resulting structure, then $A_{\tau^{-1}}$ becomes a right H_τ-comodule algebra. Moreover, if H is finite-dimensional, then the smash product $A_{\tau^{-1}} \# H_\tau^*$ is isomorphic as an algebra to the smash product $A \# H^*$.

Proof. Let us start by establishing that $A_{\tau^{-1}}$ is a right H_τ-comodule algebra via ρ_A and $\odot$. Indeed, for all $a, b, c \in A$ we have

$$\begin{aligned}
&\sum \varphi_\tau(a_{(1)}, b_{(1)}, c_{(1)}) a_{(0)} \odot (b_{(0)} \odot c_{(0)}) \\
&\quad = \sum \varphi_\tau(a_{(2)}, b_{(3)}, c_{(3)}) \tau^{-1}(b_{(2)}, c_{(2)}) \tau^{-1}(a_{(1)}, b_{(1)} c_{(1)}) a_{(0)} (b_{(0)} c_{(0)}) \\
&\quad = \sum \varphi(a_{(1)}, b_{(1)}, c_{(1)}) \tau^{-1}(a_{(2)} b_{(2)}, c_{(2)}) \tau^{-1}(a_{(3)}, b_{(3)}) a_{(0)} (b_{(0)} c_{(0)})
\end{aligned}$$

(by definition of φ_τ)

$$= \sum \tau^{-1}(a_{(1)} b_{(1)}, c_{(1)}) \tau^{-1}(a_{(2)}, b_{(2)}) (a_{(0)} b_{(0)}) c_{(0)}$$

by (2.1)

$$= (a \odot b) \odot c.$$

On the other hand,

$$\begin{aligned}
\rho_A(a \odot b) &= \sum \tau^{-1}(a_{(2)}, b_{(2)}) a_{(0)} b_{(0)} \otimes a_{(1)} b_{(1)} \\
&= \sum \tau^{-1}(a_{(1)}, b_{(1)}) a_{(0)} b_{(0)} \otimes a_{(2)} \cdot_\tau b_{(2)} \\
&= \sum a_{(0)} \odot b_{(0)} \otimes a_{(1)} \cdot_\tau b_{(1)},
\end{aligned}$$

so $A_{\tau^{-1}}$ is a right H_τ-comodule algebra having the same unit as A. Moreover, if H is finite-dimensional, we shall prove that the map

$$\eta \colon A \# H^* \to A_{\tau^{-1}} \# H_\tau^*, \qquad \eta(a \# p) = \sum a_{(0)} \# \tau(a_{(1)}, \cdot) p$$

is an algebra isomorphism. Note that the comultiplication in H_τ^* is given by

$$\begin{aligned}
&\Delta_{H_\tau^*}(p) = \sum p_{\langle 1 \rangle} \otimes p_{\langle 2 \rangle} \\
&\qquad \Leftrightarrow \sum p_{\langle 1 \rangle}(h) p_{\langle 2 \rangle}(l) = \sum \tau(h_1, l_1) p(h_2 l_2) \tau^{-1}(h_3, l_3)
\end{aligned}$$

for all $h, l \in H$, $p \in H_\tau^*$, and $A_{\tau^{-1}}$ becomes a left H_τ^*-module algebra as in Remark 2.2. Therefore, by (1.19), the multiplication in $A_{\tau^{-1}} \# H_\tau^*$ is

$$\begin{aligned}(a \# p)(b \# p') &= \sum \left(x_\tau^1 \cdot a\right) \odot \left(x_\tau^2 p_{\langle 1 \rangle} \cdot b\right) \# x_\tau^3 p_{\langle 2 \rangle} p' \\ &= \sum x_\tau^1(a_{(1)}) x_\tau^2(b_{(1)}) p_{\langle 1 \rangle}(b_{(2)}) a_{(0)} \odot b_{(0)} \# x_\tau^3 p_{\langle 2 \rangle} p' \\ &= \sum \tau^{-1}(a_{(1)}, b_{(1)}) a_{(0)} b_{(0)} \# \varphi_\tau^{-1}(a_{(2)}, b_{(2)}, \cdot) \tau(b_{(3)}, \cdot) \\ &\quad \times (p \leftharpoonup b_{(4)}) \tau^{-1}(b_{(5)}, \cdot) p',\end{aligned}$$

where $\sum x_\tau^1 \otimes x_\tau^2 \otimes x_\tau^3$ is the inverse of the reassociator in H_τ^* and for all $p \in H^*$ and $h, l \in H$ we define $p \leftharpoonup h \in H^*$ by $(p \leftharpoonup h)(l) = p(hl)$. It is easy to see that $pq \leftharpoonup h = \sum (p \leftharpoonup h_1)(q \leftharpoonup h_2)$ for all $p, q \in H^*, h \in H$. So we obtain

$$\begin{aligned}&\eta(a \# p)\eta(b \# p') \\ &= \sum \left(a_{(0)} \# \tau(a_{(1)}, \cdot) p\right)\left(b_{(0)} \# \tau(b_{(1)}, \cdot) p'\right) \\ &= \sum a_{(0)} b_{(0)} \# \tau^{-1}(a_{(1)}, b_{(1)}) \varphi_\tau^{-1}(a_{(2)}, b_{(2)}, \cdot) \tau(b_{(3)}, \cdot) \\ &\quad \times \left(\tau(a_{(3)}, \cdot) p \leftharpoonup b_{(4)}\right) p' \\ &= \sum a_{(0)} b_{(0)} \# \tau(a_{(1)} b_{(1)}, \cdot) \varphi^{-1}(a_{(2)}, b_{(2)}, \cdot)\left(\tau^{-1}(a_{(3)}, \cdot) \leftharpoonup b_{(3)}\right) \\ &\quad \times \left(\tau(a_{(4)}, \cdot) \leftharpoonup b_{(4)}\right)(p \leftharpoonup b_{(5)}) p'\end{aligned}$$

(by definition of φ_τ)

$$\begin{aligned}&= \sum a_{(0)} b_{(0)} \# \tau(a_{(1)} b_{(1)}, \cdot) \varphi^{-1}(a_{(2)}, b_{(2)}, \cdot)(p \leftharpoonup b_{(3)}) p' \\ &= \eta\left(\sum a_{(0)} b_{(0)} \# \varphi^{-1}(a_{(1)}, b_{(1)}, \cdot)(p \leftharpoonup b_{(2)}) p'\right) \\ &= \eta\left(\sum (x^1 \cdot a)(x^2 p_1 \cdot b) \# x^3 p_2 p'\right) \\ &= \eta((a \# p)(b \# p')),\end{aligned}$$

where $\sum x^1 \otimes x^2 \otimes x^3$ is the inverse of the reassociator in H^*.

It is easy to see that $\eta(1 \# \varepsilon) = 1 \# \varepsilon$ and the fact that the inverse of η is $\eta^{-1}(a \# p) = \sum a_{(0)} \# \tau^{-1}(a_{(1)}, \cdot) p$. ∎

We now define the relative Hopf modules in the context of dual quasi-bialgebras.

Definition 2.4. Let H be a dual quasi-bialgebra and A a right H-comodule algebra. A k-vector space M is called a right (H, A)-Hopf module if M is a right H-comodule and a right A-module in the tensor category $\mathscr{M}^H$, i.e., A acts on M to the right such that $m1_A = m$ and the relations

$$(ma)b = \sum \varphi(m_{(1)}, a_{(1)}, b_{(1)}) m_{(0)}(a_{(0)} b_{(0)}), \tag{2.5}$$

$$\rho_M(ma) = \sum m_{(0)} a_{(0)} \otimes m_{(1)} a_{(1)} \tag{2.6}$$

hold for all $m \in M$, $a, b \in A$, where $m \mapsto \rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$ is the right H-comodule structure of M and $m \otimes a \mapsto ma$ is the right action of A on M. We denote by $\mathscr{M}_A^H$ the category of right (H, A)-Hopf modules where the morphisms are right A-linear maps which are H-comodule maps.

Similarly, a k-vector space N is called a left (H, A)-Hopf module if N is a right H-comodule (denote by $n \mapsto \rho_N(n) = \sum n_{(0)} \otimes n_{(1)}$ the right coaction) and a left A-module in the tensor category $\mathscr{M}^H$, i.e., A acts on N to the left (denote the action by $a \otimes n \mapsto an$) such that $1_A n = n$ and the relations

$$(ab)n = \sum \varphi(a_{(1)}, b_{(1)}, n_{(1)}) a_{(0)}(b_{(0)} n_{(0)}), \tag{2.7}$$

$$\rho_N(an) = \sum a_{(0)} n_{(0)} \otimes a_{(1)} n_{(1)} \tag{2.8}$$

hold, for all $n \in N$ and $a, b \in A$. We denote by ${}_A\mathscr{M}^H$ the category of left (H, A)-Hopf modules where the morphisms are left A-linear maps which are H-comodule maps.

It follows that $A \in {}_A\mathscr{M}^H$ and $A \in \mathscr{M}_A^H$. In the same way, if A is a left H-comodule algebra, we can define the categories ${}_A^H\mathscr{M}$ and ${}^H\mathscr{M}_A$. It is easy to see that A^{op} is a right $H^{\text{op,cop}}$-comodule algebra and the following categories are isomorphic:

$${}_A^H\mathscr{M} \cong \mathscr{M}_{A^{\text{op}}}^{H^{\text{op,cop}}} \qquad \text{and} \qquad {}^H\mathscr{M}_A \cong {}_{A^{\text{op}}}\mathscr{M}^{H^{\text{op,cop}}}.$$

Now we aim to describe the above categories. In fact, as in the Hopf case, if H is a finite-dimensional quasi-Hopf algebra and A is a left H-module algebra, then the category of (H^*, A)-Hopf modules is isomorphic to the category of modules over the smash product $A \# H$. To prove this we need a generalization of Hopf algebra formulae of the type $\sum S(h_1)h_2 \otimes h_3 = 1 \otimes h$ to the quasi-coassociative setting and two lemmas. Let H be a quasi-Hopf algebra. Following [8, 9] if we define

$$q_R = \sum X^1 \otimes S^{-1}(\alpha X^3) X^2, \qquad p_R = \sum x^1 \otimes x^2 \beta S(x^3), \tag{2.9}$$

$$q_L = \sum S(x^1) \alpha x^2 \otimes x^3, \qquad p_L = \sum X^2 S^{-1}(X^1 \beta) \otimes X^3, \tag{2.10}$$

then for all $h \in H$ the relations

$$\sum (S(h_1) \otimes 1) q_L \Delta(h_2) = (1 \otimes h) q_L, \tag{2.11}$$

$$\sum \Delta(q_L^2) p_L \big(S^{-1}(q_L^1) \otimes 1\big) = 1 \otimes 1, \tag{2.12}$$

$$\sum \Delta(q_R^1) p_R \big(1 \otimes S(q_R^2)\big) = 1 \otimes 1, \tag{2.13}$$

$$(1 \otimes q_L)(\mathrm{Id} \otimes \Delta)(q_L)\Phi = \sum (S(x^2) \otimes S(x^1) \otimes 1)(f \otimes 1)(\Delta \otimes \mathrm{Id})\big(q_L \Delta(x^3)\big), \tag{2.14}$$

hold, where $q_L = \sum q_L^1 \otimes q_L^2$, $q_R = \sum q_R^1 \otimes q_R^2$, and f is the element defined by (1.11).

LEMMA 2.5. *Let H be a quasi-Hopf algebra, A an H-module algebra, and M a k-vector space. Then:*

(i) *M is a left $A \# H$-module if and only if M is a left H-module (with action denoted by $h \otimes m \mapsto h\cdot m$), A acts weakly on M to the left (i.e., there exists a k-linear map $A \otimes M \to M$, denoted by $a \otimes m \mapsto a.m$, such that $1.m = m$) and the compatibility relations*

$$a.(b.m) = \sum \big[(x^1 \cdot a)(x^2 \cdot b)\big].(x^3 \cdot m) \qquad \textit{for all } a, b \in A \textit{ and } m \in M, \tag{2.15}$$

$$h\cdot(a.m) = \sum (h_1 \cdot a).(h_2 \cdot m) \qquad \textit{for all } h \in H,\ a \in A,\ m \in M. \tag{2.16}$$

hold.

(ii) *M is a right $A \# H$-module if and only if M is a left H-module (denote the action by $h \otimes m \mapsto h \triangleright m$), A acts weakly on M to the right (denote the weak action by $m \otimes a \mapsto ma$) and the compatibility relations*

$$(ma)b = \sum S^{-1}(x^3) \triangleright \big\{m\big[(x^1 \cdot a)(x^2 \cdot b)\big]\big\} \qquad \textit{for all } a, b \in A,\ m \in M \tag{2.17}$$

$$(S^{-1}(h) \triangleright m)a = \sum S^{-1}(h_2) \triangleright \big[m(h_1 \cdot a)\big] \qquad \textit{for all } h \in H,\ a \in A,\ m \in M \tag{2.18}$$

hold.

Proof. The statement (i) is shown in [3, Proposition 2.16]. The proof of (ii) is similar, although slightly more complicated, so we include it. We only define the correspondences and leave the verification of some details to the reader.

Let M be a right $A \# H$-module with $A \# H$-module structure given by

$$m \otimes (a \# h) \mapsto m(a \# h).$$

Since j: $H \to A \# H$, $j(h) = 1 \# h$, is an algebra map, M becomes a right H-module by $m \otimes h \mapsto mj(h)$, thus a left H-module by $h \otimes m \mapsto h \triangleright m = mj(S(h))$. In general, the application i: $A \to A \# H$, $i(a) = a \# 1$, is not an algebra morphism (i.e., it is not multiplicative). But it is clear that A acts weakly on M to the right by $m \otimes a \mapsto ma = mi(a)$. Then we can check that (2.17) and (2.18) hold.

Conversely, if M is a left H-module such that A acts weakly on M to the right and (2.17) and (2.18) hold, then with the structure

$$M \otimes (A \# H) \to M, \qquad m \otimes (a \# h) \mapsto S^{-1}(h) \triangleright (ma)$$

M becomes a right $A \# H$-module. ∎

In order to construct the desired category isomorphism, the above lemma suggests another description for the category of left or right Hopf modules. Let H be a finite-dimensional quasi-Hopf algebra and A a left H-module algebra. If we denote by ${}_A({}_H\mathscr{M})$ the category of left H-modules which are also left A-modules in the tensor category ${}_H\mathscr{M}$ and where the morphisms are left H-linear maps which are also left A-linear then it is easy to see that ${}_A\mathscr{M}^{H^*} \cong {}_A({}_H\mathscr{M})$. Similarly, we can define the category $({}_H\mathscr{M})_A$ and $\mathscr{M}_A^{H^*} \cong ({}_H\mathscr{M})_A$.

LEMMA 2.6. *Let H be a quasi-Hopf algebra and $\delta, f^{-1} \in H \otimes H$ the elements defined in the Preliminaries. If we set $\delta = \sum \delta^1 \otimes \delta^2$ and $f^{-1} = \sum g^1 \otimes g^2$ then the following relations hold*:

(i) $\sum \delta^2 \alpha S^{-1}(\delta^1) = S^{-1}(\beta)$, $\sum g^2 \alpha S^{-1}(g^1) = \varepsilon(\alpha) S^{-1}(\beta)$,

(ii) $\sum S(\delta^1)\alpha\delta^2 = S(\beta)$, $\sum S(g^1)\alpha g^2 = \varepsilon(\alpha)S(\beta)$.

Proof. From the definitions of δ and f^{-1} and using (1.2) we obtain:

$$\sum \delta^1 \otimes \delta^2 = \sum X_1^1 x^1 \beta S(X^3) \otimes X_2^1 x^2 \beta S(X^2 x^3)$$

$$= \sum x^1 \beta S\left(x_2^3 X^3\right) \otimes x^2 X^1 \beta S\left(x_1^3 X^2\right)$$

$$\sum g^1 \otimes g^2 = \sum S(x^1)_1 \alpha_1 x_1^2 \delta^1 S\left(x_2^3\right) \otimes S(x^1)_2 \alpha_2 x_2^2 \delta^2 S\left(x_1^3\right).$$

Using the above relations we can obtain the desired identities. ∎

Now we can prove the following:

PROPOSITION 2.7. *Let H be a finite-dimensional quasi-Hopf algebra and A a left H-module algebra. Then*:

(i) *The categories ${}_A\mathscr{M}^{H^*}$ and ${}_{A\#H}\mathscr{M}$ are isomorphic.*

(ii) *The categories $\mathscr{M}_A^{H^*}$ and $\mathscr{M}_{A\#H}$ are isomorphic.*

Proof. First, by Remark 2.2, A becomes a right H^*-comodule algebra.

(i) follows by ${}_A\mathscr{M}^{H^*} \cong_A ({}_H\mathscr{M})$ and Lemma 2.5(i).

(ii) To show (ii) we begin with some generalities. If F is a gauge transformation for H, there is a canonical monoidal isomorphism between the tensor categories ${}_H\mathscr{M}$ and ${}_{H_F}\mathscr{M}$. This functor is the identity on objects and morphisms with the monoidal structure c given by the multiplication by F^{-1}, that is, for any two left H-modules V, W, $c_{V,W}: V \otimes W \to V \otimes W$, $c_{V,W}(v \otimes w) = \sum G^1 \cdot v \otimes G^2 \cdot w$, where $v \in V$, $w \in W$, and $F^{-1} = \sum G^1 \otimes G^2$. Under this isomorphism the algebra object (i.e., module algebra) A corresponds to an H_F-module algebra $A_{F^{-1}}$. Note that the multiplication of $A_{F^{-1}}$ is given by

$$a \circ b = \sum (G^1 \cdot a)(G^2 \cdot b),$$

and the algebra $A \# H$ is isomorphic to $A_{F^{-1}} \# H_F$ (cf. Proposition 2.3; see also [3, Proposition 2.17]).

Now, let the gauge transformation be the element f defined in (1.11). The formulae (1.13) and (1.15) express the fact that the antipode S gives an isomorphism of quasi-bialgebras $S: H^{\text{op,cop}} \to H_f$. This implies:

(1) A_f is a right H^{cop}-module algebra via S

(2) S induces a monoidal isomorphism between the tensor categories $\mathscr{M}_{H^{\text{cop}}}$ and ${}_{H_f}\mathscr{M}$.

These observations and (i) yield that the following categories are isomorphic with one another:

$$\mathscr{M}_A^{H^*} \cong ({}_H\mathscr{M})_A \cong \left({}_{H_f}\mathscr{M}\right)_{A_{f^{-1}}} \cong (\mathscr{M}_{H^{\text{cop}}})_{A_{f^{-1}}} \cong_{A_{f^{-1}}^{\text{op}}} ({}_{H^{\text{op}}}\mathscr{M})$$
$$\cong \mathscr{M}_{(A_{f^{-1}}^{\text{op}} \# H^{\text{op}})^{\text{op}}}.$$

Therefore one has only to show that the algebras $(A_{f^{-1}}^{\text{op}} \# H^{\text{op}})^{\text{op}}$ and $A \# H$ are isomorphic. Note that $A_{f^{-1}}^{\text{op}}$ is a left H^{op}-module algebra with $h \triangleright a = S(h) \cdot a$ and the multiplication in $(A_{f^{-1}}^{\text{op}} \# H^{\text{op}})^{\text{op}}$ is

$$(a \# h)(b \# h') = \sum \left(g^1 S\left(h'_1 X^2\right) \cdot a\right)\left(g^2 S(X^1) \cdot b\right) \# h h'_2 X^3, \quad (2.19)$$

for all $a, b \in A$, $h, h' \in H$, where $f^{-1} = \sum g^1 \otimes g^2$. Now, if q_L is the element defined by (2.10) then the map $\mu: A \# H \to (A_{f^{-1}}^{\text{op}} \# H^{\text{op}})^{\text{op}}$ given by

$$\mu(a \# h) = \sum f^2 S^{-1}\left(q_L^1 h_1 g^1\right) \cdot a \# S^{-1}\left(f^1 S^{-1}\left(q_L^2 h_2 g^2\right)\right), \quad (2.20)$$

for all $a \in A, h \in H$, is an algebra isomorphism. We first check that μ is an algebra map. We shall denote by $q_L = \Sigma q_L^1 \otimes q_L^2 = \Sigma Q_L^1 \otimes Q_L^2$, $f = \Sigma f^1 \otimes f^2 = \Sigma F^1 \otimes F^2 = \Sigma \mathbf{f}^1 \otimes \mathbf{f}^2 = \Sigma \mathbf{F}^1 \otimes \mathbf{F}^2 = \Sigma \mathscr{F}^1 \otimes \mathscr{F}^2$, and $f^{-1} = \Sigma g^1 \otimes g^2 = \Sigma G^1 \otimes G^2 = \Sigma \mathbf{g}^1 \otimes \mathbf{g}^2 = \Sigma \mathbf{G}^1 \otimes \mathbf{G}^2 = \Sigma \mathscr{G}^1 \otimes \mathscr{G}^2$. Then, for all $a, b \in A$, $h, h' \in H$ we have:

$$\begin{aligned}
&\mu((a \# h)(b \# h')) \\
&\quad = \sum \mu\big((x^1 \cdot a)\big(x^2 h_1 \cdot b\big) \# x^3 h_2 h'\big) \\
&\quad = \sum f^2 S^{-1}\big(q_L^1 x_1^3 (h_2)_1 h'_1 g^1\big)\big[(x^1 \cdot a)\big(x^2 h_1 \cdot b\big)\big] \\
&\qquad \# S^{-1}\big(f^1 S^{-1}\big(q_L^2 x_2^3 (h_2)_2 h'_2 g^2\big)\big) \\
&\quad = \sum \big[f_1^2 S^{-1}\big(S(x^1) F^2 (q_L^1)_2 \big(x_1^3\big)_2 ((h_2)_1 h'_1)_2 g_2^1 G^2\big) \cdot a\big] \\
&\qquad \times \big[f_2^2 S^{-1}\big(S(h_1) S(x^2) F^1 (q_L^1)_1 \big(x_1^3\big)_1 ((h_2)_1 h'_1)_1 g_1^1 G^1\big) \cdot b\big] \\
&\qquad \# S^{-1}\big(f^1 S^{-1}\big(q_L^2 x_2^3 (h_2)_2 h'_2 g^2\big)\big)
\end{aligned}$$

by (1.17) and (1.13)

$$\begin{aligned}
&= \sum \big[f_1^2 S^{-1}\big(q_L^1 \big(Q_L^2\big)_1 X^2 ((h_2)_1)_2 (h'_1)_2 g_2^1 G^2\big) \cdot a\big] \\
&\quad \times \big[f_2^2 S^{-1}\big(S(h_1) Q_L^1 X^1 ((h_2)_1)_1 (h'_1)_1 g_1^1 G^1\big) \cdot b\big] \\
&\quad \# S^{-1}\big(f^1 S^{-1}\big(q_L^2 \big(Q_L^2\big)_2 X^3 (h_2)_2 h'_2 g^2\big)\big)
\end{aligned}$$

by (2.14)

$$\begin{aligned}
&= \sum \big[f_1^2 S^{-1}\big(q_L^1 \big(Q_L^2 (h_2)_2\big)_1 X^2 (h'_1)_2 g_2^1 G^2\big) \cdot a\big] \\
&\quad \times \big[f_2^2 S^{-1}\big(S(h_1) Q_L^1 (h_2)_1 X^1 (h'_1)_1 g_1^1 G^1\big) \cdot b\big] \\
&\quad \# S^{-1}\big(f^1 S^{-1}\big(q_L^2 \big(Q_L^2 (h_2)_2\big)_2 X^3 h'_2 g^2\big)\big)
\end{aligned}$$

by (1.1)

$$\begin{aligned}
&= \sum \big[f_1^2 S^{-1}\big(q_L^1 h_1 \big(Q_L^2\big)_1 (h'_2)_1 X^2 g_2^1 G^2\big) \cdot a\big] \\
&\quad \times \big[f_2^2 S^{-1}\big(Q_L^1 h'_1 X^1 g_1^1 G^1\big) \cdot b\big] \\
&\quad \# S^{-1}\big(f^1 S^{-1}\big(q_L^2 h_2 \big(Q_L^2\big)_2 (h'_2)_2 X^3 g^2\big)\big)
\end{aligned}$$

by (1.1) and (2.11)

$$\begin{aligned}
&= \sum \big[f_1^2 X^2 S^{-1}\big(q_L^1 h_1 \big(Q_L^2\big)_1 (h'_2)_1 g_1^2 G^1\big) \cdot a\big] \\
&\quad \times \big[f_2^2 X^3 S^{-1}\big(Q_L^1 h'_1 g^1\big) \cdot b\big] \\
&\quad \# S^{-1}\big(f^1 X^1 S^{-1}\big(q_L^2 h_2 \big(Q_L^2\big)_2 (h'_2)_2 g_2^2 G^2\big)\big)
\end{aligned}$$

by (1.15).

On the other hand,

$$
\begin{aligned}
&\mu(a \# h)\,\mu(b \# h') \\
&\quad = \sum \left[f^2 S^{-1}\big(q_L^1 h_1 g^1\big) \cdot a \# S^{-1}\big(f^1 S^{-1}\big(q_L^2 h_2 g^2\big)\big)\right] \\
&\qquad \times \left[F^2 S^{-1}\big(Q_L^1 h_1' G^1\big) \cdot b \# S^{-1}\big(F^1 S^{-1}\big(Q_L^2 h_2' G^2\big)\big)\right] \\
&\quad = \sum \Big[\mathbf{g}^1 S\Big(S^{-1}\Big(\mathbf{F}^2 F_2^1 S^{-1}\big(\mathscr{F}^1 (Q_L^2)_1 (h_2')_1 G_1^2 \mathscr{G}^1\big)\mathbf{G}^2\Big) X^2\Big) \\
&\qquad\qquad\qquad\qquad \times f^2 S^{-1}\big(q_L^1 h_1 g^1\big) \cdot a\Big] \\
&\qquad \times \left[\mathbf{g}^2 S(X^1) F^2 S^{-1}\big(Q_L^1 h_1' G^1\big) \cdot b\right] \# S^{-1}\big(f^1 S^{-1}\big(q_L^2 h_2 g^2\big)\big) \\
&\qquad\qquad\qquad \times S^{-1}\Big(\mathbf{F}^1 F_1^1 S^{-1}\big(\mathscr{F}^2 (Q_L^2)_2 (h_2')_2 G_2^2 \mathscr{G}^2\big)\mathbf{G}^1\Big) X^3
\end{aligned}
$$

by (2.19) and (1.13)

$$
\begin{aligned}
&= \sum \left[\mathbf{g}^1 S(X^2) \mathbf{F}^2 F_2^1 S^{-1}\big(q_L^1 h_1 g^1 \mathscr{F}^1 (Q_L^2)_1 (h_2')_1 G_1^2 \mathscr{G}^1\big) \cdot a\right] \\
&\quad \times \left[\mathbf{g}^2 S(X^1) F^2 S^{-1}\big(Q_L^1 h_1' G^1\big) \cdot b\right] \\
&\quad \# S^{-1}\Big(S(X^3) \mathbf{F}^1 F_1^1 S^{-1}\big(q_L^2 h_2 g^2 \mathscr{F}^2 (Q_L^2)_2 (h_2')_2 G_2^2 \mathscr{G}^2\big)\Big) \\
&= \sum \left[f_1^2 X^2 S^{-1}\big(q_L^1 h_1 (Q_L^2)_1 (h_2')_1 G_1^2 \mathscr{G}^1\big) \cdot a\right] \\
&\quad \times \left[f_2^2 X^3 S^{-1}\big(Q_L^1 h_1' G^1\big) \cdot b\right] \\
&\quad \# S^{-1}\Big(f^1 X^1 S^{-1}\big(q_L^2 h_2 (Q_L^2)_2 (h_2')_2 G_2^2 \mathscr{G}^2\big)\Big)
\end{aligned}
$$

by (1.13)

$$
= \mu((a \# h)(b \# h')).
$$

It follows that $\mu(1 \# 1) = 1 \# 1$, thus one has only to show that μ is bijective. For this we define $\mu^{-1}\colon (A_{f^{-1}}^{\mathrm{op}} \# H^{\mathrm{op}})^{\mathrm{op}} \to A \# H$ by

$$
\mu^{-1}(a \# h) = \sum \big(1 \# S\big(G^1 S(h)\big)\big)\big(g^1 S\big(q_R^2\big) G^2 \cdot a \# g^2 S\big(q_R^1\big)\big), \quad (2.21)
$$

for all $a \in A$, $h \in H$, where $q_R = \sum q_R^1 \otimes q_R^2$ is the element defined by (2.9). Finally we check that μ and μ^{-1} are inverses. Now

$$
\begin{aligned}
&\mu\mu^{-1}(a \# h) \\
&\quad = \sum \mu\big(1 \# S\big(G^1 S(h)\big)\big)\mu\big(g^1 S\big(q_R^2\big) G^2 \cdot a \# g^2 S\big(q_R^1\big)\big) \\
&\quad = \sum \varepsilon(\alpha)\big(1 \# h S^{-1}(G^1)\big) \\
&\qquad \times \Big[f^2 S^{-1}\big(q_L^1 g_1^2 \mathscr{G}^1 S\big((q_R^1)_2\big)\big) g^1 S\big(q_R^2\big) G^2 \cdot a \\
&\qquad\qquad\qquad \# S^{-1}\Big(f^1 S^{-1}\big(q_L^2 g_2^2 \mathscr{G}^2 S\big((q_R^1)_1\big)\big)\Big)\Big]
\end{aligned}
$$

by (2.20) and (1.13)

$$= \sum f^2 S^{-1}\big(q_L^1 g_1^2 \mathscr{G}^1 S\big((q_R^1)_2\big)\big) g^1 S\big(q_R^2\big) G^2 \cdot a$$

$$\# \, h S^{-1}\big(f^1 S^{-1}\big(q_L^2 g_2^2 \mathscr{G}^2 S\big((q_R^1)_1\big)\big) G^1\big)$$

by (2.19) and since $\sum \varepsilon(g^1) g^2 = \varepsilon(\beta)1$, $\varepsilon(\alpha)\varepsilon(\beta) = 1$

$$= \sum f^2 \big(q_R^1\big)_2 x^2 S^{-1}\big(S(X^1) q_L^1 X^2 g_2^1 \mathscr{G}^2\big) g_1^1 \mathscr{G}^1 S\big(q_R^2 x^3\big) G^2 \cdot a$$

$$\# h S^{-1}\big(f^1 \big(q_R^1\big)_1 x^1 S^{-1}\big(q_L^2 X^3 g^2\big) G^1\big)$$

by (1.13)

$$= \sum f^2 \big(q_R^1\big)_2 x^2 S^{-1}\big(\alpha g_2^1 \mathscr{G}^2\big) g_1^1 \mathscr{G}^1 S\big(q_R^2 x^3\big) G^2 \cdot a$$

$$\# \, h S^{-1}\big(f^1 \big(q_R^1\big)_1 x^1 S^{-1}\big(g^2\big) G^1\big)$$

by (2.10)

$$= \sum f^2 \big(q_R^1\big)_2 x^2 \beta S(x^3) S\big(q_R^2\big) G^2 \cdot a \, \# \, h S^{-1}\big(f^1 \big(q_R^1\big)_1 x^1 G^1\big)$$

by (1.5) and Lemma 2.6(ii)

$$= \sum f^2 \big(q_R^1\big)_2 p_R^2 S\big(q_R^2\big) G^2 \cdot a \, \# \, h S^{-1}\big(f^1 \big(q_R^1\big)_1 p_R^1 G^1\big)$$

by (2.9)

$$= a \# h$$

by (2.13), and similarly,

$$\mu^{-1}\mu(a \# h)$$

$$= \sum \mu^{-1}\big(f^2 S^{-1}\big(q_L^1 h_1 g^1\big) \cdot a \# S^{-1}\big(f^1 S^{-1}\big(q_L^2 h_2 g^2\big)\big)\big)$$

$$= \sum \big(1 \# q_L^2 h_2 g^2\big)\big(G^1 S\big(q_R^2\big) S^{-1}\big(q_L^1 h_1 g^1\big) \cdot a \# G^2 S\big(q_R^1\big)\big)$$

by (2.21)

$$= \sum \big(q_L^2\big)_1 (h_2)_1 g_1^2 G^1 S\big(q_R^2\big) S^{-1}\big(q_L^1 h_1 g^1\big) \cdot a$$

$$\# \, \big(q_L^2\big)_2 (h_2)_2 g_2^2 G^2 S\big(q_R^1\big)$$

$$= \sum \big(q_L^2\big)_1 (h_2)_1 X^2 g_2^1 G^2 S\big(q_R^2 x^2\big) x^3 S^{-1}\big(q_L^1 h_1 X^1 g_1^1 G^1\big) \cdot a$$

$$\# \big(q_L^2\big)_2 (h_2)_2 X^3 g^2 S\big(q_R^1 x^1\big)$$

by (1.13)

$$= \sum (q_L^2)_1 X^2 (h_1)_2 g_2^1 G^2 \alpha S^{-1}\big(q_L^1 X^1 (h_1)_1 g_1^1 G^1\big) \cdot a \\ \# \, (q_L^2)_2 X^3 h_2 g^2$$

by (1.1) and (2.9)

$$= \sum (q_L^2)_1 X^2 (h_1)_2 S^{-1}\big(q_L^1 X^1 (h_1)_1 \beta\big) \cdot a \# (q_L^2)_2 X^3 h_2$$

by Lemma 2.6(i) and (1.5)

$$= \sum (q_L^2)_1 p_L^1 S^{-1}(q_L^1) \cdot a \# (q_L^2)_2 p_L^2 h$$

by (1.5) and (2.10)

$$= a \# h$$

by (2.12).

If we compute explicitly the above isomorphisms we obtain the following correspondence:

- If M is a right $A \# H$-module then M becomes a right (H^*, A)-Hopf module with the structures:

$$m \mapsto \sum_{i=1}^{n} m(1 \# S(e_i)) \otimes e^i$$

$$\big(\{e_i\}_{i=\overline{1,n}} \text{ and } \{e^i\}_{i=\overline{1,n}} \text{ are dual bases in } H \text{ and } H^*\big),$$

$$ma = \sum m\big(g^1 S(q_R^2) \cdot a \# g^2 S(q_R^1)\big).$$

- Conversely, if M is a right (H^*, A)-Hopf module then if we regard M as an object in $({}_H\mathcal{M})_A$ then M becomes a right $A \# H$-module with

$$m \leftarrow (a \# h) = S^{-1}(h)\big[\big(S^{-1}(q_L^2 g^2)m\big)\big(S^{-1}(q_L^1 g^1) \cdot a\big)\big].$$

∎

Remark 2.8. (i) It follows that if we start with a finite-dimensional dual quasi-Hopf algebra H and a right H-comodule algebra A, then the categories ${}_A\mathcal{M}^H$ and $\mathcal{M}_A^H$ are isomorphic to the categories ${}_{A \# H^*}\mathcal{M}$ and $\mathcal{M}_{A \# H^*}$ respectively. Moreover, if we let $\tau \in (H \otimes H)^*$ be a gauge transformation, then by the above proposition and Proposition 2.3 we obtain that the categories ${}_A\mathcal{M}^H$ and ${}_{A_{\tau^{-1}}}\mathcal{M}^{H_\tau}$ are isomorphic, respectively $\mathcal{M}_A^H$ and $\mathcal{M}_{A_{\tau^{-1}}}^{H_\tau}$ are isomorphic.

(ii) Let A be a finite-dimensional right comodule algebra over a dual quasi-Hopf algebra H. Then A^*, the linear dual space of A, becomes a coalgebra in the tensor category of left H-comodules ${}^H\mathcal{M}$ with left

H-comodule structure $a^* \mapsto \sum a^*(a_{i_{(0)}})a_{i_{(1)}} \otimes a^i$, comultiplication $\Delta(a^*) = \sum a^*(a_i a_j)a^i \otimes a^j$, and counit $\varepsilon(a^*) = a^*(1_A)$, for all $a^* \in A^*$, where $\{a_i\}$ is a basis in A with dual basis $\{a^i\}$ in A^*. Under the circumstances, it was proved in [4] that the categories ${}_A\mathscr{M}^H$ and $\mathscr{M}_A^H$ are isomorphic to the categories of comodules over the smash coproduct $A^* \rtimes H$, namely $\mathscr{M}^{A^* \rtimes H}$ and ${}^{A^* \rtimes H}\mathscr{M}$ respectively.

The proof of the following results is slightly different from the one in the case where H is a Hopf algebra [5, Theorem 1].

PROPOSITION 2.9. *Let A be a right H-comodule algebra such that there exists a right comodule map λ: $H \to A$ with $\lambda(1) = 1$. Then every right (H, A)-Hopf module is injective as an H-comodule.*

Proof. Let $M \in \mathscr{M}_A^H$. As in the Hopf case, $M \otimes H$ is a right H-comodule with structure map given by $\mathrm{Id} \otimes \Delta$. We show that there is an H-comodule map $\bar{\lambda}$: $M \otimes H \to M$ with $\bar{\lambda}\rho_M = \mathrm{Id}$. Thus M is injective since it is isomorphic to a direct summand of $M \otimes H$, an injective H-comodule. So, for all $m \in M$, $h \in H$ define

$$\bar{\lambda}(m \otimes h) = \sum \varphi\big(m_{(1)}, S(m_{(3)}), h_2\big)\beta(m_{(2)})\,\alpha(m_{(5)})m_{(0)}\lambda\big(S(m_{(4)})h_1\big).$$

By (1.24) and (1.25) it follows that $\bar{\lambda}\rho_M = \mathrm{Id}$. Next, we shall prove that $\bar{\lambda}$ is an H-comodule map. Indeed, for all $m \in M$ and $h \in H$, we have

$$\begin{aligned}
&\sum \bar{\lambda}(m \otimes h)_{(0)} \otimes \bar{\lambda}(m \otimes h)_{(1)} \\
&\quad = \sum \varphi\big(m_{(1)}, S(m_{(3)}), h_2\big)\beta(m_{(2)})\,\alpha(m_{(5)})\big(m_{(0)}\lambda\big(S(m_{(4)})h_1\big)\big)_{(0)} \\
&\qquad \otimes \big(m_{(0)}\lambda\big(S(m_{(4)})h_1\big)\big)_{(1)} \\
&\quad = \sum \varphi\big(m_{(2)}, S(m_{(4)}), h_3\big)\beta(m_{(3)})\,\alpha(m_{(7)})m_{(0)}\lambda\big(S(m_{(6)})h_1\big) \\
&\qquad \otimes m_{(1)}\big(S(m_{(5)})h_2\big)
\end{aligned}$$

(since λ is a comodule map)

$$\begin{aligned}
&\quad = \sum \varphi\big(m_{(1)}, S(m_{(5)}), h_2\big)\alpha(m_{(7)})m_{(0)}\lambda\big(S(m_{(6)})h_1\big) \\
&\qquad \otimes \big(m_{(2)}\beta(m_{(3)})S(m_{(4)})\big)h_3
\end{aligned}$$

by (1.20)

$$= \sum \varphi\big(m_{(1)}, S(m_{(3)}), h_2\big)\beta(m_{(2)})\,\alpha(m_{(5)})m_{(0)}\lambda\big(S(m_{(4)})h_1\big) \otimes h_3$$

by (1.24)

$$= \sum \lambda(m \otimes h_1) \otimes h_2.$$

∎

The proof of the next result is the same as in [5].

COROLLARY 2.10. *Let H be a dual quasi-Hopf algebra and A a right H-comodule algebra. The following statements are equivalent*:

(i) *A is an injective H-comodule.*

(ii) *There is a right H-comodule map λ: $H \to A$ with $\lambda(1) = 1$.*

(iii) *Every object in $\mathcal{M}_A^H$ is an injective H-comodule.*

Let H be a dual quasi-Hopf algebra and A a right H-comodule algebra. Define the H-invariant subspace of A to be the set

$$A_0 = \{a \in A \mid \rho_A(a) = a \otimes 1\}.$$

It is clear that A_0 is an associative subalgebra of A in the tensor category $\mathcal{M}^H$.

Let V be a right A_0-module. One easily checks that $V \otimes_{A_0} A$ is a right (H, A)-Hopf module with the right H-comodule structure and the right A-action given for all $v \in V$, $a, b \in A$ by

$$\rho: v \otimes_{A_0} a \mapsto \sum (v \otimes_{A_0} a_{(0)}) \otimes a_{(1)}, \qquad (v \otimes_{A_0} a)b = v \otimes_{A_0} ab.$$

Similarly, for $M \in \mathcal{M}_A^H$ define the H-invariant subspace of M to be the set

$$M_0 = \{m \in M \mid \rho_M(m) = m \otimes 1\}.$$

For any $m \in M_0$ and $a \in A_0$ we have $ma \in M_0$ and thus M_0 is a right A_0-module. Define

$$u: M_0 \otimes_{A_0} A \to M, \qquad u(m \otimes_{A_0} a) = ma \text{ for all } m \in M_0, a \in A.$$

It is easy to see that u is an H-comodule map and an A-linear map, hence a morphism in $\mathcal{M}_A^H$. Now we can prove the main result of this section which generalizes [5, Theorem 3]:

THEOREM 2.11. *Let H be a dual quasi-Hopf algebra and A a right H-comodule algebra. If there is a right H-comodule map λ: $H \to A$ which is an algebra map (i.e., is multiplicative and $\lambda(1) = 1$) then for every $M \in \mathcal{M}_A^H$ the map u defined above is an isomorphism of (H, A)-Hopf modules.*

Proof. Let P: $M \to M$ be defined for any $m \in M$ by

$$P(m) = \sum m_{(0)} \beta(m_{(1)}) \lambda(S(m_{(2)})).$$

We claim $P(M) \subset M_0$:

$$\rho_M(P(m)) = \sum \beta(m_{(1)})\big(m_{(0)}\lambda\big(S(m_{(2)})\big)\big)_{(0)} \otimes \big(m_{(0)}\lambda\big(S(m_{(2)})\big)\big)_{(1)}$$
$$= \sum \beta(m_{(2)})m_{(0)}\lambda\big(S(m_{(4)})\big) \otimes m_{(1)}S(m_{(3)})$$
$$= \sum m_{(0)}\lambda\big(S(m_{(4)})\big) \otimes m_{(1)}\,\beta(m_{(2)})S(m_{(3)})$$
$$= \sum m_{(0)}\,\beta(m_{(1)})\,\lambda\big(S(m_{(2)})\big) \otimes 1$$

by (1.24)

$$= P(m) \otimes 1.$$

Thus $P\colon M \to M_0$ and therefore the map

$$v\colon M \to M_0 \otimes_{A_0} A \qquad v(m) = \sum P(m_{(0)}) \otimes \alpha(m_{(1)})\,\lambda(m_{(2)})$$

is well defined. We will show $uv = \mathrm{Id}$, $vu = \mathrm{Id}$

$$uv(m) = \sum P(m_{(0)})\,\alpha(m_{(1)})\,\lambda(m_{(2)})$$
$$= \sum \alpha(m_{(3)})\,\beta(m_{(1)})\big(m_{(0)}\lambda\big(S(m_{(2)})\big)\big)\lambda(m_{(4)})$$
$$= \sum \varphi\big(m_{(1)}, S(m_{(3)}), m_{(7)}\big)\alpha(m_{(5)})\,\beta(m_{(2)})$$
$$\times m_{(0)}\big(\lambda\big(S(m_{(4)})\big)\lambda(m_{(6)})\big)$$

by (2.5)

$$= \sum \varphi\big(m_{(1)}, S(m_{(3)}), m_{(7)}\big)\beta(m_{(2)})m_{(0)}\lambda\big(S(m_{(4)})\,\alpha(m_{(5)})m_{(6)}\big)$$
$$= \sum \varphi\big(m_{(1)}, S(m_{(3)}), m_{(5)}\big)\beta(m_{(2)})\,\alpha(m_{(4)})m_{(0)}$$

by (1.24)

$$= m$$

by (1.25).

For $m \in M_0$ and $a \in A$ we have:

$$vu\big(m \otimes_{A_0} a\big) = \sum P(ma_{(0)}) \otimes_{A_0} \alpha(a_{(1)})\,\lambda(a_{(2)})$$
$$= \sum m\big(a_{(0)}\,\beta(a_{(1)})\,\lambda\big(S(a_{(2)})\big)\big) \otimes_{A_0} \alpha(a_{(3)})\,\lambda(a_{(4)})$$

(by (2.5), because $m \in M_0$)

$$= \sum m \otimes_{A_0} \big(a_{(0)}\,\beta(a_{(1)})\,\lambda\big(S(a_{(2)})\big)\big)\alpha(a_{(3)})\,\lambda(a_{(4)})$$

(because $\sum a_{(0)}\beta(a_{(1)})\lambda(S(a_{(2)})) \in A_0$)

$$= m \otimes_{A_0} \sum \beta(a_{(2)})\alpha(a_{(5)})\varphi\big(a_{(1)}, S(a_{(3)}), a_{(7)}\big)$$
$$\times a_{(0)}\big(\lambda\big(S(a_{(4)})\big)\lambda(a_{(6)})\big)$$

by (2.1)

$$= m \otimes_{A_0} \sum \beta(a_{(2)})\varphi\big(a_{(1)}, S(a_{(3)}), a_{(7)}\big)$$
$$\times a_{(0)}\lambda\big(S(a_{(4)})\alpha(a_{(5)})a_{(6)}\big)$$
$$= m \otimes_{A_0} \sum \beta(a_{(2)})\alpha(a_{(4)})\varphi\big(a_{(1)}, S(a_{(3)}), a_{(5)}\big)a_{(0)}$$

by (1.24)

$$= m \otimes_{A_0} a$$

by (1.25). ∎

3. THE DUAL CASE: $[C, H]$-HOPF MODULES

In this section we shall dualize the notion of (H, A)-Hopf modules defined in Section 2 for a dual quasi-Hopf algebra H and a right H-comodule algebra A. We shall only describe the dual versions of Proposition 2.3, Proposition 2.7, Proposition 2.9, and Corollary 2.10. Since the proof of Theorem 2.11 is not so easily dualized we include a sketch of the proof for Theorem 3.5.

Now, let H be a quasi-bialagebra over a field k and C a k-linear space. We say that C is a right H-module coalgebra if C is a coalgebra in the tensor category $\mathscr{M}_H$ (note that in $\mathscr{M}_H$ the associativity constraint $\underline{a}$ is given via Φ^{-1}), that is, if C has a comultiplication Δ and a usual counit ε satisfying the following conditions:

$$(\Delta \otimes \mathrm{Id})(\Delta(c))\Phi^{-1} = (\mathrm{Id} \otimes \Delta)(\Delta(c)), \tag{3.1}$$

$$\Delta(ch) = \sum c_1 h_1 \otimes c_2 h_2, \tag{3.2}$$

$$\varepsilon(ch) = \varepsilon(c)\varepsilon(h), \tag{3.3}$$

for all $c \in C$, $h \in H$, where we still denote $\Delta(c) = \sum c_1 \otimes c_2$ and where ω_C: $c \otimes h \mapsto ch$ is the right H-module structure of C.

Note that if H is finite-dimensional, then C is a coalgebra in $\mathscr{M}_H$ if and only if C is a left H^*-comodule coalgebra, i.e., C is a coalgebra in the tensor category ${}^{H^*}\mathscr{M}$. Now, let H be a finite-dimensional dual quasi-Hopf algebra. If we define on H a new comultiplication given by

$$\overline{\Delta}(h) = \sum \varphi(h_1, S(h_3), h_7S(h_9))\alpha(h_5)\varphi^{-1}(S(h_4), h_6, S(h_{10}))h_2 \otimes h_8,$$

for all $h \in H$, and we denote this new structure on H by $\overline{H}$, then $\overline{H}$ becomes a left H-comodule coalgebra with counit $\overline{\varepsilon}(h) = \beta(h)$ and with the left adjoint coaction, that is, $\rho_{\overline{H}}(h) = \sum h_1S(h_3) \otimes h_2$ for all $h \in H$ (see [4]). So, $\overline{H}$ is a right H^*-module coalgebra, where the H^*-module structure is $h \triangleleft p = \sum p(h_1S(h_3))h_2$ for all $h \in H$, $p \in H^*$.

Dual, if we start with a finite-dimensional quasi-Hopf algebra H then $\tilde{H}^*$ is a right H-module coalgebra, where $\tilde{H}^* = H^*$ as linear spaces, but the comultiplication is given by

$$\tilde{\Delta}(p) = \sum S(x^1X^2)\alpha x^2X^3_1 \rightharpoonup p_1 \leftharpoonup X^1 \otimes S(x^3X^3_2) \rightharpoonup p_2,$$

where for all $p \in H^*$ and $h, l \in H$ we define $h \rightharpoonup p \in H^*$ by $(h \rightharpoonup p)(l) = p(lh)$ and where $\Delta(p) = \sum p_1 \otimes p_2$ is the natural comultiplication on H^*.

Let H be a quasi-bialgebra and C a right H-module coalgebra. Similarly to Proposition 2.3 we can prove that a twist preserves the class of right H-module coalgebras. First, observe that C^*, the linear dual space of C, is a left H-module with $h \otimes c^* \mapsto h \rightharpoonup c^*$, where $(h \rightharpoonup c^*)(c) = c^*(ch)$, for all $h \in H$, $c \in C$, $c^* \in C^*$. Moreover, if for all $c^*, d^* \in C^*$ we put $c^*d^* = (c^* \otimes d^*) \circ \Delta$, then C^* becomes a left H-module algebra, so we can define the smash product $C^* \# H$.

PROPOSITION 3.1. *Let $F \in H \otimes H$ be a gauge transformation on H. If we introduce on C a new comultiplication by $\Delta_{F^{-1}}(c) = \Delta(c)F^{-1}$, where F^{-1} is the inverse of F, and we denote by $C_{F^{-1}}$ the resulting structure, then $C_{F^{-1}}$ becomes a right H_F-module coalgebra. Moreover,*

$$\gamma : C^* \# H \to C^*_{F^{-1}} \# H_F, \qquad \gamma(c^* \# h) = \sum F^1 \rightharpoonup c^* \# F^2h$$

is an algebra isomorphism with inverse $\gamma^{-1}(c^* \# h) = \sum G^1 \rightharpoonup c^* \# G^2h$, where $F = \sum F^1 \otimes F^2$, $F^{-1} = \sum G^1 \otimes G^2$ (formal notations). Note that the multiplication in $C^*_{F^{-1}}$ is $c^* \cdot d^* = \sum(G^1 \rightharpoonup c^*)(G^2 \rightharpoonup d^*)$ for all c^*, $d^* \in C^*$.

Now, a k-vector space is called a right $[C, H]$-Hopf module if N is a right H-module (denote the structure map by ω_N: $n \otimes h \mapsto nh$) and a right C-comodule in the tensor category $\mathscr{M}_H$, i.e., C coacts weakly on N to the right (denote the structure map by ρ_N: $n \mapsto \sum n_{[0]} \otimes n_{[1]} \in N \otimes C$) such that for all $n \in N$ and $h \in H$, $\sum \varepsilon(n_{[1]})n_{[0]} = n$ and the following relations hold:

$$\sum n_{[0]_{[0]}}x^1 \otimes n_{[0]_{[1]}}x^2 \otimes n_{[1]}x^3 = \sum n_{[0]} \otimes n_{[1]_1} \otimes n_{[1]_2}, \tag{3.4}$$

$$\rho_N(nh) = \sum n_{[0]}h_1 \otimes n_{[1]}h_2. \tag{3.5}$$

We denote by $\mathscr{M}_H^C$ the category of right $[C, H]$-Hopf modules where the morphisms are right H-linear maps which are right C-comodule maps (just as in the Hopf case). Similarly, we can define the category ${}^C\mathscr{M}_H$ of left $[C, H]$-Hopf modules (for $N \in {}^C\mathscr{M}_H$ we will denote the C-weak coaction by $n \mapsto \sum n_{[-1]} \otimes n_{[0]}$).

Now, throughout this section H will be a quasi-Hopf algebra and C a right H-module coalgebra. We can prove that if C is finite-dimensional (we will denote by $\{c_i\}$ and $\{c^i\}$ dual bases in C and C^*), then the above categories are isomorphic to the categories of modules over the smash product $C^* \# H$. In fact, we have the following:

Proposition 3.2. *Let C be a finite-dimensional H-module coalgebra. Then*

(i) *The categories ${}^C\mathscr{M}_H$ and $\mathscr{M}_{C^* \# H}$ are isomorphic.*

(ii) *The categories $\mathscr{M}_H^C$ and ${}_{C^* \# H}\mathscr{M}$ are isomorphic.*

Proof. (i) Let N be a right $C^* \# H$-module. By Lemma 2.5(ii), N is a left H-module and C^* acts weakly on N to the right such that (2.17) and (2.18) hold. Then N becomes a right H-module with $nh = S^{-1}(h) \triangleright n$ for all $n \in N$, $h \in H$, and if we define

$$\rho_N(n) = \sum c_i \otimes nc^i$$

then we can easily check that N is a left $[C, H]$-Hopf module.

Conversely, if $N \in {}^C\mathscr{M}_H$ then N is a left H-module with $h \triangleright n = nS(h)$ and C^* acts weakly on N to the right by $nc^* = \sum c^*(n_{[-1]})n_{[0]}$ such that the conditions (2.17) and (2.18) hold; therefore N is a right $C^* \# H$-module. The correspondence described above defines two functors that provide category isomorphisms.

(ii) Let N be a right $[C, H]$-Hopf module. So, N is a right H-module and hence a left H-module with $hn = nS^{-1}(h)$. Moreover, if we define

$$c^* n = \sum c^*\left(n_{[1]} S^{-1}\left(S(x^1)\alpha x^2 g^1\right)\right) n_{[0]} S^{-1}\left(x^3 g^2\right),$$

for all $c^* \in C^*$, $n \in N$ (where $f^{-1} = \sum g^1 \otimes g^2$ is the element defined by (1.12)), then by computations similar to those in Proposition 2.7(ii) we can check that the conditions (2.15) and (2.16) hold and N becomes a left $C^* \# H$-module.

Conversely, let N be a left $C^* \# H$-module. If we define

$$\rho_N(n) = \sum \left(g^1 S(X^2)\alpha X^3 \rightharpoonup c^i \# g^2 S(X^1)\right) n \otimes c_i,$$

for all $n \in N$, then N becomes a right $[C, H]$-Hopf module with the H-module structure $nh = (1 \# S(h))n$. By computations similar to those in Proposition 2.7(ii) we can check that the correspondence described above defines two functors (which act as the identity on morphisms) being inverse to each other. ∎

Note that if H is finite-dimensional then C becomes a coalgebra in the tensor category ${}^{H^*}\mathscr{M}$ with the same comultiplication and counit and with the H^*-comodule structure $c \mapsto \sum e^i \otimes ce_i$, where $c \in C$ and $\{e_i\}$ is a basis in H with dual basis $\{e^i\}$ in H^*. Under these circumstances, it was proved in [4] that the categories ${}^C\mathscr{M}_H$ and $\mathscr{M}_H^C$ are isomorphic to the categories of comodules over the smash coproduct $C \rtimes H^*$, namely ${}^{C \rtimes H^*}\mathscr{M}$ and $\mathscr{M}^{C \rtimes H^*}$ respectively.

Now, let N be a right $[C, H]$-Hopf module and suppose that there exists a right H-module map Ψ: $C \to H$ with $\varepsilon\Psi = \varepsilon$. If we define $\overline{\Psi}$: $N \to N \otimes H$ by

$$\overline{\Psi}(n) = \sum n_{[0]} X^1 \beta S\left(\Psi(n_{[1]})_1 X^2\right)\alpha \otimes \Psi(n_{[1]})_2 X^3,$$

for any $n \in N$, then $\overline{\Psi}$ is an H-module map with $\omega_N \overline{\Psi} = \mathrm{Id}$ where $N \otimes H$ has the right H-module structure given via the multiplication on H. Therefore, N is a projective H-module since it is isomorphic to a direct summand on $N \otimes H$, a free H-module. We summarize this in the following:

Proposition 3.3. *Let C be a right H-module coalgebra. Then the following statements are equivalent.*

(i) *C is a projective H-module.*

(ii) *There is a right H-module map Ψ: $C \to H$ with $\varepsilon\Psi = \varepsilon$,*

(iii) *Every object in $\mathscr{M}_H^C$ is a projective H-module.*

Finally, we dualize Theorem 2.11. Let C be a right H-module coalgebra. If H^+ denotes the kernel of ε: $H \to k$ then it follows that $\Delta(CH^+) \subseteq CH^+ \otimes C + C \otimes CH^+$ and $\varepsilon(CH^+) = 0$, hence $\overline{C} = C/CH^+$ has a unique coalgebra structure in $\mathscr{M}_H$ such that the projection p: $C \to \overline{C}$ is a coalgebra map in $\mathscr{M}_H$. Let N be a right $[C, H]$-Hopf module. Then N is a right $\overline{C}$-comodule in $\mathscr{M}_H$ via $(\mathrm{Id} \otimes p)\rho_N$ and NH^+ is a $\overline{C}$-subcomodule of N (that is, $(\mathrm{Id} \otimes p)\rho_N(NH^+) \subseteq NH^+ \otimes \overline{C}$). Thus $\overline{N} = N/NH^+$ has a unique $\overline{C}$-comodule structure $\bar{\rho}$: $\overline{N} \to \overline{N} \otimes \overline{C}$ making the projection π: $N \to \overline{N}$ a $\overline{C}$-comodule map in $\mathscr{M}_H$. Note that we have

$$\pi(nh) = \varepsilon(h)\pi(n), \qquad \text{for all } n \in N, h \in H. \tag{3.6}$$

Now, we need the following:

DEFINITION 3.4. Let C be a right H-module coalgebra and $V \in \mathscr{M}_H^C$, $W \in {}^C\mathscr{M}_H$ with structure comodule maps ρ_V, ρ_W respectively. Then the cotensor product $V \square_C W$ is the equalizer of

$$\underline{a}_{V,C,W}(\rho_V \otimes \mathrm{Id}), \qquad \mathrm{Id} \otimes \rho_W : V \otimes W \to V \otimes C \otimes W,$$

that is, $\sum_i v_i \otimes w_i \in V \square_C W \subseteq V \otimes W$ if and only if

$$\sum_i (v_i)_{[0]} x^1 \otimes (v_i)_{[1]} x^2 \otimes w_i x^3 = \sum_i v_i \otimes (w_i)_{[-1]} \otimes (w_i)_{[0]}. \quad (3.7)$$

It is not hard to see that $C \in {}^{\overline{C}}\mathscr{M}_H$ via $(p \otimes \mathrm{Id})\Delta$. In these terms, if we define

$$\theta : N \to \overline{N} \otimes C, \qquad \theta(n) = \sum \pi(n_{[0]}) \otimes n_{[1]}$$

then it is easy to see that $\theta(N) \subseteq \overline{N} \square_{\overline{C}} C$ and, moreover, that θ is a $[C, H]$-Hopf module map, where $\overline{N} \square_{\overline{C}} C$ is a right $[C, H]$-Hopf module via $\mathrm{Id} \otimes \omega_C$ and $\mathrm{Id} \otimes \Delta$.

Now, we can generalize [5, Theorem 5]. The proof is similar, we just sketch it for the sake of the reader.

THEOREM 3.5. *Let C be a right H-module coalgebra. If there is a right H-module map $\Psi : C \to H$ which is a coalgebra map (as in the Hopf case) then for every right $[C, H]$-Hopf module N, the map θ defined above is an isomorphism of $[C, H]$-Hopf modules.*

Proof. Define the map

$$Q : N \to N, \qquad Q(n) = \sum n_{[0]} \beta S(\Psi(n_{[1]})),$$

for all $n \in N$. Because Ψ is right H-linear, by (3.5) and (1.5), we have that $Q(nh) = \varepsilon(h)Q(n)$ for all $h \in H, n \in N$, hence Q vanishes on NH^+. Thus there is a map $\overline{Q} : \overline{N} \to N$ such that $\overline{Q}\pi = Q$.

In particular, if we define $Q_0 : C \to C$ by $Q_0(c) = \sum c_1 \beta S(\Psi(c_2))$ then Q_0 factors through $\overline{C}$, that is, there exists a map $\overline{Q}_0 : \overline{C} \to C$ with $Q_0 = \overline{Q}_0 p$. Moreover, by (3.1), (1.6), and the hypothesis we have

$$\sum Q_0(c_1) \alpha \Psi(c_2) = c, \qquad \text{for any } c \in C. \quad (3.8)$$

Now, let $\theta^{-1} : \overline{N} \square_{\overline{C}} C \to N$, $\theta^{-1}(\pi(n) \square_{\overline{C}} c) = \overline{Q}(\pi(n)) \alpha \Psi(c)$ for all $n \in N$ and $c \in C$. It follows that $\theta^{-1}(\pi(n) \square_{\overline{C}} c) = Q(n) \alpha \Psi(C)$ and by (3.4), (1.6), and the hypothesis we obtain that $\theta^{-1}\theta = \mathrm{Id}$.

Also, by (3.4) and (3.6), we have that

$$\theta(Q(n) \alpha \Psi(c)) = \sum \pi(n_{[0]}) \otimes Q_0(n_{[1]}) \alpha \Psi(c),$$

for all $n \in N$, $c \in C$, thus, if we take $\pi(n) \square_{\overline{C}} c \in \overline{N} \square_{\overline{C}} C$, using (3.6), (3.7), and (3.8), we have that $\theta(Q(n)\alpha\Psi(c)) = \pi(n) \otimes c$. Hence we have shown that $\theta\theta^{-1}$ is the identity on $\overline{N} \square_{\overline{C}} C$ and this finishes the proof. ∎

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