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THE GEOMETRIC MEAN PROCEDURE FOR ESTIMATING THE SCALE OF A JUDGMENT MATRIX†

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Abstract—Typically the literature has advocated the use of the dominant right eigenvector and an associated consistency ratio "C.R." We give reasons why the geometric mean (GM) (also known as the LLSM or logarithmic least-squares method) may be preferable as an estimator of the unknown underlying scale u. We also develop an index of consistency and related rules to judge the consistency of a matrix when using the GM as an estimator. The rules for the index of consistency are closely related to the commonly used rule that the C.R. should be <0.1.

INTRODUCTION

We assume a collection of entities E_i, $i = 1, \ldots, n$, and the existence of an unknown underlying scale $u = (u_1, \ldots, u_n)^T$, where u_i/u_j is the relative worth of entity E_i to entity E_j.

We assume a (reciprocal symmetric) judgment matrix $A = [a_{ij}]$, where a_{ij} is an estimate of the relative worth of E_i to E_j. We are concerned here with the choice of an appropriate procedure to yield estimates v of the underlying scale u.

Typically the literature has advocated the use of the dominant right eigenvector (EV) and an associated consistency ratio "C.R." [1]. We give reasons why the geometric mean (GM) (also known as the LLSM, or logarithmic least-squares method) may be preferable as an estimator of u and suggest related rules to judge the consistency of a matrix. These rules are closely related to the commonly used rule that the C.R. should be <0.1.

In general the GM and the EV are numerically quite close. Compared with the experimental error inherent in A the differences in the GM or the EV will be small, especially if the dimension is small (the two scales will be equal if the dimension is $\leqslant 3$) or if A is almost consistent (the scales will be equal if A is consistent).

Given the numerical similarity of the EV and the GM procedures, it is not surprising that they suffer some of the same drawbacks. Jensen [2, pp. 320–321] shows that if the respondent uses a bounded response scale the EV and GM procedures may make consistency adjustments outside those bounds. Further, neither the EV nor the GM (regardless of the distribution of errors) are necessarily optimal if rank preservation is the sole criterion. Also the use of either method may be philosophically at odds with commonly used respondent scales.

In the overall structure of an Analytic Hierarchy Process (AHP) the analyst must make extra-mathematical decisions that may substantially affect the results of the analysis. These decisions may have more important effects on the results than the choice between the EV and the GM estimating procedure.

Analyzing an AHP requires estimating the underlying scale, and we believe there are theoretical and practical reasons for preferring the GM. The qualities of the EV have received much attention in the literature. In this paper we have concentrated on properties of the GM. A possible shortcoming of the GM has been the lack of a test for consistency comparable to "C.R. $<$ 0.1". The principal new result in this paper is the section on "An Index of Consistency" where a test for consistency is developed that is in keeping with the spirit of the GM. The test has been developed from the C.R. $<$ 0.1 test for consistency. It is comparable to, and in practice usually equivalent to, the C.R. consistency for the EV procedure.

† Adapted in part and reprinted from: G. Crawford and C. Williams, A note on the analysis of subjective judgment matrices. *J. math. Psychol.* **29**, 387–405 (1985). Reprinted by permission of the publisher, Copyright © 1985 Academic Press.

Elsewhere [3] the empirical performances of the two procedures have been compared with a series of Monte Carlo experiments where the underlying scale is known and the ratios of the components of the scale are perturbed by random errors to yield judgment matrices. These experiments have shown that in all the scenarios considered the GM does as well as, or better than, the EV by several measures, including rank preservation. As might be expected, the procedures are very close in performance under the likely scenarios where the dimension is small or the judgment matrices are almost consistent. As the dimension or the variance of the errors increases, the performance of the GM exceeds that of the EV by all of the measures considered, including rank preservation.

There are examples in the literature specifically constructed to show that the GM and the EV may give different rankings. We reject the suggestion that a difference highlights shortcomings of the GM any more than it highlights shortcomings of the EV. Of the practical examples given in the literature we have noticed only one (the school selection example [1]) where the two procedures suggest a different resolution. In that case the utilities of the two entities are nearly equal under the GM and the EV estimates. The EV gives a weak preference for school B, but the GM gives preference to school A. (Having personal experience with the erratic judgment of adolescents, we will not put excessive emphasis on the ultimate decision that was made—to attend school A—by the respondent, Saaty's son.†)

For most estimation problems, the wealth of statistical literature on estimation procedures and their properties has enhanced understanding of the problem. Below we relate the estimation of u, and a related quantification of consistency, to well-known statistical models. The geometric mean vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$, defined by $v_i = \pi_j a_{ij}^{1/n}$, satisfies the continuity and consistency criteria used to defend the dominant eigenvector, and has several other desirable traits. In certain circumstances, it is statistically optimal and gives rise to an estimate of scales and a measure of consistency with known statistical distributions. In empirical studies [3] it seems to do as well as, or better than, the eigenvector in preserving rank order. In addition, it satisfies several criteria that might reasonably be expected of a method for estimating multiplicative scales. It is also supported by a literature describing methods of handling a wealth of variations of the problem, including missing data and multiple judges.‡

The scale corresponding to a judgment matrix is only determined to within a multiplicative factor. To make the scale unique different normalizations may be used, the most common being to normalize so that the sum of the components of the scale and the estimate add up to 1. Here, in keeping with the multiplicative nature of the problem, and to simplify the notation, we will normalize so that the product of the components of the scale will be 1.

THE GM SCALE

For $n \times n$ judgment matrices $A = [a_{ij}]$ and $C = [c_{ij}]$, define

$$m(A, C) = \left[\sum_i \sum_{j>i} (\ln a_{ij} - \ln c_{ij})^2 \right]^{1/2}.$$

It is not difficult to verify that m satisfies the triangle inequality and is a metric on the space of

† In fairness it should be mentioned that in Saaty's comments [1] he remarks that he did not interfere in this decision to attend school A although he believed the C.R. was too high. He also notes that although costs were not considered by his son or spouse, the costs of school B were much higher than for school A. There are clear advantages to being a pragmatic analyst.

‡ In an interesting twist, Narasimhan [4] gives a procedure to adjust a judgment matrix and make it consistent before using the EV procedure to estimate the scale. Budescu [5] shows that the Narasimhan procedure yields precisely the same estimates as using the GM on the unadjusted judgment matrix.

$n \times n$ judgment matrices.† Theorem 3 shows that for any $n \times n$ judgment matrix A, there is a consistent matrix C that is m-closest to A. Such a consistent matrix is given by $c_{ij} = v_i/v_j$, where $v_i = \pi_j a_{ij}^{1/n}$; i.e. v_i is the GM of the elements of the ith row of A. We will use the vector $\mathbf{v}$, suitably normalized, as the estimate of the underlying ratio scale corresponding to A.

The following two invariance properties show that m is a suitable choice of metric for the space of judgment matrices. Their proofs follow from the definition of m.

Theorem 1 (invariance under transpose)

(i) Let $A = [a_{ij}]$ and $C = [c_{ij}]$ be $n \times n$ judgment matrices. Then A^T and C^T are also judgment matrices, and $m(A^T, C^T) = m(A, C)$.

(ii) Let $A = [a_{ij}]$ be an $n \times n$ judgment matrix, and suppose that $C = [c_{ij}]$ is the consistent matrix that is m-closest to A. Then C^T is the consistent matrix that is m-closest to A^T.

In the following we denote the componentwise or Hadamard product of two matrices A and B (or vectors $\mathbf{A}$ and $\mathbf{B}$) of the same dimension by $A \cdot B$.

Theorem 2 (invariance under multiplication)

Let A, B and C be n-dimensional judgment matrices. Then $A \cdot B$ is a judgment matrix,

(i) $m(A, C) = m(A \cdot B, C \cdot B)$

and

(ii) if C is the m-closest consistent matrix to A, and B is consistent, then $C \cdot B$ is the m-closest consistent matrix to $A \cdot B$.

Recall that we seek a procedure for associating ratio scales with judgment matrices in such a way that the ratio scales capture the subjective information inherent in the corresponding matrices. Let A be an $n \times n$ judgment matrix. Let $C = [c_{ij}]$ be a consistent matrix that is m-closest to A, and suppose that $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ is a ratio scale for C; i.e. $c_{ij} = v_i/v_j$. We choose $(v_1, v_2, \ldots, v_n)^T$ as the estimator of the ratio scale corresponding to A.

Under this association, Theorem 1 guarantees that the scale $(1/v_1, 1/v_2, \ldots, 1/v_n)^T$ is the estimator of the scale corresponding to A^T. The appeal of this invariance arises in a natural way. Suppose a respondent put his estimate of u_i/u_j in the position r_{ji} instead of r_{ij}. (There appears to be nothing intrinsically right or wrong about recording the estimates this way.) In that case, the estimation procedure should return estimates of $1/u_i$ instead of u_i. With this convention the estimated value of u_i should not depend on an artifact of the way the data are recorded. It follows from Theorem 1 that the GM procedure has this invariance property. The EV procedure does not have this property although in practice it is usually close.‡

If the judgment matrix components are the product of multiplicative errors and the ratios of the respective components of the true scale, then Theorem 2 guarantees that our choice of ratio scale is invariant under a scale change in the judgment matrix. We do not argue that this invariance is indispensable in dealing with subjective judgment, nor that the numerical differences are large (see the remarks following Lemma 1) but it is another example of invariance of the GM scale that is not satisfied by the EV scale.

Theorem 3 guarantees that the GM scale gives the m-closest consistent matrix to any judgment matrix.

† Fichtner [6] has developed a metric m with the property that the m-closest consistent matrix to a subjective judgment matrix is the consistent matrix corresponding to the maximal eigenvector. Typically a metric is of interest because it provides an intuitive understanding of the topology of a space—i.e. it tells you what points are close to (neighborhoods of) other points, hence what sequences converge, and to what points. On the space of inconsistent judgment matrices Fichtner's metric yields the discrete topology, a pathological topology wherein every point is an open set, and the only sequences that converge are those whose terms are all the same from some point on. It is the same topology given by the metric that calls the distance from two points 1 if the points are not equal and 0 if they are.

‡ See Johnson [7] for an excellent discussion of this lack of symmetry and its relation to the dominant right or left EV.

Theorem 3

Let $A = [a_{ij}]$ be an $n \times n$ judgment matrix. Let $C = [c_{ij}]$ be the consistent matrix given by $c_{ij} = v_i/v_j$, where v_i is the GM of the elements of the ith row of A; i.e. $v_i = \pi_j \alpha_{ij}^{1/n}$, $i = 1, 2, \ldots, n$. Then $m(A, C)$ is the minimal m-distance from A to any $n \times n$ consistent matrix.

Proof. For any consistent C, $C = [c_{ij}]$, we can write $c_{ij} = w_i/w_j$, where $\mathbf{w} = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is a ratio scale. We seek a scale that minimizes the sum of squares $S = \sum_i \sum_{j>1} [\ln a_{ij} - (\ln w_i - \ln w_j)]^2$. As mentioned, we normalize by imposing the side condition $\pi_j w_j = 1$. Let $Y_{ij} = \ln a_{ij}, i,j = 1, 2, \ldots, n$; $b_i = \ln w_i, i = 1, 2, \ldots, n$. Then the problem is to minimize $\sum_i \sum_{j>i} [y_{ij} - (b_i - b_j)]^2$ under the side condition $\sum_i b_i = 0$. S is strictly convex in the differences $b_i - b_j$ and therefore strictly convex in the vector $\mathbf{b}$, so it has a unique minimum at the point where its first partials with respect to b_i are zero. Setting these partial derivatives equal to zero, for $k = 1, 2, \ldots, n$,

$$\partial S/\partial b_k = -2\sum (y_{kj} - b_k + b_j) = -2\left(\sum_j y_{kj} - nb_k + \sum_j b_j\right) = 0$$

and therefore, since $\sum_j b_j = 0$, it follows that $\sum_j y_{kj} = nb_k$. Thus S is minimized by $b_k = \sum_j y_{kj}/n$; i.e. $\ln w_k = \sum_j a_{kj}/n$, $k = 1, 2, \ldots, n$; and consequently the m-distance from A to C is minimized by the vector $\mathbf{v}$ given by $v_k = \pi_j a_{kj}^{1/n}$. This completes the proof of Theorem 3.

Lemma 1

Let A and B be judgment matrices and A' and B' be, respectively, their closest consistent matrices. Then $A' \cdot B'$ is the closest consistent matrix to the judgment matrix $A \cdot B$.

Proof. That $A \cdot B$ is a judgment matrix, and that $A' \cdot B'$ is consistent follow from Theorem 2. Let a_{ij}, b_{ij}, a_i'/a_j', b_i'/b_j' be the elements of A, B, A', B', respectively. Let g_i/g_j be the elements of the closest consistent matrix to $A \cdot B$. It follows from Theorem 3 that

$$\begin{aligned} \ln g_i &= (1/n)\sum(\ln a_{ij} + \ln b_{ij}) \\ &= \ln a_i' + \ln b_i', \end{aligned}$$

as was to be shown.

Lemma 1 implies that the mapping from judgment matrices (using componentwise multiplication) into their GM scale is a homomorphism; i.e. the image of the product of A and B is the product of the image of A and the image of B.

The mapping of judgment matrices into their EV scale is homomorphic if one of the matrices is consistent: Vargas' formula [1, pp. 196, 197] shows that if A and B are judgment matrices and w is the image under the EV mapping of A, then the image under the EV mapping of the product of $A \cdot B$ is the product of the image of A and the image of $A \cdot V \cdot B$, where V is the transpose of the consistent matrix $[w_i/w_j]$.† If A is consistent then V consists of the elementwise reciprocals of A, and therefore $A \cdot V$ is equal to the Hadamard identity matrix; hence $A \cdot V \cdot B$ reduces to B. Thus, in this case the EV image of $A \cdot B$ is the EV image of A times the EV image of B. If A is almost consistent, then $A \cdot V$ is almost equal to the identity matrix, and the mapping is "almost" homomorphic.

Barzilai *et al.* [8] show that among all mappings that take a consistent matrix into its corresponding scale and are invariant under permutations in the ordering of the entities, the requirement that the mapping be homomorphic uniquely determines the GM mapping. Thus, in this sense, Lemma 1 is a necessary and sufficient condition that the mapping be the GM.

† In reconciling this form of Vargas' theorem, recall we are normalizing so that the product of the components of the scale is 1.

THE GM VECTOR AND THE MAXIMUM LIKELIHOOD ESTIMATOR

We have shown that, given an arbitary judgment matrix A, the GM vector gives rise to the m-closest consistent matrix to A. The problem of using a judgment matrix to estimate a ratio scale can also be cast in the framework of the general linear statistical regression model. Since the m-closest matrix C is obtained by minimizing the quadratic form S, it is not surprising that under suitable assumptions on the distribution of errors in the expert's judgment, the GM vector is the maximum likelihood estimator for the ratio scale corresponding to the judgment matrix.

Explicitly, letting $A = [a_{ij}]$ be an $n \times n$ judgment matrix, we assume that there is an underlying scale $(w_1, w_2, \ldots, w_n)$ whose ratios are perturbed (by inconsistent human judgment) to give the elements of A, namely $a_{ij} = (w_i/w_j)(e_{ij})$, and thus $\ln a_{ij} = \ln w_i - \ln w_j + \ln e_{ij}$, $i = 1, 2, \ldots, n; j > i$.

Regarding the choice of distribution of the error term e_{ij}, the context of the AHP approach assumes a multiplicative model—if $u = (u_1, \ldots, u_n)^T$ is a scale for the entities $\{E_i\}$, then the value of E_i relative to E_j is given by u_i/u_j. Accordingly, we have assumed that errors are multiplicative. Further, this context assumes that if a_{ij} estimates u_i/u_j, then a_{ji} is a comparable estimate of u_j/u_i; hence it is appropriate that the distribution of e_{ij} be reciprocal symmetric in the sense that $P(a < e_{ij} \leqslant b) = P(a < 1/e_{ij} \leqslant b)$.

Just as the normal distribution is a common model for additive errors, the lognormal distribution, for similar reasons, is a common mathematical model for multiplicative errors [9]. Additionally, the lognormal distribution is reciprocal symmetric. We assume that model here.

Assuming the judgment matrix is the componentwise product of ratios of elements of the true scale and lognormal errors, and making the substitution $Y = (\ln a_{1,1}, \ln a_{1,2}, \ldots, \ln a_{n-1,n})^T$, $B = (\ln w_1, \ldots, \ln w_n)^T$ and $E = (\ln e_{1,1}, \ln e_{1,2}, \ldots, \ln e_{n-1,n})^T$, the equation $A = \mathbf{w} \cdot E$ can be written as the general linear equation $Y = XB + E$, where the matrix X has components -1, 0, $+1$. In this framework it is well-known [10] that the maximum likelihood estimate for $B = [\ln w_i]$ is the estimate that minimizes the quadratic form S and is given by the least-squares estimate $\beta_i = (1/n) \sum_j \ln a_{ij}$. The estimate has all of the usual desirable properties of least-squares estimates under the general linear hypothesis (unbiased, minimum variance etc.). Taking exponentials, the maximum likelihood estimate of w_i is given by: $\omega_i = \exp(\beta_i) = \pi_j a_{ij}^{1/n}$. (The same estimate is derived above from the metric m on the space of judgment matrices.)

This argument shows that the GM is optimal under certain conditions on the distribution of errors. The optimality does not require that the estimate of w_i/w_j be independent of all other estimates, but it does make certain requirements of the multiplicative errors. A sufficient condition for optimality, commonly assumed in regression analysis, is that the errors are independent with identical variances. Here, as in most applications of the least-squares method, the independence assumption may be suspect. McElry [11] has shown these estimates to be optimal in the sense of being the minimum variance estimate among all unbiased estimates in the more general case where errors are normal (lognormal in our application) with a constant covariance matrix (a covariance matrix with 1s on the major diagonal and all other entries constant). McElry also shows that with this covariance matrix the condition of normality may be relaxed and the estimates will be the minimum variance among all linear unbiased estimates.

If the variances or the covariance matrix are known, and not of the form considered by McElry, the least-squares procedure, modified to take advantage of this information, will still be optimal. For details of this and other comments on the robustness of the least-squares procedure, see Scheffe [10].

The procedure outlined above can be modified to solve more general estimation problems. Suppose that instead of a single comparison for each pair of objects E_i and E_j, there are n_{ij} comparisons, a_{ijk}, $k = 1, \ldots, n_{ij}$, where n_{ij} may be 0 (reflecting missing data) or >1 (reflecting multiple comparisons, perhaps by different judges). The problem is then to find a vector $\mathbf{w}$ that minimizes the sum of squares: $S = \sum_i \sum_{j>i} \sum_{k=1} [\ln a_{ijk} - (\ln w_i - \ln w_j)]^2$. This generalization does not yield a simple closed-form solution such as the geometric mean vector, but in practice S can be minimized and $\mathbf{w}$ determined using standard least-squares regression packages.

De Jong [12] treats the statistical qualities of the GM under other assumptions regarding dependence in extensive detail. He shows it to be the minimum variance estimate among all linear unbiased estimates. His treatment yields the covariance matrix for the components of the estimate of the scale. It is shown that the components of the GM estimate are optimal estimates of the corresponding components of the underlying scale.

AN INDEX OF CONSISTENCY

Regardless of the values of n_{ij}, the GM estimation procedure leads to a natural measure of consistency for judgment matrices that is well-grounded in statistical theory and can be used in hypothesis testing. Let s^2 be the residual mean square $s^2 = S/\text{d.f.}$, where d.f. is the number of independent observations minus the number of linearly independent parameters. (Note that if $n_{ij} = 1$, then d.f. $= [n(n-1)/2] - (n-1) = (n-1)(n-2)/2$.) Then s^2 is an unbiased estimator of σ^2 (the variance of the perturbations) and hence is a natural measure of consistency of A.

Recall that if $n_{ij} \equiv 1$, then s^2 can be viewed as the squared distance from A to the m-closest consistent matrix. Therefore s^2 is zero when A is consistent, is close to zero when A is close to consistent and is removed from zero as A becomes increasingly inconsistent. Moreover, because s^2 depends entirely on ratios, it is invariant under scale changes and transposes in the sense of Theorems 1 and 2.

A practical test to determine when a judgment matrix is "too inconsistent" is important. The commonly accepted benchmark for consistency has been the rule that C.R. < 0.1. This rule has been widely accepted and found useful. We will follow that line of reasoning here to develop a comparable measure for the rule s^2.

The C.R. is computed from a consistency index, C.I., given by C.I. $= (\lambda - n)/(n - 1)$, where λ is the maximal right EV of the judgment matrix. The C.R. is defined as C.I./$f(n)$, where $f(n)$ is an empirically determined function of n, the dimension of the scale; $f(n)$, called a random index, is the average C.I. resulting from a number of randomly generated judgment matrices. The commonly used table for $f(n)$ [1, p. 21] resulted from matrices generated by randomly selecting, with a uniform distribution, an integer from the response scale of 1–9. That number was randomly placed in a judgment matrix, and its reciprocal was appropriately placed to preserve the reciprocal symmetric nature of a judgment matrix [1, p. 21]. Repeating this experiment yielded $f(n)$, the average C.I. of the matrices so generated for values of n between 3 and 15:

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(n)$	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49	1.51	1.48	1.56	1.57	1.59

[The last four average C.I.s were computed on the basis of a small sample size by different researchers, possibly explaining the seeming irregularity at $f(12)$.]

Budescu *et al.* [13] have extended this approach and used this procedure, and others, to generate random matrices and measure their average C.I. and average s^2. They generated matrices using a 9, 13 and 50 integer response scale. In addition they generated judgment matrices using the constant sum method suggested by Torgerson [14]. The results are shown to depend on the dimension, as above, in addition to the procedure (constant sum vs response scale) and on the size of the response scale used. They calibrated the results for both the EV and the GM procedures. Additionally, they provide analytic approximations to the results to allow the user to interpolate and substitute his own values of dimension and response scale and achieve a consistency test comparable to the C.R. < 0.1 test for both the EV and the GM.

In addition to the differences noted by Budescu *et al.* [13], it is to be expected that the average C.I.s of randomly generated judgment matrices will also depend on the choice of distribution of the integers of the response scale. Distributions other than the uniform may be appropriate, small integers may occur more commonly than large ones in judgment matrices. It has been shown, for instance, that the occurrence of integers in a surprising variety of tables and natural and artificial collections [15, 16] follow a logarithmic frequency, not a uniform frequency. Theoretical justification, based on an invariance argument, is given for the logarithmic frequency.

Further, the use of a 9 (or 13 or 50) integer response scale, with qualitative descriptions for what each digit signifies, while common and empirically justified, may be at odds with the interpretation of the judgment matrix either by the EV or the GM procedure. Whereas the respondent may be operating on the premise that "3" indicates a "weak importance of one over another", we note the analyst uses this entry in a judgment matrix as an estimate of the ratio of the values of the two entities—i.e. one entity is three times as valuable as the other. The degree of consistency of a matrix, commonly considered important as a measure of the merit of a judgment matrix for the job at hand, depends on the interpretation that the elements are estimates of ratios. (Despite this seeming irregularity, we cannot ignore an important fact: the analysis of an AHP—as advocated by Saaty and others works [1, pp. 17–26]—has been useful, important and widely used.)

In the justification of the GM procedure we have found it helpful to consider the elements of the judgment matrix to be the ratio of the scales perturbed by a multiplicative random error. We continue that approach here and develop a test for the s^2 measure of consistency comparable to the C.R. test.

We note that the C.I. [1, p. 180] may be expressed as a sum of the multiplicative errors:

$$\text{C.I.} = -1 + [1/n(n-1)] \sum (e_{ij} + 1/e_{ij}),$$

where the sum is over $1 \leqslant i < j \leqslant n$. Rearranging terms:

$$\text{C.I.} = [1/n(n-1)] \sum (e_{ij} + 1/e_{ij} - 2).$$

Assuming that the distribution of the error terms is reciprocal symmetric, it follows that the expected value of the C.I. is given by $\text{Ex}(X - 1)$, where X is distributed as the error terms e_{ij}. Given a suitable distribution for the error terms, this relationship may be exploited to give a region of acceptance for s^2 based on the acceptance region C.I. $< (0.1)f(n)$ that follows from the C.R. test for consistency.

Explicitly, if errors are assumed to be lognormally distributed, $\text{Ex}[\ln(X) = 0$, and $\text{var}[\ln(X)] = \sigma^2$, then the above expression for C.I. gives $\text{Ex.C.I.} = \exp[(1/2)\sigma^2] - 1$.

It follows that if it is reasonable to accept a matrix with C.I. $< (0.1)f(n)$, then under the assumption of lognormal errors, the comparable test for s^2 would be $s^2 < 2\ln[(0.1)f(n) + 1]$. Abbreviating the rejection region $s^2 < g(n)$, we have:

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$g(n)$	0.113	0.172	0.212	0.234	0.248	0.264	0.271	0.278	0.281	0.276	0.290	0.292	0.295

Summarizing, for a test comparable to C.R. > 0.1, the matrix should be considered satisfactory if it is of dimension 3 and $s^2 < 0.1$, if the dimension is 4–7 and $s^2 < 0.2$, and if the dimension is >7 and $s^2 < 0.3$.

If errors are multiplicative with a reciprocal symmetric distribution and a relatively small second moment, both the C.I. and s^2 have small variances. [This will generally be true even without the assumption of independent errors. The distribution of s^2 can be shown to be fairly robust with respect to a moderate relaxation of the independence assumption (see Scheffe [10).]

Although in practice the variance of the multiplicative errors seems to be small when the dimension is small and increases as it gets larger, the variance of both the C.I. and s^2, for fixed variance of the error term, decreases as the square of dimension.

For these reasons it is to be expected that both the C.I. and s^2 will be close to their expected values. It follows from the construction of the acceptance region for s^2 that generally the tests are equivalent; that is, one of the tests will result in accepting a matrix if, and only if, the other test would also result in acceptance.

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