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THE PRODUCT OF $\langle \alpha_i \rangle$-SPACES

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Dedicated to Professor Keiô Nagami on his 60th birthday

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The purpose of this paper is to give answers to the following problems posed by A.V. Arhangel'skii: Is the product of $\langle \alpha_i \rangle$-spaces $\langle \alpha_i \rangle$ for $i = 1, 2, 3, 4, 5$? Is every countably compact sequential space $\langle \alpha_i \rangle$? We also give, under CH, a negative answer to the following problem: If a subspace of the product of finitely many strongly Fréchet spaces is Lašnev, then is it metrizable?

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$\langle \alpha_i \rangle$-space   Franklin compact   Fréchet
Lašnev   sequential   strongly Fréchet

1. Introduction

A topological space X is said to be strongly Fréchet [17] (=countably-bisequential in the sense of E. Michael [11]) if, for every decreasing sequence $\{A_i: i = 1, 2, \ldots\}$ accumulating at $x \in X$, there exists a convergent sequence B of X with $x \in \overline{B \cap A_i}$ for each i. If $A_i = A_j$ for each i and j, then such a space is said to be Fréchet.

1.1. Problem. When is the product of given Fréchet spaces Fréchet?

This problem has been studied by many mathematicians. Some of the significant results for this problem are as follows:

1.2. Example (J. Isbell and R.C. Olson [15]). There exist two strongly Fréchet spaces X and Y such that $X \times Y$ is not Fréchet.

In the construction of the above Example 1.2, Isbell and Olson used the hypothesis $2^{\aleph_0} < 2^{\aleph_1}$ but the hypothesis is only used to construct an (ω_1, ω_1^*)-gap which was done by Hausdorff [8] without extra set-theoretic assumption.

1.3. Examples (T.K. Bohme and M. Rosenfeld [3] (under the continuum hypothesis CH), V.I. Malyhin and B.È. Šapirovskii [10] (under Martin's axiom MA), P. Simon [16] (without extra set-theoretic assumptions)). There exist two compact Fréchet spaces X and Y such that $X \times Y$ is not Fréchet.

R.C. Olson [14] showed that every countably compact Fréchet space is strongly Fréchet, therefore Examples 1.3 give Example 1.2.

On the other hand, A.V. Arhangel'skii [1, 2] introduced the classes $\langle\alpha_i\rangle$ and $\langle\alpha_i\text{-FU}\rangle$ for $i = 1, 2, 3, 4, 5$ (in [2], $\langle\alpha_i\rangle$ and $\langle\alpha_i\text{-FU}\rangle$ are denoted by $\langle i\rangle$ and $\langle i\text{-FU}\rangle$, respectively. For the definitions of $\langle\alpha_i\rangle$ and $\langle\alpha_i\text{-FU}\rangle$, see Section 2 below). It seems that these classes are important. He showed in [2] that:

1.4. Theorem. *The product of a countably compact Fréchet space with an $\langle\alpha_3\text{-FU}\rangle$-space is Fréchet, where, for a class $\mathscr{C}$ of spaces, an element of $\mathscr{C}$ is said to be a $\mathscr{C}$-space.*

Since the class of w-spaces in the sense of G. Gruenhage [5] coincides with the class $\langle\alpha_2\text{-FU}\rangle$ [14], Theorem 1.4 implies the following theorem of J. Gerlits and Zs. Nagy [7]: The product of a compact Fréchet space with a w-space is Fréchet.

A.V. Arhangel'skii [2] also showed that a Fréchet space is strongly Fréchet if and only if it is an $\langle\alpha_4\rangle$-space. Hence the notion of strongly Fréchetness is generalized to two notions, i.e. $\langle\alpha_4\rangle$ and Fréchetness. Examples 1.2 and 1.3 show that, under the product operation, the Fréchetness is not preserved in the class of strongly Fréchet spaces.

In this paper we study the product of $\langle\alpha_i\rangle$-spaces and give answers in a way to the following series of problems posed by A.V. Arhangel'skii [2, 5.29-i].

1.5. Problem. Is the product of two $\langle\alpha_i\rangle$-spaces an $\langle\alpha_i\rangle$-space for $i = 1, 2, 3, 4, 5$?

We also show that a sequential version of R.C. Olson's theorem is not true, i.e., we give a counterexample to the following problem posed by A.V. Arhangel'skii [2, 5.29-i].

1.6. Problem. Is every countably compact sequential space an $\langle\alpha_4\rangle$-space?

Another purpose of this paper is to give, under CH, a counterexample to the following problem:

1.7. Problem. If a subspace of the product of finitely many strongly Fréchet spaces is Lašnev, then is it metrizable?

In this paper all spaces are assumed to be Hausdorff topological spaces.

2. Definitions and the product of $\langle\alpha_i\rangle$-spaces

Let X be a space. A collection $\mathscr{A}$ of convergent sequences of X is said to be a sheaf in X if all members of $\mathscr{A}$ converge to the same point of X, which is said to be the vertex of the sheaf $\mathscr{A}$. In this paper all sheaves are assumed to be countable.

We consider the following five properties of X which were introduced by A.V. Arhangel'skii [1, 2].

Let $\mathscr{A}$ be a sheaf in X with the vertex $x \in X$. Then there exists a sequence B converging to x such that:

(α_1) if $A \in \mathscr{A}$, then $|A - B| < \aleph_0$, where, for a set C, $|C|$ denotes the cardinality of C,

(α_2) if $A \in \mathscr{A}$, then $A \cap B$ is an infinite subsequence of A and B,

(α_3) $|\{A \in \mathscr{A}: A \cap B$ is an infinite subsequence of A and $B\}| = \aleph_0$,

(α_4) $|\{A \in \mathscr{A}: A \cap B \neq \emptyset\}| = \aleph_0$,

(α_5) $A \cap B \neq \emptyset$ for each $A \in \mathscr{A}$.

We say B satisfies (α_i) with respect to $\mathscr{A}$, if B satisfies the property (α_i). The class of spaces satisfying the property (α_i) for every sheaf $\mathscr{A}$ and vertex $x \in X$ is denoted by $\langle\alpha_i\rangle$ for $i = 1, 2, 3, 4, 5$. We denote by $\langle\alpha_i$-FU$\rangle$ the intersection of the class of Fréchet spaces and the class $\langle\alpha_i\rangle$ for $i = 1, 2, 3, 4, 5$.

Clearly each $\langle\alpha_i\rangle$-space is an $\langle\alpha_{i+1}\rangle$-space for $i = 1, 2, 3$, and each $\langle\alpha_2\rangle$-space is an $\langle\alpha_5\rangle$-space. We first show the following easy theorem.

2.1. Theorem. $\langle\alpha_2\rangle = \langle\alpha_5\rangle$.

Proof. We show each $\langle\alpha_5\rangle$-space is an $\langle\alpha_2\rangle$-space.

Let $\mathscr{A} = \{A_n: n \in N\}$ be a sheaf in an $\langle\alpha_5\rangle$-space X with the vertex x, where N denotes the integers. We put $A_n = \{a^n_m: m \in N\}$ and $A_{n,m} = \{a^n_k: k > m\}$. Since X is an $\langle\alpha_5\rangle$-space, there exists a convergent sequence B such that B satisfies (α_5) with respect to $\{A_{n,m}: (n, m) \in N^2\}$. Clearly B satisfies (α_2) with respect to $\mathscr{A}$. The proof is completed.

In a previous paper [14] we showed that if X and Y are $\langle\alpha_i$-FU$\rangle$-spaces and if $X \times Y$ is Fréchet, then $X \times Y$ is an $\langle\alpha_i$-FU$\rangle$-space for $i = 1, 2, 3$. But the proof implies that if X and Y are $\langle\alpha_i\rangle$-spaces, then $X \times Y$ is an $\langle\alpha_i\rangle$-space for $i = 1, 2, 3$. Moreover, we also showed that the classes $\langle\alpha_i\rangle$, $i = 1, 2, 3$, are almost countably productive. (A class $\mathscr{C}$ of spaces is said to be almost countably productive if $\prod_{i=1}^n X_i \in \mathscr{C}$ for each $n \in N$, then $\prod_{i=1}^\infty X_i \in \mathscr{C}$.) We obtain the following theorem.

2.2. Theorem. *The classes* $\langle\alpha_i\rangle$, $i = 1, 2, 3, 5$, *are countably productive.*

2.3. Remark. The spaces X and Y in Example 1.2 are $\langle\alpha_2\rangle$-spaces. Hence, by the above theorem, $X \times Y$ is an $\langle\alpha_2\rangle$-space. The author does not know if the space $X \times Y$ in Examples 1.3 are $\langle\alpha_4\rangle$ or not.

Before we prove that the product of two strongly Fréchet spaces (=$\langle\alpha_4$-FU$\rangle$-spaces) need not be an $\langle\alpha_4\rangle$-space, we study when the product of $\langle\alpha_4\rangle$-spaces is again $\langle\alpha_4\rangle$.

2.4. Theorem. *Let X be an $\langle\alpha_3\rangle$-space and Y be an $\langle\alpha_4\rangle$-space. Then $X \times Y$ is an $\langle\alpha_4\rangle$-space.*

Proof. Let $\mathscr{C}=\{C_n: n\in N\}$ be a sheaf in $X\times Y$ with the vertex $z=(x,y)$. Let π_X and π_Y be the projections from $X\times Y$ to X and to Y, respectively. If $|\{n\in N: \pi_X^{-1}(x)\cap C_n$ is an infinite set$\}|=\aleph_0$ or $|\{n\in N: \pi_Y^{-1}(y)\cap C_n$ is an infinite set$\}|=\aleph_0$, then the arguments are completed trivially. Therefore we assume, without loss of generality, that $\pi_X^{-1}(x)\cap C_n=\emptyset$ and $\pi_Y^{-1}(y)\cap C_n=\emptyset$ for all $n\in N$. Since $\mathscr{A}=\{\pi_X(C_n): n\in N\}$ is a sheaf in X with the vertex x, there exists a convergent sequence A which satisfies (α_3) with respect to $\mathscr{A}$. We put $M=\{n\in N: A\cap\pi_X(C_n)$ is an infinite subsequence of $\pi_X(C_n)\}$ and, for $n\in M$, put $D_n=C_n\cap\pi_X^{-1}(A\cap\pi_X(C_n))$. Then $\mathscr{B}=\{\pi_Y(D_n): n\in M\}$ is a sheaf in Y with the vertex y. Let B be a convergent sequence in Y which satisfies (α_4) with respect to $\mathscr{B}$. Put $L=\{n\in M: B\cap\pi_Y(D_n)\neq\emptyset\}$. For each $n\in L$ and $b\in B\cap\pi_X(D_n)$, choose $a(b)\in\pi_Y^{-1}(b)\cap D_n$. Then clearly $C=\{(a(b),b): b\in B\}$ is a convergent sequence in $X\times Y$ which satisfies (α_4) with respect to $\mathscr{C}$. The proof is completed.

2.5. Theorem. *Let X be an $\langle\alpha_4\rangle$-space and Y be a regular countably compact space. If $X\times Y$ is Fréchet, then it is an $\langle\alpha_4\rangle$-space.*

Proof. Let $\mathscr{C}=\{C_n: n\in N\}$ be a sheaf in $X\times Y$ with the vertex $z=(x,y)$. By the same reason of the above proof, we assume $\pi_X^{-1}(x)\cap C_n=\emptyset$ and $\pi_Y^{-1}(y)\cap C_n=\emptyset$ for all $n\in N$. Let $\mathscr{V}=\{V\}$ be a neighborhood base of y in Y. We put $C_n(V)=C_n\cap X\times V$ for each $V\in\mathscr{V}$. Then $\mathscr{A}(V)=\{\pi_X(C_n(V)): n\in N\}$ is a sheaf in X with the vertex x. Let $A(V)$ be a convergent sequence which satisfies (α_4) with respect to $\mathscr{A}(V)$. We can also assume that $A(V)$ satisfies the following property: For each $a\in A(V)$, there exists $b(a(V))\in V$ and $n(a)\in N$ such that $(a(V),b(a(V)))\in C_{n(a)}(V)$ and if $a'(V)\in A(V)$, $a'(V)\neq a(V)$, then $n(a)\neq n(a')$.

We choose such $b(a(V))\in V$ for each $a(V)\in A(V)$. Since Y is countably compact and Fréchet, there exist a subsequence $\{b(a_i(V)): i\in N\}$ of $\{b(a(V)): a(V)\in A(V)\}$ and $y(V)\in\bar{V}$ such that $\lim b(a_i(V))=y(V)$. Note that the sequence $\{(a_i(V),b(a_i(V))): i\in N\}$ is a convergent sequence with the limit point $(x,y(V))$. By the regularity of Y, $y\in\overline{\{y(V): V\in\mathscr{V}\}}$. Since Y is Fréchet, we can choose a convergent sequence $\{y(V_i): i\in N\}$ of $\{y(V): V\in\mathscr{V}\}$ with the limit point y. We put $D_n=\{(a_i(V_n),b(a_i(V_n))): i\in N\}-\bigcup\{C_k: k\leq n\}$. Then $(x,y)\in\overline{\bigcup\{D_n: n\in N\}}$. Since $X\times Y$ is Fréchet, there exists a convergent sequence $C\subset\bigcup\{D_n: n\in N\}$ with the limit point (x,y). Now we show that C satisfies (α_4) with respect to $\mathscr{C}$. Assume $C\subset\bigcup\{C_k: k\leq n\}$ for some $n\in N$. Then, since $D_m\cap(\bigcup\{C_k: k\leq n\})=\emptyset$ for $n<m$, $C\subset\bigcup\{D_k: k\leq n\}$. The last implication is impossible since D_k is a convergent sequence with the limit point $(x,y(V_k))$ for $k\leq n$. The proof is completed.

3. Examples and problems

We denote by βN the Stone–Čech compactification of N. For a subset A of N, we denote $A^*=\mathrm{Cl}_{\beta N}A-A$. Let F be a closed subset of N^*. Put $X=N\cup\{F\}$ and topologize X as follows: Points of N are isolated. The set of the $U \cup \{F\}$ is a neighborhood base of $\{F\}$ in X, where U is a subset of N such that $F \subset U^*$. The following facts are well known.

3.1. Fact. Let Z be a non-empty zero set in N^*. Then $\mathrm{Int}_{N^*} Z \neq \emptyset$.

3.2. Fact. Two disjoint cozero sets in N^* have disjoint closures.

3.3. Fact. Let $X = N \cup \{F\}$. A subset A of N^* converges to $\{F\}$ if and only if $A^* \subset F$.

3.4. Lemma ([19]). *Let* $X = N \cup \{F\}$.

(i) *X is Fréchet if and only if F is a regular closed subset of N^*.*

(ii) *X is strongly Fréchet if and only if X is Fréchet and, for each zero set Z of N^*, $F \cap Z \neq \emptyset$ implies $F \cap \mathrm{Int}_{N^*} Z \neq \emptyset$.*

3.5. Lemma. *Let $X = N \cup \{F\}$ and $Y = N \cup \{G\}$. Then the subspace $\Delta \cup \{F\} \times \{G\}$ of the product space $X \times Y$ is homeomorphic to $N \cup \{F \cap G\}$, where $\Delta = \{(n, n): n \in N\}$ is the diagonal of N^2.*

Proof. The natural correspondence $f: N \cup \{F \cap G\} \to \Delta \cup \{F\} \times \{G\}$ defined as $f(n) = (n, n)$, $f(\{F \cap G\}) = \{F\} \times \{G\}$ gives a homeomorphism. The proof is completed.

3.6. Lemma. *Let $\{U_n: n \in N\}$ be a countable collection of pairwise disjoint non-empty clopen subsets of N^* and K be a nowhere dense closed subset of N^*. Then $X = N \cup \{K \cup \overline{\bigcup \{U_n: n \in N\}}\}$ does not satisfy (α_4).*

Proof. Let A_n be a subset of N such that $A_n^* = U_n$ for each $n \in N$. Then A_n converges to the point $\{K \cup \overline{\bigcup \{U_n: n \in N\}}\}$ by Fact 3.3. Choose $a_n \in A_n - \bigcup \{A_i: i < n\}$, and let $B = \{a_n: n \in N\}$. Then $B \cap A_n$ is finite for each $n \in N$. Therefore $B^* \cap \overline{\bigcup \{U_n: n \in N\}} = \emptyset$. This shows that B^* is not a subset of $K \cup \overline{\bigcup \{U_n: n \in N\}}$ since K is nowhere dense. By Fact 3.3, B does not converge to $\{K \cup \overline{\bigcup \{U_n: n \in N\}}\}$. The proof is completed.

3.7. Remark. Let $A = \{0\} \cup \{1/n: n \in N\}$ be a convergent sequence and $T = \bigoplus \{A_i: i \in N\}$, where $\bigoplus$ denotes the disjoint union and A_i denotes a copy of A for each $i \in N$. Let S be the quotient space obtained from T by identifying $\{0_i: i \in N\}$ to a point. Then S is a Lašnev space (=a closed continuous, onto image of a metric space). It is easy to show that the space X in the above lemma is homeomorphic to S if $K = \emptyset$.

A collection $\mathcal{P}$ of infinite subsets of a set A is said to be almost disjoint if $|P \cap Q| < \aleph_0$ for $P \neq Q$, $P \in \mathcal{P}$ and $Q \in \mathcal{P}$.

Let $\mathscr{P}$ be an almost disjoint collection of subsets of N. The Franklin compact $\mathscr{F}(\mathscr{P})$ is a topological space whose underlying set is $N \cup \mathscr{P} \cup \{\infty\}$ and whose topology is given as follows: N is a set of isolated points, a basic neighborhood of a point $P \in \mathscr{P}$ is $\{P\} \cup$ cofinite subset of P, $\{\infty\}$ is a point distinct from all $n \in N$ and $P \in \mathscr{P}$, which compactifies the space $N \cup \mathscr{P}$. Note that the subspace $N \cup \{\infty\}$ of $\mathscr{F}(\mathscr{P})$ is homeomorphic to $N \cup \{N^* - \bigcup \{P^*: P \in \mathscr{P}\}\}$. Hence, by Lemma 3.4, $\mathscr{F}(\mathscr{P})$ is Fréchet if and only if $N^* - \bigcup \{P^*: P \in \mathscr{P}\}$ is regular closed set in N^*. Let $\mathscr{P}$ be an almost disjoint collection of N. We denote by $\mathscr{P}|A = \{A \cap P: A \cap P$ is an infinite set and $P \in \mathscr{P}\}$. We consider the following property $(*)$ of $\mathscr{P}$:

$(*)$ For each infinite subset A of N, if $\mathscr{P}|A$ is a maximal almost disjoint collection of A, then $|\mathscr{P}|A| < \aleph_0$.

The following lemma is obvious.

3.8. Lemma. *Let $\mathscr{P}$ be an almost disjoint collection of N. Let $\mathscr{A} = \{A_i: i \in N\}$ be a countable subcollection of $\mathscr{P}$. If $\mathscr{P}$ has the property $(*)$, then $\mathscr{P} - \mathscr{A}$ also has property $(*)$.*

Now we shall construct two compact Fréchet (hence strongly Fréchet, see Section 1) spaces whose product is not an $\langle\alpha_4\rangle$-space. The following lemma, proved by P. Simon [16], is important for our construction.

3.9. Lemma. *There exist an infinite maximal almost disjoint collection $\mathscr{P}$ of N and its partition $\mathscr{P}_1 \cup \mathscr{P}_2 = \mathscr{P}$ such that P_i satisfies $(*)$ for $i = 1, 2$.*

Note that if the cardinality of one of $\mathscr{P}_1$ or $\mathscr{P}_2$ is $\aleph_0$, then the other does not satisfy $(*)$. Hence both $\mathscr{P}_1$ and $\mathscr{P}_2$ are uncountable.

3.10. Theorem. *There exist two compact Fréchet spaces whose product is not an $\langle\alpha_4\rangle$-space.*

Proof. Let $\mathscr{P}$, $\mathscr{P}_1$ and $\mathscr{P}_2$ be the collections mentioned in the above lemma. Let $\mathscr{A}$ be a countable infinite subcollection of $\mathscr{P}_1$. Then, by Lemma 3.8, $\mathscr{P}_1 - \mathscr{A}$ also satisfies $(*)$. Let $X = \mathscr{F}(\mathscr{P}_1 - \mathscr{A})$ and $Y = \mathscr{F}(\mathscr{P}_2)$. Then X and Y are compact Fréchet. Let us consider the subspace $Z = \Delta \cup \{(\infty, \infty)\}$ of $X \times Y$. Then Z is homeomorphic to

$$N \cup \{(N^* - \bigcup \{P^*: P \in \mathscr{P}_1 - \mathscr{A}\}) \cap (N^* - \bigcup \{P^*: P \in \mathscr{P}_2\})\}$$

by Lemma 3.5. Note that

$$(N^* - \bigcup \{P^*: P \in \mathscr{P}_1 - \mathscr{A}\}) \cap (N^* - \bigcup \{P^*: P \in \mathscr{P}_2\}) =$$
$$= (N^* - \bigcup \{P^*: P \in \mathscr{P}\}) \cup \bigcup \{P^*: P \in \mathscr{A}\}.$$

Since $N^* - \bigcup \{P^*: P \in \mathscr{P}\}$ is a nowhere dense closed subset of N^*, Z is not an $\langle\alpha_4\rangle$-space by Lemma 3.6. The proof is completed.

A subset U of a space X is said to be sequentially open if each sequence converging to a point in U is eventually in U. X is said to be a sequential space [4] if each sequentially open subset of X is open. Clearly each Fréchet space is sequential.

3.11. Corollary. *There exists a compact sequential space which is not an $\langle\alpha_4\rangle$-space.*

Proof. Let X and Y be the spaces constructed in the above theorem. Then $X\times Y$ is sequential since the product of countably many countably compact sequential spaces is sequential [12]. Therefore the space $X\times Y$ is a compact sequential space which is not an $\langle\alpha_4\rangle$-space. The proof is completed.

E. Michael [11] showed that a Lašnev space X is metrizable if and only if X is strongly Fréchet. K. Tamano [18] showed that, under the product operations, 'something nice' is preserved in the class of Lašnev spaces. Is the following generalization of E. Michael's theorem true?

3.12. Problem. Let Z be a subspace of the product of Lašnev spaces X and Y. If Z is strongly Fréchet, then is Z metrizable?

The following example shows that an analogous one of the above problem is not true under CH.

3.13. Theorem (CH). *There exist two strongly Fréchet spaces X, Y and a subspace W of $X\times Y$ such that W is a non-metrizable Lašnev space.*

Proof. We shall construct, under CH, strongly Fréchet spaces $X=N\cup\{F\}$ and $Y=N\cup\{G\}$ such that $F\cap G=\bigcup\{U_i\colon i\in N\}$, where $\{U_i\colon i\in N\}$ is a countable collection of pairwise disjoint non-empty clopen subsets in N^*. Then, by Lemma 3.5 and Remark 3.7, $W=\Delta\cup\{F\}\times\{G\}$ is a Lašnev space which is not metrizable.

Let $\{U_i\colon i\in N\}$ is a countable collection of pairwise disjoint non-empty clopen subsets in N^*. We put $Z=N^*-\bigcup\{U_i\colon i\in N\}$. Then Z is a zero set in N^* with non-empty boundary in N^*. Let H be the boundary of Z in N^*. We construct two regular closed sets F_1 and G_1 in N^* such that

(i) $F_1\subset Z$ and $G_1\subset Z$,

(ii) $\mathrm{Bdy}_{N^*} F_1=\mathrm{Bdy}_{N^*} G_1=H$,

(iii) $\mathrm{Int}_{N^*} F_1\cap \mathrm{Int}_{N^*} G_1=\emptyset$,

(iv) for each zero set K of N^* such that $H\cap\mathrm{Bdy}_{N^*} K\neq\emptyset$, $K\cap\mathrm{Int}_{N^*} F_1\neq\emptyset$ and $K\cap\mathrm{Int}_{N^*} G_1\neq\emptyset$.

After constructing such F_1 and G_1, we put $F=F_1\cup\bigcup\{U_i\colon i\in N\}$ and $G=G_1\cup\bigcup\{U_i\colon i\in N\}$. Then $X=N\cup\{F\}$ and $Y=N\cup\{G\}$ are strongly Fréchet by (iv) and $F\cap G=\bigcup\{U_i\colon i\in N\}$ by (ii) and (iii).

Now we construct such F_1 and G_1 by transfinite induction. Note that the cardinality of the set of all zero sets in N^* equals the cardinality of the continuum. Let $\{Z_\alpha: \alpha < \omega_1\}$ be a family of all zero sets in Z such that $H \cap \mathrm{Bdy}_{N^*} Z_\alpha \neq \emptyset$ for each $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal number. Let $\{W_\alpha: \alpha < \omega_1\}$ be zero sets in Z such that $W_\alpha \subsetneqq W_\beta$ for $\alpha > \beta$, $W_\alpha = \bigcap \{W_\beta: \beta < \alpha\}$ if α is limit and $\bigcap \{W_\alpha: \alpha < \omega_1\} = H$. We choose $\mathbb{O}_1$ and V_1, non-empty disjoint clopen subsets of Z_1, and inductively we suppose that we have defined, for each $\beta < \alpha$, non-empty clopen subsets $\mathbb{O}_\beta$ and V_β of N^* such that:

(1) $\bigcup \{\mathbb{O}_\gamma: \gamma < \beta\} \subset \mathbb{O}_\beta \subset \overline{\mathrm{Int}_{N^*} Z}, \bigcup \{V_\gamma: \gamma < \beta\} \subset V_\beta \subset \overline{\mathrm{Int}_{N^*} Z}$,

(2) $\mathbb{O}_\beta \cap Z_\beta \neq \emptyset, V_\beta \cap Z_\beta \neq \emptyset, (\mathbb{O}_\beta - \bigcup \{\mathbb{O}_\gamma: \gamma < \beta\}) \cup (V_\beta - \bigcup \{V_\gamma: \gamma < \beta\}) \subset W_\beta$,

(3) $\mathbb{O}_\gamma \cap V_\delta = \emptyset$ for γ, $\delta < \alpha$.

We define $\mathbb{O}_\alpha$ and V_α. We first define $\mathbb{O}'_\alpha$ and V'_α as follows: If α is limit, then since $\bigcup \{U_n: n \in N\}, \bigcup \{\mathbb{O}_\beta: \beta < \alpha\}$ and $\bigcup \{V_\beta: \beta < \alpha\}$ are disjoint cozero sets in N^*, by fact 3.2 and $W = \bigcap \{W_\beta: \beta < \alpha\}$, there exist disjoint clopen subsets $\mathbb{O}'_\alpha$ and V'_α such that $\mathbb{O}'_\alpha \cup V'_\alpha \subset Z$, $\bigcup \{\mathbb{O}_\beta: \beta < \alpha\} \subset \mathbb{O}'_\alpha$, $\bigcup \{V_\beta: \beta < \alpha\} \subset V'_\alpha$. and $(\mathbb{O}'_\alpha - \bigcup \{\mathbb{O}_\beta: \beta < \alpha\} \cup (V'_\alpha - \bigcup \{V_\beta: \beta < \alpha\}) \subset W_\alpha$. If α is isolated, put $\mathbb{O}'_\alpha = \mathbb{O}_{\alpha-1}$ and $V'_\alpha = V_{\alpha-1}$. Note that $Z - \mathbb{O}'_\alpha$ and $Z - V'_\alpha$ are zero sets in N^* whose boundary in N^* are H. Since $Z_\alpha \cap (Z - \mathbb{O}'_\alpha) \cap (Z - V'_\alpha) \neq \emptyset$, $\mathrm{Int}_{N^*} (Z_\alpha \cap (Z - \mathbb{O}'_\alpha) \cap (Z - V'_\alpha)) \neq \emptyset$ by Fact 3.1. Let S_α and T_α be non-empty disjoint clopen subsets of N^* such that

$$S_\alpha \cup T_\alpha \subset \mathrm{Int}_{N^*} (Z_\alpha \cap (Z - \mathbb{O}'_\alpha) \cap (Z - V'_\alpha)).$$

Let $\mathbb{O}_\alpha = \mathbb{O}'_\alpha \cup S_\alpha$ and $V_\alpha = V'_\alpha \cup T_\alpha$.

We have chosen $\mathbb{O}_\alpha$ and V_α ($\alpha < \omega_1$) satisfying the conditions (1), (2) and (3). Put $F_1 = \bigcup \{\mathbb{O}_\alpha: \alpha < \omega_1\}$ and $G_1 = \bigcup \{V_\alpha: \alpha < \omega_1\}$. Then clearly F_1 and G_1 satisfy (i), (iii) and (iv). We show (ii). Let U be any clopen subset of N^* with $U \cap H \neq \emptyset$. Then $U \cap Z$ is a non-empty zero set with $U \cap Z \cap H \neq \emptyset$. Hence $U \cap F_1 \neq \emptyset$ and $U \cap G_1 \neq \emptyset$ by (iv). This implies that H is the boundary of F_1 and G_1. Similarly we can show that H is the boundary of F and G. The proof is completed.

3.14. Remarks. Every Lašnev subspace of a regular countably compact space with countable tightness (a space X is said to be a space with countable tightness if, for each subset A of X and $x \in \bar{A}$, there exists a countable subset B of A such that $x \in \bar{B}$) is metrizable [13]. Hence every Lašnev subspace of the product space $X \times Y$ in Example 1.3 or Theorem 3.10 is metrizable.

Note that $X \times Y$ in Theorem 3.12 is not Fréchet by Theorem 2.5. But the author cannot prove that the space $X \times Y$ in Theorem 3.13 is Fréchet or not. The following problems naturally arise.

3.15. Problem. Let X and Y be strongly Fréchet spaces. If $X \times Y$ is Fréchet, then is $X \times Y$ an $\langle \alpha_4 \rangle$-space?

3.16. Problem. Let X be a (countably) compact Fréchet space. If X^2 is Fréchet, then is X^3 Fréchet? More generally, is there a (countably) compact Fréchet space X such that X^n is Fréchet but X^{n+1} is not Fréchet for $n \geq 2$?

3.17. Remarks. Under MA, G. Gruenhage [6] constructed a strongly Fréchet space X such that X^n is Fréchet for all $n \in N$ but X^ω is not Fréchet. Note that Problem 3.12 is yes if one of X or Y is countably compact Fréchet and a negative answer of Problem 3.16 implies a negative answer of Problem 3.15.

3.18. Problem. Let X and Y be strongly Fréchet (or compact Fréchet) spaces. If $X \times Y$ is an $\langle\alpha_4\rangle$-space, then is $X \times Y$ Fréchet?

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