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Packing 3-vertex paths in claw-free graphs and related topics

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ABSTRACT

A Λ-*factor* of a graph G is a spanning subgraph of G whose every component is a 3-vertex path. Let $v(G)$ be the number of vertices of G and $\gamma(G)$ the domination number of G. A *claw* is a graph with four vertices and three edges incident to the same vertex. A graph is *claw-free* if it does not have an induced subgraph isomorphic to a claw. Our results include the following. Let G be a 3-connected claw-free graph, $x \in V(G)$, $e = xy \in E(G)$, and L a 3-vertex path in G. Then $(a1)$ if $v(G) \equiv 0 \mod 3$, then G has a Λ-factor containing (avoiding) e, $(a2)$ if $v(G) \equiv 1 \mod 3$, then $G - x$ has a Λ-factor, $(a3)$ if $v(G) \equiv 2 \mod 3$, then $G - \{x, y\}$ has a Λ-factor, $(a4)$ if $v(G) \equiv 0 \mod 3$ and G is either cubic or 4-connected, then $G - L$ has a Λ-factor, $(a5)$ if G is cubic with $v(G) \geq 6$ and E is a set of three edges in G, then $G - E$ has a Λ-factor if and only if the subgraph induced by E in G is not a claw and not a triangle, $(a6)$ if $v(G) \equiv 1 \mod 3$, then $G - \{v, e\}$ has a Λ-factor for every vertex v and every edge e in G, $(a7)$ if $v(G) \equiv 1 \mod 3$, then there exist a 4-vertex path Π and a claw Y in G such that $G - \Pi$ and $G - Y$ have Λ-factors, and $(a8)\gamma(G) \leq \lceil v(G)/3 \rceil$ and if in addition G is not a cycle and $v(G) \equiv 1 \mod 3$, then $\gamma(G) \leq \lfloor v(G)/3 \rfloor$. We also explore the relations between packing problems of a graph and its line graph to obtain some results on different types of packings and discuss relations between Λ-packing and domination problems.

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1. Introduction

We consider undirected graphs with no loops and no parallel edges unless stated explicitly. All notions and facts on graphs, that are used but not described here, can be found in [1,2,23].

Given a graph G and a family $\mathcal{F}$ of non-isomorphic graphs, an $\mathcal{F}$-*packing* of G is a subgraph of G whose every component is isomorphic to a member of $\mathcal{F}$. An $\mathcal{F}$-packing P of G is called an $\mathcal{F}$-*factor* if $V(P) = V(G)$. The $\mathcal{F}$-*packing problem* is the problem of finding in G an $\mathcal{F}$-packing having the maximum number of vertices.

If $\mathcal{F}$ consists of one graph F, then an $\mathcal{F}$-packing and an $\mathcal{F}$-factor are called simply an F-*packing* and an F-*factor*, respectively. Accordingly, the F-*packing problem* is the problem of finding in G an F-packing having the maximum number of vertices or, equivalently, the maximum number of components.

If F is a 2-vertex connected graph, then the F packing problem is the classical matching problem and a very beautiful and deep theory has been developed about this problem and its generalizations (see, for example, [20] as well as [5,7,19]). In particular, it is known that there is a polynomial-time algorithm for finding a maximum matching. It turns out that if F is a connected graph with at least three vertices, then the F-packing problem is already NP-hard [4]. Moreover, if P_k is the k-vertex path, then for every $k \geq 3$ the P_k-packing problem turns out to be also NP-hard for cubic graphs [8].

Let Λ denote a 3-vertex path. We will consider mainly the Λ-packing problem. This problem is interesting for various reasons. Here are some of them.

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($R1$) Path Λ is the smallest graph F, for which the F-packing problem is NP-hard (even in the class of cubic graphs). Although the Λ-packing problem is NP-hard, i.e. possibly intractable in general, it would be interesting to find some natural and non-trivial classes of graphs, for which the problem is tractable, i.e. solvable in polynomial time (e.g. **3.4**, **3.5**, **3.12** and **3.14** below). It is also interesting to find polynomial-time algorithms that provide a good approximation solution for the problem (e.g. **3.1–3.5** and **3.13**).

($R2$) Probably, one of the first non-trivial results in matching theory is Petersen's theorem (1891) stating that every cubic connected graph with at most two bridges has a perfect matching (see [20]). There are indications that a result of similar nature may also be true for the Λ-packing problem in the class of 3-connected graphs (see Problem **3.7** and theorems **3.8–3.11**).

($R3$) It is known [5] that there is a polynomial-time algorithm for the $\{P_3, P_4, P_5\}$-packing problem. It can also be shown that a cubic 3-connected graph has a $\{P_3, P_4, P_5\}$-factor. This fact for $\{P_3, P_4, P_5\}$-factors is analogous to Petersen's theorem for matchings mentioned above. However, the complexity status of an $\{A, B\}$-packing problem for $A, B \in \{P_3, P_4, P_5\}$ and $A \neq B$ is not known. Some results in [9] (see also **3.8–3.11**) show that the Λ-packing problem for cubic 3-connected graphs is related to an $\{A, B\}$-packing problem with $A = P_3 = \Lambda$ and $B \in \{P_4, P_5\}$.

($R4$) The Λ-packing problem is also related to the minimum domination problem in a graph (which is known to be NP-hard). Namely, the size of a maximum Λ-packing in a graph G can be used to give an upper bound for its domination number (see Section **5**).

($R5$) The Λ-packing problem is also related to the various problems on whether a graph G has a spanning subgraph H of special type. In the graph hamiltonicity theory H is usually a Hamiltonian cycle or a Hamiltonian path. Obviously, the existence of such a subgraph H in a graph G implies the existence of a Λ-packing with $\lfloor v(G)/3 \rfloor$ components. For that reason, various Hamiltonicity conjectures give rise to the corresponding Λ-factor problems or conjectures. (This was the original motivation to consider Problem **3.7**.) For example, in 1984 Mathews and Summer [21] conjectured that every 4-connected claw-free graph has a Hamiltonian cycle. Some results in the paper support this conjecture.

($R6$) Let $L(G)$ denote the line graph of G. Then a vertex disjoint packing in $L(G)$ corresponds to an edge disjoint packing in G and a vertex disjoint packing in G corresponds to an induced vertex disjoint packing in $L(G)$. Since $L(G)$ is a claw-free graph, the study of the Λ-packing problem for claw-free graphs may allow to solve some problems on vertex and/or edge disjoint packings in graphs (see Section **6**).

($R7$) The problem of packing induced 3-vertex paths in a claw-free graph is also related to the Hadwiger conjecture (see Section **6**).

In Section **2** we give main notions and notation we use. In Section **3** we describe some known results and open questions and outline main results of the paper. The formulations and proofs of the main results on packings in claw-free graphs are given in Section **4**. In Section **5** we discuss the relation between Λ-packing and domination problems and provide some bounds on the graph domination numbers based on some Λ-packing results. Finally, in Section **6** we explore the relation between packing problems of a graph and its line graph to obtain some results on different types of packings. We also discuss the induced Λ-packing problem and its relation with the Hadwiger conjecture.

2. Main notions, notation, and simple observations

As usual, $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively, and $v(G) = |V(G)|$, $e(G) = |E(G)|$. If P is a path with the end-vertices x and y, we put $\text{End}(P) = \{x, y\}$. Given $X \subseteq E(G)$, let $\dot{X}$ denote the subgraph of G induced by X. Given $x \in V(G)$, let $N(x, G) = N(x)$ denote the set of vertices in G adjacent to x. Let $Cmp(G)$ denote the set of components of G and $cmp(G) = |Cmp(G)|$. Let $\lambda(G)$ denote the maximum number of disjoint 3-vertex paths in G. A vertex subset X of G is called a *domination set* in G, if every vertex in $V(G) \setminus X$ is adjacent to a vertex in X. Let $\gamma(G)$ denote the size of a minimum domination set in G; $\gamma(G)$ is called the *domination number* of G. A *leaf* in a graph is a vertex of degree one. Let $Lv(G)$ denote the set of leaves in G and $lv(G) = |Lv(G)|$.

A *claw* is a graph isomorphic to $K_{1,3}$, i.e. the graph with four vertices and three edges having a common end-vertex. A graph is called *claw-free* if it contains no induced claw. A *net* is a graph obtained from a triangle with three vertices x_1, x_2, and x_3 by adding three new vertices z_1, z_2, and z_3 and three new edges x_1z_1, x_2z_2, and x_3z_3. A graph with one edge and two vertices is called a *match*.

A graph G is *minimal 2-connected* if G is 2-connected but $G-e$ is not 2-connected for every $e \in E(G)$. A 2-*frame* (or simply, a *frame*) of G is a minimal 2-connected spanning subgraph of G.

Given a subgraph S of a graph G, a vertex $x \in V(S)$ is a *boundary vertex* of S if x is adjacent to a vertex in $G - S$ and an *inner vertex* of S, otherwise.

A *block* of a connected graph G is a maximal connected subgraph H of G such that $H - v$ is connected for every vertex v of H, and so H is either 2-connected or a match. If B has at most one boundary vertex, then B is called an *end-block* of G. Let $eb(G)$ denote the number of end-blocks of connected graph G, and so if $eb(G) = 1$, then G is either 2-connected or a match.

We call a graph H a *chain* if H is connected and has at most two end-blocks. An *end-chain* of G is a maximal proper subgraph H of G such that H is a chain, every block of H is a block of G with at most two boundary vertices in G, and H contains an end-block of G. Obviously, a connected graph G has an end-chain if and only if G has at least three end-blocks. Also if G has end-chains, then every end-block of G is a subgraph (moreover, an end-block) of exactly one end-chain of G and every end-chain of G contains exactly one end-block of G.

We call a graph G a Δ-*graph* if G is cubic and every vertex of G belongs to exactly one triangle, and so a Δ-graph is a claw-free graph.

We call a graph G a *cactus* if G is connected, G has at least three end-blocks, and each end-chain of G is a match.

Given a graph G, we write $G = AxB$ if A and B are graphs, $V(A) \cap V(B) = \{x\}$, and $G = A \cup B$, and so if A and B are connected graphs with at least two vertices, then G is connected and x is a cut-vertex of G.

We recall that a Λ-*packing* in a graph G is a subgraph of G whose every component is a 3-vertex graph and a Λ-*factor* in G is a spanning Λ-packing of G. In addition, a Λ-*packing* P in G is called a Λ-*quasi-factor* of G if $v(G) - v(P) \le 2$.

We will use the following simple facts.

2.1. *Let $G = AxB$, where A and B are connected graphs with at least two vertices. Suppose that G is claw-free. Then the following holds.*

($a1$) *$N(x, A)$ and $N(x, B)$ induce complete subgraphs in A and B, respectively.*
($a2$) *If A is a block of G and $v(A) \ge 3$, then $B - x$ is either 2-connected or a match, and so in this case $eb(B - x) = 1$.*
($a3$) *If A is a block of G, $v(A) \ge 4$, and $xy \in E(A)$, then $A - \{x, y\}$ is a chain, and so $eb(A - x) \le 2$.*
($a4$) *If A is a chain and $v(A) \ge 3$, then $A - x$ is also a chain, and so $eb(A - x) \le 2$.*
($a5$) *If A is an end-chain of G, $v(A) \ge 4$, and $xy \in E(A)$, then $A - \{x, y\}$ is connected and $eb(A - \{x, y\}) \le 3$.*

3. Preliminaries and an outline of new results

In [10,16] we gave an answer to the following natural question:

How many disjoint 3-vertex paths must a cubic n-vertex graph have?

Obviously, $\lambda(G) \le \lfloor v(G)/3 \rfloor$.

3.1. *If G is a cubic graph, then $\lambda(G) \ge \lceil v(G)/4 \rceil$ and at least $v(G)/4$ disjoint 3-vertex paths in G can be found in polynomial time.*

Obviously, if every component of G is K_4, then $\lambda(G) = v(G)/4$. Therefore the bound in **3.1** is sharp.

Let $\mathcal{G}_2^3$ denote the set of graphs with each vertex of degree 2 or 3. In [10] we gave (among other things) an answer to the following question:

How many disjoint 3-vertex paths must an n-vertex graph from $\mathcal{G}_2^3$ have?

3.2. *Suppose that $G \in \mathcal{G}_2^3$ and G has no 5-vertex components. Then $\lambda(G) \ge v(G)/4$ and at least $v(G)/4$ disjoint 3-vertex paths in G can be found in polynomial time.*

From **3.2** it follows that every cubic graph G has at least $v(G)/4$ disjoint 3-vertex paths [16] because if G is a cubic graph, then $G \in \mathcal{G}_2^3$ and G has no 5-vertex components.

In [10] we also gave a construction that allowed to prove the following:

3.3. *There are infinitely many 2-connected graphs in $\mathcal{G}_2^3$ (and even subdivisions of cubic 3-connected graphs) for which the bound in* **3.2** *is attained.*

Here are some packing results on regular graphs.

3.4. [11] *Let G be a d-regular graph with $d \ge 4$. Then $\lambda(G) \ge v(G)/4$ and at least $v(G)/4$ disjoint 3-vertex paths in G can be found in polynomial time.*

3.5. [17] *Let T be a tree on t vertices and let $\epsilon > 0$. Suppose that G is a d-regular graph and $d \ge \delta \ln \delta$, where $\delta = \frac{128t^3}{\epsilon^2}$. Then G contains at least $(1 - \epsilon)n/t$ vertex disjoint copies of T and they can be found in polynomial time.*

There are infinitely many 2-connected cubic graphs G having no Λ-quasi-factors. Some of such graphs were constructed in [12] to provide 2-connected counterexamples to Reed's domination conjecture (see Section **5**). In particular, a graph sequence $(R_k : k \ge 3)$ in [12] is such that each R_k is a cubic graph of connectivity two, $v(R_k) = 20k$, and $\gamma(R_k) = \left(\frac{1}{3} + \frac{1}{60}\right) v(R_k)$. Obviously, $\gamma(G) \le v(G) - 2\lambda(G)$. Therefore $\lambda(R_k) \le \frac{13}{40} v(R_k)$. Questions arise whether there are 2-connected cubic graphs with some additional properties and without Λ-quasi-factors. For example,

3.6 Problem. *Does every 2-connected, cubic, bipartite, and planar graph have a Λ-quasi-factor?*

In [13] we answered the question in **3.6** by giving a construction that provides infinitely many 2-connected, cubic, bipartite, and planar graphs without Λ-quasi-factors.

As to cubic 3-connected graphs, an our old open question here is:

3.7 Problem. *Is the following claim true?*

(P) Every cubic 3-connected graph G has a Λ-quasi-factor, i.e. $\lambda(G) = \lfloor v(G)/3 \rfloor$.

In [9] we discuss Problem **3.7** and show, in particular, that claim (P) in **3.7** is equivalent to some seemingly much stronger claims. Here are some results of this kind.

3.8. [9] *The following are equivalent for cubic 3-connected graphs G:*

(z) $v(G) \equiv 0 \bmod 6 \Rightarrow G$ *has a* Λ*-factor,*
(t) $v(G) \equiv 2 \bmod 6 \Rightarrow G - \{x, y\}$ *has a* Λ*-factor for some* $x, y \in V(G)$, $x \neq y$, *and*
(f) $v(G) \equiv 4 \bmod 6 \Rightarrow G - x$ *has a* Λ*-factor for some* $x \in V(G)$.

3.9. [9] *The following are equivalent for cubic 3-connected graphs G with* $v(G) \equiv 0 \bmod 6$*:*

$(z0)$ *G has a* Λ*-factor,*
$(z1)$ *for every* $e \in E(G)$ *there is a* Λ*-factor of G avoiding e, i.e.* $G - e$ *has a* Λ*-factor,*
$(z2)$ *for every* $e \in E(G)$ *there is a* Λ*-factor of G containing e,*
$(z3)$ $G - X$ *has a* Λ*-factor for every* $X \subseteq E(G)$, $|X| = 2$, *and*
$(z4)$ $G - L$ *has a* Λ*-factor for every 3-vertex path L in G.*

3.10. [9] *The following are equivalent for cubic 3-connected graphs G with* $v(G) \equiv 2 \bmod 6$*:*

$(t0)$ $G - \{x, y\}$ *has a* Λ*-factor for some* $x, y \in V(G)$, $x \neq y$,
$(t1)$ $G - \{x, y\}$ *has a* Λ*-factor for some* $xy \in E(G)$,
$(t2)$ $G - \{x, y\}$ *has a* Λ*-factor for every* $xy \in E(G)$, *and*
$(t3)$ *there exists a 5-vertex path W such that* $G - W$ *has a* Λ*-factor, and so G has a* $\{P_3, P_5\}$*-factor.*

3.11. [9] *The following are equivalent for cubic 3-connected graphs G with* $v(G) \equiv 4 \bmod 6$*:*

$(f0)$ $G - x$ *has a* Λ*-factor for some* $x \in V(G)$,
$(f1)$ $G - x$ *has a* Λ*-factor for every* $x \in V(G)$,
$(f2)$ $G - \{x, e\}$ *has a* Λ*-factor for every* $x \in V(G)$ *and every* $e \in E(G)$, *and*
$(f3)$ *there exists a 4-vertex path Z such that* $G - Z$ *has a* Λ*-factor, and so G has a* $\{P_3, P_4\}$*-factor.*

There are some interesting results on the Λ-packing problem for claw-free graphs. Recall that a graph is called *claw-free* if it contains no induced subgraph isomorphic to a claw.

3.12. [6] *Suppose that G is a 2-connected claw-free graph. Then*

$(a1)$ *if* $v(G) \equiv 0 \bmod 3$, *then G has a* Λ*-factor,*
$(a2)$ *if* $v(G) \equiv 1 \bmod 3$, *then* $G - x$ *has a* Λ*-factor for some* $x \in V(G)$, *and*
$(a3)$ *if* $v(G) \equiv 2 \bmod 3$, *then* $G - \{x, y\}$ *has a* Λ*-factor for some* $x, y \in V(G)$, $x \neq y$.

In every case a maximum Λ*-packing can be found in polynomial time.*

3.13. [6] *Suppose that G is a connected claw-free graph and* $eb(G) \geq 2$. *Then* $\lambda(G) \geq \lfloor (v(G) - eb(G) + 2)/3 \rfloor$, *this lower bound is sharp, and* $\lfloor (v(G) - eb(G) + 2)/3 \rfloor$ *disjoint 3-vertex paths in G can be found in polynomial time.*

From **3.13** we have, in particular:

3.14. [6] *Suppose that G is a connected claw-free graph having exactly two end-blocks. Then* $\lambda(G) = \lfloor v(G)/3 \rfloor$ *and a maximum* Λ*-packing can be found in polynomial time.*

As we have mentioned in Section **1**, the Λ-packing problem remains *NP*-hard in the class of all graphs and even in the class of cubic graphs [4,8]. It would be interesting to answer the following question.

3.15 Problem. *Is the* Λ*-packing problem NP-hard in the class of claw-free graphs?*

In this paper (see Section **4**) we give some more results on the Λ-packings in claw-free graphs showing, in particular, to what extent the claims in **3.8–3.11** are true for claw-free graphs. Here are some of these results.

$(c1)$ If G is a 2-connected claw-free graph and $v(G) \equiv 0 \bmod 3$, then for every edge e in G there exists a Λ-factor of G avoiding e, i.e. $G - e$ has a Λ-factor (see **4.13** and compare with **3.9**$(z1)$ and **3.12**$(a1)$).

$(c2)$ If G is a 3-connected claw-free graph and $v(G) \equiv 0 \bmod 3$, then for every edge e in G there exists a Λ-factor of G containing e (see **4.20** and compare with **3.9**$(z2)$ and **3.12**$(a1)$).

$(c3)$ If G is a cubic 2-connected claw-free graph with every vertex belonging to exactly one triangle and E is a set of two edges in G, then $G - E$ has a Λ-factor (see **4.23** and compare with **3.9**$(z3)$).

$(c4)$ If G is a cubic 3-connected claw-free graph with $v(G) \geq 6$ and E is a set of three edges in G, then $G - E$ has a Λ-factor if and only if the subgraph induced by E in G is not a claw and not a triangle (see **4.22**).

($c5$) If G is a cubic 3-connected claw-free graph or a 4-connected claw-free graph with $v(G) \equiv 0 \bmod 3$, then for every 3-vertex path L in G there exists a Λ-factor containing L, i.e. $G - L$ has a Λ-factor (see **4.19** and **4.21** and compare with **3.9**($z4$)).

($c6$) If G is a 2-connected claw-free graph and $v(G) \equiv 2 \bmod 3$, then for every vertex x in G there exist two edge xy and xz in G such that $G - \{x, y\}$ and $G - \{x, z\}$ have Λ-factors (see **4.13** and **4.15** and compare with **3.10**(t1) and **3.12**($a3$)).

($c7$) If G is a 3-connected claw-free graph and $v(G) \equiv 2 \bmod 3$, then $G - \{x, y\}$ has a Λ-factor for every edge xy in G (see **4.16** and compare with **3.10** (t2) and **3.12**($a3$)).

($c8$) If G is a 3-connected claw-free graph and $v(G) \equiv 2 \bmod 3$, then G has a 5-vertex path W such that $G - W$ has a Λ-factor, and so G has a $\{P_3, P_5\}$-factor (see **4.13**($a2$) and compare with **3.10**(t3)).

($c9$) If G is a 2-connected claw-free graph and $v(G) \equiv 1 \bmod 3$, then $G - x$ has a Λ-factor for every vertex x in G (see **4.25** and compare with **3.11**($f1$) and **3.12**($a2$)).

($c10$) If G is a 3-connected claw-free graph and $v(G) \equiv 1 \bmod 3$, then $G - \{x, e\}$ has a Λ-factor for every vertex x and every edge e in G (see **4.26** and compare with **3.11**($f2$)).

($c11$) If G is a 2-connected claw-free graph and $v(G) \equiv 1 \bmod 3$, then there exist a 4-vertex path Π and a claw Y in G such that $G - \Pi$ and $G - Y$ have Λ-factors, and so G has a $\{P_3, P_4\}$-factor and $\{P_3, Y\}$-factor (see **4.13**($a2$) and **4.14** and compare with **3.11**($f3$) and **5.1**).

($c12$) We show that the Λ-packing problem for a claw-free graph G can be reduced in polynomial time to that for a special claw-free graph K (called a cactus) with $v(K) \le v(G)$ (see **4.5** and **4.6**).

($c13$) If G is a 2-connected claw-free graph, then $\gamma(G) \le \lceil v(G)/3 \rceil$ and if in addition G is not a cycle and $v(G) \equiv 1 \bmod 3$, then $\gamma(G) \le \lfloor v(G)/3 \rfloor$ (see **5.3**).

4. Main results on claw-free graphs

We will often use the following combination of **3.12** and **3.14**.

4.1. [6] *Suppose that G is a connected claw-free graph having at most two end-blocks. Then $\lambda(G) = \lfloor v(G)/3 \rfloor$ and a maximum Λ-packing can be found in polynomial time.*

Recall that $G = AxB$ is the union of graphs A and B with $V(A) \cap V(B) = \{x\}$. We need the following bounds on $\lambda(AxB)$.

4.2. *Let $G = AxB$, where A and B are connected graphs with at least two vertices. Then*

($a0$) *if* $v(A) \equiv 0 \bmod 3$, *then* $\lambda(G) \le v(A)/3 + \lambda(B - x)$,
($a1$) *if* $v(A) \equiv 1 \bmod 3$, *then* $\lambda(G) \le (v(A) - 1)/3 + \lambda(B)$, *and*
($a2$) *if* $v(A) \equiv 2 \bmod 3$, *then* $\lambda(G) \le (v(A) - 2)/3 + \lambda(B \cup xy)$, *where* $xy \in E(A)$.

Proof. Let S be a maximum Λ-packing in G.

(**p1**) Suppose that $v(A) \equiv 0 \bmod 3$. Let $S_1 = S \cap A$ and $S_2 = S - S_1$. Then $\lambda(S_1) \le v(A)/3$.

Suppose that $x \in V(S_1)$. Then $\lambda(S_2) = \lambda(B - x)$, and so $\lambda(S) = \lambda(S_1) + \lambda(S_2) \le v(A)/3 + \lambda(B - x)$.

Now suppose that $x \notin V(S_1)$. Then $\lambda(S_1) \le v(A)/3 - 1$. Let $S'_2 = S_2 \cap (B - x)$ and $S''_2 = S_2 - S'_2$. Then $\lambda(S'_2) \le \lambda(B - x)$. If L is a 3-vertex path in S''_2, then $x \in V(L)$. Hence $\lambda(S''_2) \le 1$. Therefore $\lambda(S_2) = \lambda(S'_2) + \lambda(S''_2) \le \lambda(B - x) + 1$. Thus, $\lambda(S) = \lambda(S_1) + \lambda(S_2) \le (v(A)/3 - 1) + \lambda(B - x) + 1 = v(A)/3 + \lambda(B - x)$.

(**p2**) Suppose that $v(A) \equiv 1 \bmod 3$. Let $S_1 = S \cap (A - x)$ and $S_2 = S - S'$.

Suppose that $\lambda(S_1) = v(A - x)/3$. Then $\lambda(S_2) = \lambda(B)$, and so $\lambda(S) = \lambda(S_1) + \lambda(S_2) = v(A)/3 + \lambda(B - x)$.

Now suppose that $\lambda(S_1) \le v(A - x)/3 - 1$. Let $S'_2 = S_2 \cap (B)$ and $S''_2 = S_2 - S'_2$. Then $\lambda(S'_2) \le \lambda(B)$. If L is a 3-vertex path in S''_2, then $x \in V(L)$. Hence $\lambda(S''_2) \le 1$. Therefore $\lambda(S_2) = \lambda(S'_2) + \lambda(S''_2) \le \lambda(B) + 1$. Thus, $\lambda(S) = \lambda(S_1) + \lambda(S_2) \le (v(A)/3 - 1) + \lambda(B) + 1 = v(A)/3 + \lambda(B)$.

(**p3**) Finally, suppose that $v(A) \equiv 2 \bmod 3$. Let $S_1 = S \cap (A - \{x, y\})$ for some $xy \in E(A)$ and $S_2 = S - S_1$. Let $S'_2 = S_2 \cap (B \cup xy)$ and $S''_2 = S_2 - S'_2$. Then $\lambda(S'_2) \le \lambda(B \cup xy)$.

Suppose that $\lambda(S_1) = v(A - \{x, y\})/3$. Then $\lambda(S_2) = \lambda(B \cup xy)$, and so $\lambda(S) = \lambda(S_1) + \lambda(S_2) = v(A - \{x, y\})/3 + \lambda(B)$.

Suppose that $\lambda(S_1) = v(A - \{x, y\})/3 - 1$. If L is a 3-vertex path in S''_2, then $V(L) \subset V(A - S_1)$. Since $\lambda(S_1) = v(A - \{x, y\})/3 - 1$, we have: $|V(A - S_1)| = 5$. Hence $\lambda(S''_2) \le 1$. Therefore $\lambda(S_2) = \lambda(S'_2) + \lambda(S''_2) \le \lambda(B) + 1$. Thus, $\lambda(S) = \lambda(S_1) + \lambda(S_2) \le (v(A - \{x, y\})/3 - 1) + \lambda(B \cup xy) + 1 = v(A - \{x, y\})/3 + \lambda(B \cup xy)$.

Now suppose that $\lambda(S_1) = v(A - \{x, y\})/3 - 2$. If L is a 3-vertex path in S''_2, then $V(L) \cap \{x, y\} \ne \emptyset$. Hence $\lambda(S''_2) \le 2$. Therefore $\lambda(S_2) = \lambda(S'_2) + \lambda(S''_2) \le \lambda(B) + 2$. Thus, we have: $\lambda(S) = \lambda(S_1) + \lambda(S_2) \le (v(A - \{x, y\})/3 - 2) + \lambda(B) + 2 = v(A - \{x, y\})/3 + \lambda(B \cup xy)$. □

It turns out that the end-chains of a claw-free graph have some special Λ-packing properties.

4.3. *Let G be a connected claw-free graph, C an end-chain of G, $v(C) \ge 3$, and b the boundary vertex of C (and so $eb(C) \le 2$). Then there exists an edge bb' in C such that $\lambda(C - \{b, b'\}) = \lfloor v(C - \{b, b'\}/3) \rfloor$.*

Proof (Uses **4.1**). Let $bx \in E(C)$. Since G is claw-free, $C - \{b, x\}$ is also claw-free. If there exists an edge bb' in C such that $eb(C - \{b, b'\}) \le 2$, then we are done by **4.1**.

Let B be the end-block of C containing b, and so b is a boundary vertex of B. Since G is claw-free, B and $B - \{b, x\}$ are also claw-free for $bx \in E(B)$ and $N(x, B)$ induces a complete subgraph K in B.

(**p1**) Suppose that B has exactly one edge bb'. Since $v(C) \ge 3$, there is a (unique) block B' in C such that b' is a boundary vertex of B'. If $e(B') = 1$, then $eb(C - \{b, b'\}) \le 2$ and we are done. If $e(B') \ge 2$, then B' is 2-connected. Since $N(b', B')$ induces a complete subgraph in B', clearly $B' - b'$ is either 2-connected or a match. Therefore again $eb(C - \{b, b'\}) \le 2$ and we are done.

(**p2**) Now suppose that B has at least two edges, and so B is 2-connected. Then $eb(B - \{b, x\}) \le 2$ for every edge bx in B. Therefore if $B = C$, then we are done. So we assume that $B \ne C$, and so C has the end-block D distinct from B. If there is an edge bb' in B such that $eb(B - \{b, b'\}) = 1$, then $eb(C - \{b, b'\}) \le 2$ and we are done. So we assume that $eb(B - \{b, x\}) = 2$ for every vertex x in K. Then $v(K) \ge 2$. Let $B_1(x)$ and $B_2(x)$ be the two end-blocks of $B - \{b, x\}$, where $B_1(x)$ has no vertex adjacent to b. Since $eb(B - \{b, x\}) = 2$, clearly $B_1(x)$ and $B_2(x)$ are 2-connected for every edge bx in B. Since $B \ne D$, clearly $eb(C - \{b, x\}) = 3$. Since G is claw-free, $C_x = C - (K - x)$ is also claw-free. Since bx is an end-block of C_x, clearly $N(x, C_x - b)$ induces in $C_x - b$ a complete subgraph K_x. Now since $B_1(x)$ has a vertex adjacent to x, clearly $K_x \subseteq B_1(x)$. Since $B_2(x)$ is 2-connected, $K - x$ has an inner vertex z of $B_2(x)$. Since $K - x$ is a complete graph and $B_2(x)$ is 2-connected, we have: $K - x \subseteq B_2(x)$.

First we assume that $v(K) \ge 3$, and so there is vertex y in $K - \{x, z\}$. Then $y \in B_2(x)$, and so in $B - \{b, z\}$, vertex x is adjacent to $B_1(x)$ and $B_2(x) - z$. Therefore $B - \{b, z\}$ is 2-connected, and so $eb(B - \{b, z\}) = 1$, a contradiction.

Now we assume that $v(K) = 2$, say, $V(K) = \{b_1 = x, b_2 = y\}$. Let B_1 be the subgraph of B induced by $B_1(x) \cup x$ and $B_2 = B_2(x)$. Then $N(b_1, C - b - b_2) = N(b_1, B_1)$, $N(b_2, C - b - b_1) = N(b_2, B_2)$, and each $N(b_i, B_i)$ induces in B_i a complete subgraph. Let $B'_i = B_i - b_i$. Clearly, $B'_1 = B_1(b_1) = B_1(x)$. Since $B_1(x)$ is 2-connected, B'_1 is 2-connected. Since $B_2 = B_2(x)$ is 2-connected, B'_i is either 2-connected or a match. Obviously, B'_1, B'_2, and D are the three end-blocks of $C - \{b, b_1, b_2\}$. Let C_i be the end-chain in $C - \{b, b_1, b_2\}$ containing B'_i and c_i be the boundary vertex of C_i. Let $C^i = C - \{b, b_i\}$ and B'_i be the subgraph of C induces by $C_i \cup b_i$.

(**p2.1**) Suppose that $v(C_i) \equiv 0 \mod 3$ for some $i \in \{1, 2\}$, say, for $i = 1$. Obviously, $C^1 - (C_1 - c_1)$ has exactly two end-blocks (namely, B'_2 and D).

Since c_1 is a vertex in $C^1 - (C_1 - c_1)$ and the boundary vertex of C_i, the neighborhood of c_1 in $C^1 - (C_1 - c_1)$ induces a complete subgraph in $C^1 - (C_1 - c_1)$. Therefore $C^1 - C_1$ has also two end-blocks (namely, B'_2 and D). By **4.1**, $\lambda(C^1 - C_1) = \lfloor v(C^1 - C_1)/3 \rfloor$. Since C_1 is a chain and $v(C_1) \equiv 0 \mod 3$, by **4.1**, $\lambda(C_1) = v(C_1)/3$. Thus, $\lambda(C^1) = \lambda(C^1 - C_1) + v(C_1)/3 = \lfloor v(C^1)/3 \rfloor$.

(**p2.2**) Suppose that $v(C_1) \equiv 1 \mod 3$ for some $i \in \{1, 2\}$, say $i = 1$. Since c_1 is the boundary vertex of end-chain C_1 in C^1, the neighborhood of c_1 in C_1 induces a complete subgraph in C_1. Therefore $C_1 - c_1$ is a chain. Since $v(C_1) \equiv 1 \mod 3$, clearly $v(C_1 - c_1) \equiv 0 \mod 3$. Therefore by **4.1**, $\lambda(C_1 - c_1) = v(C_1 - c_1)/3$. Obviously, $C^1 - (C_1 - c_1)$ is a chain. By **4.1**, $\lambda(C^1 - (C_1 - c_1)) = \lfloor v(C^1 - (C_1 - c_1))/3 \rfloor$. Thus, $\lambda(C^1) = \lambda(C^1 - C_1) + v(C_1 - c_1)/3 = \lfloor v(C^1)/3 \rfloor$.

(**p2.3**) Finally, suppose that $v(C_1) \equiv 2 \mod 3$ and $v(C_2) \equiv 2 \mod 3$. Let C'_1 denote the end-chain in C^2 containing B'_1. Then $C_1 \subset C'_1$ and $v(C'_1) = v(C_1) + 1 = 0 \mod 3$. Now the arguments similar to those in (**p1**) shows that our claim is true. □

Now we can improve bounds on $\lambda(G)$ in **4.2** when $G = AxB$ is claw-free and A is an end-chain of G.

4.4. *Let $G = AxB$, where A and B are connected graphs with at least two vertices. Suppose that G is claw-free and A is an end-chain of G. Then*

($a0$) *if* $v(A) \equiv 0 \mod 3$, *then* $\lambda(G) = v(A)/3 + \lambda(B - x)$,
($a1$) *if* $v(A) \equiv 1 \mod 3$, *then* $\lambda(G) = (v(A) - 1)/3 + \lambda(B)$,
($a2$) *if* $v(A) \equiv 2 \mod 3$, *then* $\lambda(G) = (v(A) - 2)/3 + \lambda(B \cup xy)$, *where xy is an edge in A.*

Proof (Uses **4.1**, **4.2** and **4.3**). Suppose that $v(A) \equiv 0 \mod 3$. Then by **4.1**, A has a Λ-factor P. Let Q be a maximum Λ-packing in $B - x$. Then $P \cup Q$ is a Λ-packing in G and $\lambda(P \cup Q) = v(A)/3 + \lambda(B - x)$. Therefore by **4.2**($a0$), $\lambda(G) = v(A)/3 + \lambda(B - x)$.

Suppose that $v(A) \equiv 1 \mod 3$. Then by **4.1**, $A - x$ has a Λ-factor P. Let Q be a maximum Λ-packing in B. Then $P \cup Q$ is a Λ-packing in G and $\lambda(P \cup Q) = (v(A) - 1)/3 + \lambda(B)$. Therefore by **4.2**($a1$), $\lambda(G) = (v(A) - 1)/3 + \lambda(B)$.

Finally, suppose that $v(A) \equiv 2 \mod 3$. Then by **4.3**, there exists an edge xy in A such that $A - \{x, y\}$ has a Λ-factor P. Let Q be a maximum Λ-packing in B. Then $P \cup Q$ is a Λ-packing in G and $\lambda(P \cup Q) = (v(A) - 2)/3 + \lambda(B \cup xy)$. Therefore by **4.2**($a2$), $\lambda(G) = (v(A) - 2)/3 + \lambda(B \cup xy)$. □

Theorem **4.4** suggests the following reduction procedure for claw-free graphs.

Let G be a connected claw-free graph, C an end-chain of G, and c the boundary vertex of C. Let us define a graph $\lfloor C \rfloor$ as follows:

if $v(C) \equiv 0 \mod 3$ and $v(C) \ge 3$, then $\lfloor C \rfloor = C$,
if $v(C) \equiv 1 \mod 3$ and $v(C) \ge 4$, then $\lfloor C \rfloor = C - c$, and

if $v(C) \equiv 2 \mod 3$ and $v(C) \geq 5$, then $\lfloor C \rfloor = C - \{c, c'\}$, where cc' is an edge in C such that $\lambda(C - \{c, c'\}) = \lfloor v(C - \{c, c'\}/3) \rfloor$ (see **4.3**).

Obviously, $v(\lfloor C \rfloor)/3 = \lfloor v(C)/3 \rfloor$.

Recall that a graph G is called a *cactus* if G is connected and has at least three end-chains and each end-chain has exactly two vertices.

The following procedure for claw-free graphs allows to either find a Λ-factor in a graph G or to reduce the Λ-packing problem for G to that for a cactus K with $v(K) \leq v(G)$.

4.5 Reduction. *Let G be a connected claw-free graph.*

($s1$) *If C_1 is an end-chain of G with $v(C_1) \geq 3$, then put $D_1 = \lfloor C_1 \rfloor$ and $G_1 = G - D_1$.*
($s2$) *We assume that G_i and the sequence $(D_1, \ldots, D_i)$ has already been defined for some $i \geq 1$.*

If G_i has less than three end-chains or every end-chain of G_i has exactly two vertices, then stop and put $i = k$. Otherwise, let C_{i+1} be an end-chain of G with $v(C_{i+1}) \geq 3$. Put $D_{i+1} = \lfloor C_{i+1} \rfloor$ and $G_{i+1} = G_i - D_{i+1}$.

The output of this procedure is $(D_1, \ldots, D_k)$ and G_k.

Obviously, Reduction **4.5** is a polynomial-time procedure.

Let $D^k = \cup\{D_i : i \in \{1, \ldots, k\}\}$. Clearly all D_i's are disjoint, and so $\lambda(D^k) = \sum\{\lambda(D_i) : i \in \{1, \ldots, k\}\}$.

It follows that G_k in Reduction **4.5** is either a claw-free chain or a claw-free cactus. It is easy to show that if G_k is a cactus, then D^k and G_k are uniquely defined; in this case let us denote D_k by $D(G)$ and G_k by $R(G)$.

4.6. *Let G be a connected claw-free graph and $(D_1, \ldots, D_k)$ and G_k be the output of Reduction* **4.5** *applied to G. Let Q be a maximum Λ-packing in G_k. Then*

($a1$) *each D_i has a Λ-factor P_i, and so $\lambda(P_i) = \lambda(D_i) = v(D_i)/3$,*
($a2$) *if G_k is a chain, then P is a Λ-factor of G,*
($a3$) *$P = Q \cup \{P_i : i \in \{1, \ldots, k\}\}$ is a maximum Λ-packing in G,*
($a4$) *if G_k is not a chain, then*

$$\lambda(G) = \lambda(R(G)) + v(D(G))/3 \geq \lfloor (v(R(G)) - eb(R(G)) + 2)/3 \rfloor + v(D(G))/3 = l,$$

this lower bound is sharp, and l disjoint 3-vertex paths in G can be found in polynomial time.

Proof (Uses **3.13**, **4.1**, **4.3**, and **4.5**). We prove ($a1$) and ($a2$). By Reduction **4.5**, each D_i has at most two end-blocks and $v(D_i) \equiv 0 \mod 3$. Since G is claw-free, each D_i is also claw-free. By **4.1**, each D_i has a Λ-factor P_i, and so $\lambda(P_i) = \lambda(D_i) = v(D_i)/3$. Therefore ($a1$) holds. If G_k is a chain, then by the same reason, Q is a Λ-factor of G_k. Then $P = Q \cup \{P_i : i \in \{1, \ldots, k\}\}$ is a Λ-factor of G, and so ($a2$) holds. Now ($a3$) follows from **4.4** and ($a4$) follows from ($a3$) and **3.13**. □

From **4.5** and **4.6** it follows that Problem **3.15** is equivalent to

4.7 Problem. *Is Λ-packing problem NP-hard for claw-free cacti?*

Now we describe an infinite class of sub-cubic claw-free graphs with no Λ factors. This class includes infinitely many cacti. We will use this description to establish some Λ-packing properties of Δ-graphs (see **4.22**).

Let $\mathcal{S}$ denote the set of graphs S with the following properties:

($\alpha 1$) S is connected,
($\alpha 2$) every vertex in S has degree at most 3,
($\alpha 3$) every vertex in S of degree 2 or 3 belongs to exactly one triangle, and
($\alpha 4$) S has at least three leaves.

4.8. *If $S \in \mathcal{S}$, then S has no Λ-factor.*

Proof. Let $S \in \mathcal{S}$. If $v(S) \not\equiv 0 \mod 3$, then our claim is obviously true. So we assume that $v(S) \equiv 0 \mod 3$. By ($\alpha 3$), $v(S) \equiv lv(S) \mod 3$, and so $lv(S) \equiv 0 \mod 3$. Obviously, it is sufficient to prove our claim for $S \in \mathcal{S}$ with property ($\alpha' 4$): $lv(A) = 3$. We prove our claim by induction on $v(G)$. The smallest graph in $\mathcal{S}$ is a net N with $v(N) = 6$ and our claim is obviously true for N. So let $v(S) \geq 9$. Suppose, on the contrary, that S has a Λ-factor P. Let v be a leaf of S and vx the edge incident to v. Since P is a Λ-factor in S, it has a component $L = vxy$, and so $P - L$ is a Λ-factor in $S - L$ and $d(x, S) \geq 2$. By property ($c3$), x belongs to a unique triangle xyz in A and $d(x, a) = 3$. If $d(z, S) = 2$, then z is an isolated vertex in $S - L$, and so P is not a Λ-factor in S, a contradiction. Therefore by ($c2$), $d(z, S) = 3$. Hence z is a leaf in $S - L$, and so $lv(S - L) = 3$. Therefore $S - L$ satisfies ($\alpha 2$), ($\alpha 3$), and ($\alpha' 4$).

Suppose that $G - L$ is not connected and that the three leaves do not belong to a common component. Then $S - L$ has a component C with $v(C) \not\equiv 0 \mod 3$, and so $S - L$ has no Λ-factor, a contradiction.

Finally, suppose that $S - L$ has a component C containing all three leaves of $S - L$. Then $C \in \mathcal{S}$ and $v(C) < v(S)$. By the induction hypothesis, C has no Λ-factor. Therefore $S - L$ also has no Λ-factor, a contradiction. □

Recall that a *frame* of G is a minimal 2-connected spanning subgraph of G.

We need the following procedure from [6] that provides a frame of a 2-connected graph. This procedure was used in [6] to prove **3.12**.

4.9 Procedure $\mathcal{E}$. *Let G be a 2-connected graph. We define sequences $\mathcal{A} = (A_0, \ldots, A_r)$ and $\mathcal{G} = (G_0, \ldots, G_r)$ recursively, where each A_i and each G_i is a subgraph of G:*

($s1$) *Let A_0 be a longest cycle in G and $G_0 = A_0$.*
($s2$) *Assuming that the sequences $(A_0, \ldots, A_{i-1})$ and $(G_0, \ldots, G_{i-1})$ are already defined, let A_i be a longest path in G with the property*
$(\mathbf{E_i}) : e(A_i) \geq 2$ *and* $G_{i-1} \cap A_i = \mathrm{End}(A_i)$.
Put $G_i = G_{i-1} \cup A_i$.
($s3$) *Let r be the minimum positive integer such that G has no path A_{r+1} with property $(\mathbf{E_{r+1}})$.*

If G is a 2-connected graph, then we put $F(G) = G_r, A(G) = A_r$, and $\mathcal{A}(G) = \mathcal{A}$ in Procedure $\mathcal{E}$. Clearly, every 2-connected graph has a frame.

It is easy to see the following.

4.10. *Let G be a 2-connected graph. Then $F(G)$ is a frame of G and $F(G)$ is a Hamiltonian cycle of G if and only if $r = 0$.*

We will also need the following modification of Procedure $\mathcal{E}$. Recall that an edge $e = xy$ is a *chord* of a cycle C in G if $e \notin E(C)$ and $x, y \in V(C)$.

4.11 Procedure $\mathcal{E}'$. *Let G be a 2-connected graph and $e \in E(G)$. Let Procedure $\mathcal{E}'$ be obtained from Procedure $\mathcal{E}$ by replacing the first step*

($s1$) *Let $A'_0 = G'_0$ be a longest cycle in G.*
by
($s'1$) *Let A'_0 be a longest cycle among all cycles C in G such that edge e is either in C or is a chord of C and let $G'_0 = A'_0$.*

Since G is 2-connected, G has a cycle containing e. Therefore a cycle A'_0 exists.

It turns out [6] that applied to a 2-connected claw-free graph G, Procedure $\mathcal{E}$ provides a frame $F(G) = G_r$ of G and its ear-assembly with very useful properties.

Recall that a *claw-free frame* of G is a minimal 2-connected claw-free spanning subgraph of G. Clearly, every 2-connected claw-free graph has a claw-free frame.

4.12. [6] *Let G be a 2-connected claw-free graph and G not a cycle. Let $F = F(G)$ and $A = A(G)$ from Procedure $\mathcal{E}$. Then*

($f1$) *F is a frame of G with the maximum vertex degree three,*
($f2$) *G has a unique matching M such that $F_c = F \cup M$ is a claw-free frame of G with the maximum vertex degree three, and so every vertex of degree three belongs to a unique triangle in F_c and every vertex of every triangle in F_c has degree three in F_c,*
($f3$) *$F_c - A$ is a claw-free frame of $G - A$ (and so $G - A$ is 2-connected and claw-free) (put $F_c = F_c(G)$),*
($f4$) *if P is a maximum Λ-packing of A, then $G - P$ is a 2-connected claw-free graph, and*
($f5$) *the above claims are also true for Procedure $\mathcal{E}'$.*

Obviously, even the first steps in Procedures $\mathcal{E}$ and $\mathcal{E}'$ are NP-hard. However, there are modifications of these procedures which find $F(G), F_c(G), A(G)$, and $\mathcal{A}(G)$ with properties in **4.12** in polynomial time for every 2-connected claw-free graph G.

4.13. *Suppose that G is a 2-connected claw-free graph.*

($a1$) *If $v(G) \equiv 0 \bmod 3$ and $e \in E(G)$, then $G - e$ has a Λ-factor.*
($a2$) *If $v(G) \equiv k \bmod 3$, where $k \in \{1, 2\}$, then G has a k-vertex path P_k and a $(k+3)$-vertex path P_{k+3} such that $G - P_k$ and $G - P_{k+3}$ have Λ-factors, and so G has a $\{\Lambda, P_k\}$-factor and a $\{\Lambda, P_{k+3}\}$-factor.*

Proof (Uses **4.12**). We prove ($a1$) by induction on $v(G)$. If G is a cycle, then our claim is obviously true. Otherwise, consider $A = A(G)$ provided by Procedure $\mathcal{E}'$. Then $e \notin E(A)$. Let P be a maximum Λ-packing in $A(G)$. Since $e(A) \geq 2$, clearly $v(P) = 3s$ for some $s \geq 1$. Therefore $v(G - P) \equiv 0 \bmod 3$ and by **4.12**, $G - P$ is also a 2-connected claw-free graph. Since $e \notin E(A)$, clearly $e \notin E(P)$. Obviously, $v(G - P) < v(G)$. By the induction hypothesis, $G - P$ has a Λ-factor Q avoiding e. Then $P \cup Q$ is a Λ-factor of G avoiding edge e. The proof of ($a2$) is similar to that above. □

It turns out that an analogue of **4.13**($a2$) when a 4-vertex path is replaced by a claw is also true provided a graph has a claw.

4.14. *Suppose that G is a 2-connected claw-free graph, $v(G) \equiv 1 \bmod 3$, and G is not a cycle. Then G has at least two claws Y such that $G - Y$ has a Λ-factor.*

Proof (Uses **3.14** and **4.12**). Let $H = F_c(G)$ and $A = A(G)$ (see **4.12**), and so $A \subset H$.

Suppose first that A is a cycle. Then A is a Hamiltonian cycle of H. Since G is not a cycle, we have by **4.12**($f2$): $E(H) \setminus E(A) \neq \emptyset$ and every edge in $E(H) \setminus E(A)$ belongs to a unique triangle in H. Then H has at least two claws and $H - Y$ has a Λ-factor for every claw in H. Since H is a spanning subgraph of G, every Λ-factor of $H - Y$ is also a Λ-factor of $G - Y$.

Now suppose that A is a path. Let x and y be the end-vertices of A. Since H is a spanning subgraph of G, it suffices to prove the following

Claim. *For every vertex $v \in \{x, y\}$ there exist two claws Y_v an Z_v in H such that either $G - Y_v$ or $G - Z_v$ has a Λ-factor.*

Proof of Claim. By **4.12**, every end-vertex of A has degree three and belongs to a unique triangle of H. By symmetry, we can assume that $v = y$. Let Δ be the triangle containing y and $V(\Delta) = \{s, y, z\}$. Let Y and Z be the claws in H centered at y and z, respectively. Then Y contains the end-edge yy' of L and Z contains the edge zz', where $z' \notin V(\Delta)$. Let $H' = H - A$. By **4.12**($f3$), H' is a 2-connected claw-free spanning subgraph of $G - A$.

(**p1**) Suppose that $v(L) \equiv 0 \mod 3$. Let $R_0 = (A - x) \cup Y$ and $H_0 = H - R_0$. Then $A' = R_0 - Y = A - \{x, y, y'\}$ is the subpath of R_0. Since $v(L) \equiv 0 \mod 3$, also $v(A') \equiv 0 \mod 3$. Then A' has a unique Λ-factor P. Let $H'' = H - (A - x)$. Obviously, $H_0 = H'' - \{s, z\}$. Since H' is 2-connected, H'' is also 2-connected. Since s and z have degree three in H, they both have degree two in H''. Therefore $H_0 = H'' - \{s, z\}$ has exactly two end-blocks. Since H is claw-free, H_0 is also claw-free. By **3.14**, H_0 has a Λ-factor Q. Then $P \cup Q$ is a Λ-factor of $H_0 \cup (A - Y) = H - Y$.

(**p2**) Suppose that $v(A) \equiv 1 \mod 3$. Let $R_0 = L \cup Z$ and $H_0 = H - R_0$. Then $L' = R_0 - Z = L - y$ is the subpath of R_0. Since $v(A) \equiv 1 \mod 3$, clearly $v(A') \equiv 0 \mod 3$. Then A' has a unique Λ-factor P. Since s and z have degree three in H, they both have degree two in H'. Now since H' is 2-connected, $H_0 = H' - \{s, z, z'\}$ has exactly two end-blocks. Since H is claw-free, H_0 is also claw-free. By **3.14**, H_0 has a Λ-factor Q. Then $P \cup Q$ is a Λ-factor of $H_0 \cup (A - Z) = H - Z$.

(**p3**) Suppose that $v(A) \equiv 2 \mod 3$. Let $R_0 = L \cup \Delta$ and $H_0 = H - R_0$. Then $A' = R_0 - Y = A - \{y, y'\}$ is the subpath of R_0. Since $v(A) \equiv 2 \mod 3$, clearly $v(A') \equiv 0 \mod 3$. Then A' has a unique Λ-factor P. Since s and z have degree three in H, they both have degree two in H'. Now since H' is 2-connected, $H_0 = H' - \{s, z\}$ has exactly two end-blocks. Since H is claw-free, H_0 is also claw-free. By **3.14**, H_0 has a Λ-factor Q. Then $P \cup Q$ is a Λ-factor of $H_0 \cup (A - Y) = H - Y$. □

By **4.13**($a2$), every 2-connected claw-free graph with $v(G) \equiv 2 \mod 3$ has an edge xy such that $G - \{x, y\}$ has a Λ-factor. It turns that the following stronger result is true.

4.15. *Suppose that G is a 2-connected claw-free graph and $v(G) \equiv 2 \mod 3$. Then for every vertex x in G there exist at least two edges xb_1 and xb_2 in G such that each $G - \{x, b_i\}$ is connected and has a Λ-factor.*

Proof (Uses **4.1**, **4.5**, and **4.6**). Since G is 2-connected, there exists an edge xy in G such that $G - \{x, y\}$ is connected. Suppose that $G - \{x, y\}$ has no Λ-factor. Then by **4.1**, $G - \{x, y\}$ has at least three end-blocks B_i, $i \in \{1, \ldots, k\}$, $k \geq 3$. Let b'_i be the boundary vertex of block B_i in $G - \{x, y\}$. Let V_i be the set of vertices in $\{x, y\}$ adjacent to a vertex in $B_i - b'_i$ and $\mathcal{B}_v$ be the set of the end-blocks in $G - \{x, y\}$ having an inner vertex adjacent to $v \in \{x, y\}$. Since G is 2-connected, each $|V_i| \geq 1$. Since G is claw-free, each $|\mathcal{B}_v| \leq 2$. Since $k \geq 3$, $|\mathcal{B}_z| = 2$ for some $z \in \{x, y\}$, say $\mathcal{B}_z = \{B_1, B_2\}$. Let $zb_i \in E(G)$, where $b_i \in V(B_i - b'_i)$ for $i \in \{1, 2\}$. Since G is claw-free, $\{x, y, b_1, b_2\}$ does not induce a claw in G. Therefore $sb_j \in E(G)$ for $\{s, z\} = \{x, y\}$ and some $j \in \{1, 2\}$, say, for $j = 2$. Then $\mathcal{B}_s = \{B_i : i \geq 2\}$. Since $\mathcal{B}_s = 2$, we have: $k = 3$ and $\mathcal{B}_s = \{B_2, B_3\}$. Now we can assume that $z = x$ and $s = y$. Obviously, $G - \{x, b_1\}$ is claw-free, connected, and has exactly two end-blocks. By **4.1**, $G - \{x, b_1\}$ has a Λ-factor.

We want to prove that $G - \{x, b_2\}$ also has a Λ-factor. Let C_i be the end-chain of $G - \{x, y\}$ containing B_i, $i \in \{1, 2, 3\}$. Graph $G - \{x, y\}$ is claw-free and has exactly three end-blocks. Since $G - \{x, y\}$ has no Λ-factor, by **4.6**($a3$), a graph G_k obtained from G by Reduction **4.5** has exactly three end-chains and each of them has one edge. Therefore each $v(C_i) \equiv 2 \mod 3$. Graph $G - \{x, b_2\}$ is claw-free, connected, and has a leaf y and two or three end-chains. If $G - \{x, b_2\}$ has two end-chains, then by **4.1**, $G - \{x, b_2\}$ has a Λ-factor. So we assume that $G - \{x, b_2\}$ has three end-chains C'_1, C'_1, and C'_1, where $b'_1 \in V(C'_1)$ and $b'_3 \in V(C'_3)$. Then $C'_1 = C_1$, $C'_2 = C_2 - b'_2$, and C'_3 is obtained from C_3 by adding edge yb_3. Since $v(C_2) \equiv 2 \mod 3$, clearly $v(C'_2) \equiv 1 \mod 3$. Then a graph G_k obtained from G by Reduction **4.5** has two end-blocks. Therefore by **4.6**($a3$), $G - \{x, b_2\}$ has a Λ-factor. □

From **4.15** we have for 3-connected claw-free graphs the following stronger result (with a simpler proof).

4.16. *Suppose that G is a 3-connected claw-free graph and $v(G) \equiv 2 \mod 3$. Then $G - \{x, y\}$ has a Λ-factor for every edge xy in G.*

Proof (Uses **4.1**). Let $G' = G - \{x, y\}$. Since G is 3-connected, G' is connected. By **4.1**, it suffices to prove that G' has at most two end-blocks. Suppose, on the contrary, that G' has at least three end-blocks. Let B_i, $i \in \{1, 2, 3\}$, be some three blocks of G'. Since G is 3-connected, for every block B_i and every vertex $v \in \{x, y\}$ there is an edge vb_i, where b_i is an inner vertex of B_i. Then $\{v, b_1, b_2, b_3\}$ induces a claw in G, a contradiction. □

As we have seen in the proof of **4.15**, the claim of **4.16** is not true for claw-free graphs of connectivity two.

4.17. *Suppose that G is a 3-connected claw-free graph and $v(G) \equiv 0 \bmod 3$. Then for every edge xy in G there exist at least two 3-vertex paths L_1 and L_2 in G centered at y, containing xy, and such that each $G - L_i$ is connected and has a Λ-factor.*

Proof (Uses **4.1**). We need the following simple fact.

Claim. *Let G be a 3-connected graph. Then for every vertex x and every edge xy in G there exist two 3-vertex paths Λ_1 and Λ_2 in G centered at y, containing xy, and such that each $G - \Lambda_i$ is connected.*

By the above Claim, G has a 3-vertex path $L = xyz$ such that $G - L$ is connected. If every such 3-vertex path belongs to a Λ-factor of G, then we are done. Therefore we assume that $G - L$ is connected but has no Λ-factor. Obviously, $G - L$ is claw-free. Therefore by **4.1**, $G - L$ has at least three end-blocks B_i, $i \in \{1, \dots, k\}$, $k \geq 3$. Let b_i' be the boundary vertex of B_i. Let V_i be the set of vertices in L adjacent to inner vertices $G - L$ having an inner vertex adjacent to v in $V(L)$. Since G is 3-connected, each $|V_i| \geq 2$. Since G is claw-free, each $|\mathcal{B}_v| \leq 2$. It follows that $k = 3$, each $|V_i| = 2$, each $|\mathcal{B}_v| = 2$, as well as all V_i's are different and all $\mathcal{B}_v$'s are different. Let $s^1 = z$, $s^2 = x$, $s^3 = y$, and $S = \{s^1, s^2, s^3\}$. We can assume that $V_i = S - s^i$, $i \in \{1, 2, 3\}$. Then for every vertex $s^j \in V_i$ there is a vertex b_i^j in $B_i - b_i'$ adjacent to s^j, where $\{b_i^j : s^j \in V_i\}$ has exactly one vertex if and only if $B_i - b_i'$ has exactly one vertex. Let $L_i = s^2 s^3 b_i$, where $b_i = b_i^3$. By **4.1**, it suffices to show that each $G - L_i$ is connected and has at most two end-blocks.

Let $i = 1$. If $B_1 - b_1$ is 2-connected, then $B_1 - b_1$ and $G - L_1 - (B_1 - b_1')$ are the two end-blocks of $G - L_1$ and we are done. If $B_1 - b_1$ is empty, then $G - L_1$ is 2-connected. So we assume that $B_1 - b_1$ is not empty and not 2-connected. Then $B_1 - b_1$ is connected and has exactly two end-blocks, say C_1 and C_2. Let c_i' be the boundary vertex of C_i in $B_1 - b_1$. Since G is 3-connected, each $C_i - c_i'$ has a vertex adjacent to $\{s^2, s^3\}$. We can assume that a vertex c_1 in $C_1 - c_1'$ is adjacent to s^2. If there exists a vertex c_2 in $C_2 - c_2'$ adjacent to s^2, then $\{s^2, b_3^2, c_1, c_2\}$ induces a claw in G, a contradiction. So we assume that no vertex in $C_2 - c_2'$ is adjacent to s^2. Then there is a vertex c_2 in $C_2 - c_2'$ adjacent to s^3. Then $\{s^2, s^3, b_2^3, c_2\}$ induces a claw in G, a contradiction.

Finally, let $i = 2$. If $B_2 - b_2$ is 2-connected, then B_1 and $G - L_2 - (B_1 - b_1')$ are the two end-blocks of $G - L_2$ and we are done. If $B_1 - b_1$ is empty, then $G - L_2$ has two end-blocks, namely, B_1 and the subgraph of G induced by $B_3 \cup s^1$. So we assume that $B_2 - b_2$ is not empty and not 2-connected. Then $B_2 - b_2$ is connected and has exactly two end-blocks, say D_1 and D_2. Let d_i' be the boundary vertex of D_i in $B_2 - b_2$. Since G is 3-connected, each $D_i - d_i'$ has a vertex adjacent to $\{s^1, s^3\}$. We can assume that a vertex d_1 in $D_1 - d_1'$ is adjacent to s^3. If there exists a vertex d_2 in $D_2 - d_2'$ adjacent to s^3, then $\{s^3, d_1, d_2, b_1^3\}$ induces a claw in G, a contradiction. So suppose that no vertex in $D_2 - d_2'$ is adjacent to s^3. Then there is a vertex d_2 in $D_2 - d_2'$ adjacent to s^1. Then $\{s^1, s^3, b_3^1, d_2\}$ induces a claw in G, a contradiction. □

From the proof of **4.17** we have, in particular:

4.18. *Suppose that G is a 3-connected claw-free graph and $v(G) \equiv 0 \bmod 3$. If L is a 3-vertex path and the center vertex of L has degree 3 in G, then $G - L$ is connected and has a Λ-factor in G.*

From **4.18** and the proof of **4.17** we have:

4.19. *Suppose that G is a cubic 3-connected claw-free graph or 4-connected claw-free graph with $v(G) \equiv 0 \bmod 3$. Then $G - L$ is connected and has a Λ-factor for every 3-vertex path L in G.*

The claim of **4.19** may not be true for a claw-free graph of connectivity 3 if they are not cubic. Recall that a *net* is a graph obtained from a claw by replacing its vertex of degree 3 by a triangle. Let N be a net with the three leaves v_1, v_2, and v_3, T a triangle with $V(T) = \{t_1, t_2, t_3\}$, and let N and T be disjoint. Let $H = N \cup T \cup \{v_i t_j : i, j \in \{1, 2, 3\}, i \neq j\}$. Then H is a 3-connected claw-free graph, $v(H) = 9$, each $d(t_i, H) = 4$, $d(x, H) = 3$ for every $x \in V(H - T)$, and $H - T = N$ has no Λ-factor. If L is a 3-vertex path in T, then $H - L = H - T$, and so $H - L$ has no Λ-factor. There are infinitely many pairs (G, L) such that G is a 3-connected, claw-free, and non-cubic graph, $v(G) \equiv 0 \bmod 3$, L is a 3-vertex path in G, and $G - L$ has no Λ-factor. By **4.8**, such a pair can be obtained from the above pair (H, L) by replacing N by any graph A with exactly three leaves satisfying the assumptions of **4.8**.

From **4.17** we have, in particular:

4.20. *Suppose that G is a 3-connected claw-free graph and $v(G) \equiv 0 \bmod 3$. Then for every edge e of G there exists a Λ-factor in G containing e.*

The following examples show that assumption "*G is a 3-connected graph*" in **4.20** is essential. Let R be the graph obtained from two disjoint cycles A and B by adding a new vertex z, and the set of four new edges $\{a_i z, b_i z : i \in \{1, 2\}\}$, where $a = a_1 a_2 \in E(A)$ and $b = b_1 b_2 \in E(B)$. It is easy to see that R is a claw-free graph of connectivity one. Furthermore, if $v(A) \equiv 1 \bmod 3$ and $v(B) \equiv 1 \bmod 3$, then $v(R) \equiv 0 \bmod 3$ and R has no Λ-factor containing edge $e \in \{a, b\}$. Similarly, let Q be the graph obtained from two disjoint cycles A and B by adding two new vertices z_1 and z_2, a new edge $e = z_1 z_2$, and the set of eight new edges $\{a_i z_j, b_i z_j : i, j \in \{1, 2\}\}$, where $a_1 a_2 \in E(A)$ and $b_1 b_2 \in E(B)$. It is easy to see that Q is a claw-free graph of connectivity two. Furthermore, if $v(A) \equiv 2 \mod 3$ and $v(B) \equiv 2 \mod 3$, then $v(Q) \equiv 0 \mod 3$ and Q has no Λ-factor containing edge e.

Let F be a graph, $x \in V(F)$, and $X = \{x_1, x_2, x_3\}$ be the set of vertices in F adjacent to x. Let T be a triangle, $V(T) = \{t_1, t_2, t_3\}$, and $V(F) \cap V(T) = \emptyset$. Let $G = (F - x) \cup T \cup \{x_i t_i : i \in \{1, 2, 3\}\}$. We say that *G is obtained from F by replacing a vertex x by a triangle*.

Given a cubic graph F with possible parallel edges, let $F^\triangle$ denote the graph obtained from F by replacing each vertex of F by a triangle. Clearly, $F^\triangle$ is cubic and claw-free, every vertex belongs to exactly one triangle, every edge belongs to at most one triangle in $F^\triangle$, and $v(F^\triangle) \equiv 0 \mod 3$, and so $F^\triangle$ is a Δ-*graph*. Obviously, $F^\triangle$ is k-connected if and only if F is k-connected, $k \in \{1, 2, 3\}$.

4.21. *Let G be a 2-connected Δ-graph. Let L be a 3-vertex path in G. Then*

(a) *$G - L$ has a Λ-factor.*
Moreover,
($a1$) *if L induces a triangle in G, then G has a Λ-factor R containing L and such that each component of R induces a triangle*
($a2$) *if L does not induce a triangle in G, then G has a Λ-factor R containing L and such that no component of R induces a triangle, and*
($a3$) *if L does not induce a triangle in G, then G has a Λ-factor containing L and a component that induces a triangle.*

Proof. Since G is a 2-connected Δ-graph, G can be obtained from a 2-connected cubic graph G' (with possible parallel edges) by replacing each vertex of G' by a triangle. Obviously, there is a natural bijection $\alpha : E(G') \to E'$. Let E' be the set of edges in G that belong to no triangle. Let $L = xzz_1$. Since each vertex of G belongs to exactly one triangle, we can assume that xz belongs to a triangle $T = xzs$.

(**p1**) Suppose that L induces a triangle in G, and so $s = z_1$. Obviously the union of all triangles in G contains a Λ-factor, say P, of G and $L \subset P$. Therefore claim ($a1$) is true.

(**p2**) Now suppose that L does not induce a triangle in G, and so $s \neq z_1$. Let $\bar{s} = ss_1$ and $\bar{z} = zz_1$ be the edges of G not belonging to T, and therefore belonging to no triangles in G. Hence $\bar{s} = \alpha(\bar{s}')$ and $\bar{z} = \alpha(\bar{z}')$, where $\bar{s}' = s's'_1$ and $\bar{z}' = z'z'_1$ are edges in G', and $s' = z'$. Since every vertex in G belongs to exactly one triangle, clearly $s_1 \neq z_1$.

(**p2.1**) We prove ($a2$). By using Tutte's criterion for a graph to have a perfect matching (see [20]), it is easy to prove the following

Claim. *If A is a cubic 2-connected graph, then for every 3-vertex path J of A there exists a 2-factor of A containing J.*

By the above Claim, G' has a 2-factor F' containing 3-vertex path $S' = s'_1 s' z'_1$. Let C' be the (cycle) component of F' containing S'. If Q' is a (cycle) component of F', then let Q be the subgraph of G, induced by the edge subset $\{\alpha(e) : e \in E(Q')\} \cup \{E(\Delta_v) : v \in V(Q')\}$. Obviously $v(Q) \equiv 0 \mod 3$ and Q has a (unique) Hamiltonian cycle $H(Q)$. Also the union F of all Q's is a spanning subgraph of G and each Q is a component of F. Moreover, if C is the component in F, corresponding to C', then $L \subset H(C)$. Therefore each $H(Q)$ has a Λ-factor $P(Q)$, such that no component of $P(Q)$ induces a triangle, and $H(C)$ has a (unique) Λ-factor $P(C)$, such that $L \subset P(C)$ and no component of $P(C)$ induces a triangle. The union of all these Λ-factors is a Λ-factor P of G containing L and such that no component of P induces a triangle. Therefore ($a2$) holds.

(**p2.2**) Finally, we prove ($a3$). Since G' is 2-connected and cubic, there is a cycle C' in G' such that $V(C') \neq V(G')$ and C' contains $S' = s'_1 s' z'_1$. Let, as above, C be the subgraph of G, induced by the edge subset $\{\alpha(e) : e \in E(C')\} \cup \{E(\Delta_v) : v \in V(C')\}$. Obviously, $v(C) \equiv 0 \mod 3$, C has a (unique) Hamiltonian cycle H, and $L \subset H$. Therefore H has a (unique) Λ-factor $P(C)$ containing L. Since $V(C') \neq V(G')$, we have $V(G' - C') \neq \emptyset$. Therefore $G - C$ has a triangle. Moreover, every vertex v in $G - C$ belongs to a unique triangle Δ_v, and therefore as in (**p1**), $G - C$ has a Λ-factor Q whose every component induces a triangle in $G - C$. Then $P(C) \cup Q$ is a required a Λ-factor in G. □

Obviously, **4.21**(a) also follows from **4.18**.

Theorem **4.21** is not true for a cubic 2-connected claw-free graph F with an edge xy belonging to two triangles T_i with $V(T_i) = \{x, y, z_i\}$, $i \in \{1, 2\}$, because $L = z_1 x z_2$ is a 3-vertex path in F and y is an isolated vertex in $F - L$.

Now we can give a polynomial-time characterization of pairs (G, E) such that G is a 2-connected Δ-graph, $E \subset E(G)$, $|E| = 3$, and $G - \dot{E}$ has no Λ-factor. Recall that if $E \subseteq E(G)$, then $\dot{E}$ denotes the subgraph of G induced by E.

4.22. *Suppose that G is a 2-connected Δ-graph. Let $E \subset E(G)$ and $|E| = 3$. Then the following are equivalent:*

(g) *$G - \dot{E}$ has no Λ-factor and*
(e) *$\dot{E}$ satisfies one of the following conditions:*
 ($e1$) *$\dot{E}$ is a claw,*
 ($e2$) *$\dot{E}$ is a triangle,*
 ($e3$) *$\dot{E}$ has exactly two components, the 2-edge component $\dot{E}_2$ belongs to a triangle in G, the 1-edge component $\dot{E}_1$ belongs to no triangle in G, and $G - \dot{E}$ is not connected, and*
 ($e4$) *$\dot{E}$ has exactly two components, the 2-edge component $\dot{E}_2$ belongs to a triangle T and the 1-edge component $\dot{E}_1$ belongs to a triangle D in G, $\dot{E}_1$ and $\dot{E}_2$ belong to different component of $G - \{d, t\}$, where d and t are the edges in $G - E(D) - E(T)$ incident to the single vertex of $D - \dot{E}_1$ and to the isolated vertex of $T - E_2$, respectively.*

Proof (Uses **4.1**, **4.8** and **4.21**(a)). Let $E = \{a, b, c\}$, where $c = c_1c_2$.

(**p1**) We prove $(e) \Rightarrow (g)$.

Suppose that $\dot{E}$ satisfies (e1), i.e. $\dot{E}$ is a claw. Then $G - E$ has an isolated vertex and therefore has no Λ-factor. Suppose that $\dot{E}$ satisfies (e2), i.e. $\dot{E}$ is a triangle. Then by **4.8**, $G - E$ has no Λ-factor.

Now we assume (as in (e3) and (e4)) that $\dot{E}$ has exactly two components $\dot{E}_2$ and $\dot{E}_1$, the 2-edge component $\dot{E}_2$ belongs to a triangle T in G, $E_2 = \{a, b\}$, and $E_1 = \{c\}$, $t = t_1t_2$ is an edge in $G - E$, where t_1 is an isolated vertex in $T - E_2$, and so t_1 is a (unique) leaf in $G - E$. Let u be the edge in T distinct from a and b.

Suppose that $\dot{E}$ satisfies (e3), and so $\dot{E}_1$ belongs to no triangle in G and $G-E$ is not connected. Obviously, $G-E$ has exactly two components. Let S the component in $G-E$ containing edge t. Then edge u is not in S. Therefore every vertex in S distinct from the leaf t_1 belongs to exactly one triangle. Hence $v(S) \equiv 1 \bmod 3$ implying that $G - E$ has no Λ-factor.

Finally, suppose that $\dot{E}$ satisfies (e4), and so edge $c = c_1c_2$ belongs to a triangle D in G, $G - \{d, t\}$ is not connected, and E_1, E_2 belong to different components of $G - \{d, t\}$, where d is the edge in $G - E(D)$ incident to the single vertex in $D - \{c_1, c_2\}$. Suppose, on the contrary, that $G - E$ has a Λ-factor P. Since G is 2-connected and claw-free, $G - \{d, t\}$ has exactly two components. Therefore $G - \{a, b\} - (E(D) - c)$ has also two components. Let C' be the component of $G - \{a, b\} - (E(D) - c)$ containing c. Since G is a Δ-graph, $G = F^{\Delta}$ for some cubic 2-connected graph F (with possible parallel edges). Let d' and t' be the edges in F corresponding to edges d and t of G, respectively. Since F is 2-connected, $F - \{d', t'\}$ has at most two components. Since $G - \{d, t\}$ is not connected, $F - \{d', t'\}$ has exactly two components. It follows that $H = G - \{a, b\} - E(D) = G - E - E(D)$ has also exactly two components. Therefore $C = C' - c$ is connected, and so C is a component of H containing the end-vertices c_1 and c_2 of edge c.

Let C_u and C_t be the components of H containing u and t, respectively. Then $C_u \neq C_t$. Now $C_2 = C_u \cup T$ is the component in $G - \{d, t\}$ containing E_2. By (e4), $c \not\in E(C_2)$. Therefore c is an edge of $C_1 = C_t \cup D$. Thus $C_t = C$. Clearly, C has exactly three leaves c_1, c_2, and t_1 (the leaf incident to t) and every other vertex of C belongs to a unique triangle in C, and so $v(C) \equiv 0 \bmod 3$. By **4.8**, C has no Λ-factor. Therefore P has a 3-vertex path L which contains at least one edge in $D - c$. Since t is a dangling edge in $G - E$, clearly P also has a 3-vertex path L_t containing t and $L_t \subset C$. Therefore P has to contain a Λ-factor of $C - L - L_t$. However, $v(C - L - L_t) \not\equiv 0 \bmod 3$, a contradiction.

(**p2**) Finally, we prove $(g) \Rightarrow (e)$. Namely, we assume that $\dot{E}$ does not satisfy (e) and we want to show that in this case $G - E$ has a Λ-factor.

Let $X, Y \subset E(G)$ be such that X meets no triangle in G, each edge in Y belongs to a triangle in G, and no triangle in G has more than one edge from Y, and so $X \cap Y = \emptyset$. We will use the following simple observation.

Claim. *$G - X - Y$ has a Λ-factor P such that every component of P induces a triangle in G and if an edge y from Y is in a triangle T, then $T - y$ is a component of P.*

By the above Claim, we can assume that the two edges of E_2 belong to the same triangle T.

Suppose that $\dot{E}$ is connected. Since $\dot{E}$ does not satisfy (e), $\dot{E}$ is not a claw and not a triangle. Then $\dot{E}$ is a 3-edge path and $u, t \not\in E$. Let V be a 3-vertex path in G containing u and avoiding E. Then $G - V$ has no edges from E, and so $G - V = G - E - V$. By **4.21**(a), $G - V$ has a Λ-factor.

Finally, suppose that $\dot{E}$ is not connected, and so $\dot{E}$ has exactly two components $\dot{E}_1$ and $\dot{E}_2$. As in (**p1**), let $E_2 = \{a, b\}$ and $E_1 = \{c\}$, and let u be the edge of T distinct from a and b.

(**p2.1**) Suppose that c belongs to no triangle in G. Since $\dot{E}$ does not satisfy (e) (namely, (e3)), $G - E$ is connected. Clearly, $G - E$ is claw-free. Also $G - E$ has exactly two end-blocks and the block of one edge t is one of them. By **4.1**, $G - E$ has a Λ-factor.

(**p2.2**) Finally, suppose that c belongs to a triangle D in G. Then $D \neq T$. Since $\dot{E}$ does not satisfy (e) (namely, (e4)), $\dot{E}_1$ and $\dot{E}_2$ belong to the same component of $G - \{d, t\}$. Let $V(D) = \{c_1, c_2, d_1\}$ and as above $c = c_1c_2$. Let $d = d_1d_2$ and $t = t_1t_2$, where $t_1 \in V(T)$, and so t_1 is an isolated vertex in $T - \{a, b\}$.

Let $G' = G - \{c, d\}$. Then G' and $G' - E_2$ are claw-free and $v(B) \equiv 0 \bmod 3$ for every block B in G'. Obviously, $G' - E_2 = G - E - d$. Since G is 2-connected, $G - c$ is also 2-connected. Therefore $eb(G') \leq 2$ and if $eb(G') = 2$, then the end-vertices d_1 and d_2 of edge d belong to different end-blocks B_1 and B_2 of G', respectively. If $eb(G') = 1$, then G' is 2-connected. Therefore $G' - E_2$ has at most two end-blocks. By **4.1**, $G' - E_2 = G - E - d$ has a Λ-factor P, which is also a Λ-factor of $G - E$. So we assume that $eb(G') = 2$. Since E_1 and E_2 belong to the same component of $G - \{d, t\}$, clearly d_1, and E_2 belong to the same component of $G' - t$.

Suppose that t is a cut-edge of G'. Then $G' - t$ has exactly two components C_1 and C_2 containing $\{d_1, t_1\}$ and d_2, respectively, and each $v(C_i) \equiv 0 \bmod 3$. Let L be a 3-vertex path $c_1d_1d_2$ in $G - c$. Then $C'_1 = C_1 - \{c_1, d_1, t_1\}$ and $C'_2 = (C_2 - d_2) \cup \{t, t_1\}$ are the two components of $G - E - L$ containing d_1 and d_2, respectively. Also each, $v(C'_i) \equiv 0 \bmod 3$ and C'_i is claw-free and has exactly two end-blocks. By **3.12** and **3.14**, C'_i has a Λ-factor P_i, $i \in \{1, 2\}$. Then $L \cup P_1 \cup P_2$ is a Λ-factor of $G - E$.

Finally, suppose that t is not a cut-edge of G'. Then $\{a, b, t\}$ belongs to a 2-connected block R of G'. Then $R - E_2$ has at most two end-blocks. By **3.12** and **3.14**, $R - E_2$ has a Λ-factor Q. Let $\mathcal{B}$ denote the set of all 2-connected blocks of G' distinct from R. Since $v(B) \equiv 0 \bmod 3$ for every $B \in \mathcal{B}$, each B in $\mathcal{B}$ has a Λ-factor $P(B)$. Then $\{P(B) : B \in \mathcal{B}\} \cup Q$ is a Λ-factor of G', which is also a Λ-factor of $G - E$. □

From **4.22** we have, in particular:

4.23. *Suppose that G is a Δ-graph. Let $E \subset E(G)$ and $|E| = 2$. Then $G - E$ has a Λ-factor.*

From **4.22** we also have:

4.24. *Suppose that G is a 3-connected claw-free graph. Let $E \subset E(G)$ and $|E| = 3$. Then $G - E$ has a Λ-factor if and only if $\dot{E}$ is not a claw and not a triangle.*

It turns out that condition "*G is claw-free*" in **4.22** and in **4.24** is essential. Namely, we have a construction showing that for every 3-edge graph Y with no isolated vertices there are infinitely many pairs (G, E) such that G is a cubic 3-connected graph, $v(G) \equiv 0 \bmod 3$, $E \subset E(G)$, the subgraph $\dot{E}$ induced by E in G is isomorphic to Y (and so $|E| = 3$), and $G - E$ has no Λ-factor.

4.25. *Suppose that G is a 2-connected claw-free graph and $v(G) \equiv 1 \bmod 3$. Then $G - x$ has a Λ-factor for every vertex x in G.*

Proof (Uses **4.1**). Let $x \in V(G)$. Since $v(G) \equiv 1 \bmod 3$, clearly $v(G - x) \equiv 0 \bmod 3$. Since G is 2-connected, $G - x$ is connected. Since G is claw-free, $G - x$ is claw-free and has at most two end-blocks. By **4.1**, $G - x$ has a Λ-factor. □

If on step $(s1)$ of Procedure $\mathcal{E}$ we find a longest cycle in G containing a given vertex x, then this modification of Procedure $\mathcal{E}$ can also be used to prove **4.25**.

Moreover, the following strengthening of **4.25** holds for 3-connected claw-free graphs.

4.26. *Suppose that G is a 3-connected claw-free graph and $v(G) \equiv 1 \bmod 3$. Then $G - \{x, e\}$ has a Λ-factor for every vertex x and every edge e in G.*

Proof (Uses **3.12** and **4.13**). Since G is 3-connected, $G - x$ is a 2-connected claw-free graph. Since $v(G) \equiv 1 \bmod 3$, we have $v(G - x) \equiv 0 \bmod 3$. By **3.12**, $G - x$ has a Λ-factor P. If $e \notin E(G - x)$, then P is a Λ-factor of $G - \{x, e\}$. If $e \in E(G - x)$, then by **4.13**, $G - \{x, e\}$ has a Λ-factor. □

5. Packings and domination in graphs

Recall that $X \subseteq V(G)$ is called a domination set in graph G, if every vertex in $V(G) \setminus X$ is adjacent to a vertex in X and that the domination number $\gamma(G)$ is the size of a minimum domination set in G. We call a subgraph P in G a *star-packing* if every component of P is isomorphic to $K_{1,s}$ for some integer $s \geq 0$.

Obviously, X is a domination set in G if and only if there exists a star-factor $P = P(X)$ such that $Cmp(P) = \{P_x : x \in X\}$, where $x \in V(P_x)$ and x is a (unique) vertex of degree at least two if $v(P_x) \geq 3$, and so $|Cmp(P)| = |X|$. Thus X is a minimum domination set in G if and only if $P(X)$ is a star-factor in G having the minimum number of components and $\gamma(G) = cmp(P(X))$.

It is easy to show that every connected graph G with no isolated vertices has a star-factor with no isolated vertices, and so $\gamma(G) \leq v(G)/2$.

Clearly, every Λ-packing P is a star-packing in G and P can be extended to a star-factor P' in G. Then $\gamma(G) \leq cmp(P')$. For that reason, results on the maximum Λ-packings in graphs may be useful in the study of some graph domination problems. Here is an example of such correlation.

In [22] B. Reed conjectured that if G is a connected cubic graph, then $\gamma(G) \leq \lceil v(G)/3 \rceil$. It turns out that Reed's conjecture is not true for connected and even for 2-connected cubic graphs [12,18]. Obviously,

$(d1)$ if a graph G has a Λ-factor (and so $v(G) \equiv 0 \bmod 3$), then $\gamma(G) \leq v(G)/3$,
$(d2)$ if $v(G) \equiv 1 \bmod 3$ and $G - x$ has a Λ-factor for some vertex x of G, then $\gamma(G) \leq \lceil v(G)/3 \rceil$, and
$(d3)$ if $v(G) \equiv 2 \bmod 3$ and $G - \{x, y\}$ has a Λ-factor for some edge xy of G, then again $\gamma(G) \leq \lceil v(G)/3 \rceil$.

Now if claim (P) in Problem **3.7** is true, then from **3.8** and **3.10** it follows, in particular, that $(d1)$, $(d2)$, and $(d3)$ above are true, and so Reed's conjecture is true for 3-connected cubic graphs.

The following packing result is also related with Reed's domination conjecture.

5.1. [14] *If G is a cubic Hamiltonian graph with $v(G) \equiv 1 \bmod 3$, then G has a claw Y such that $G - Y$ has a Λ-factor, and so G has $\{\Lambda, Y\}$-factor.*

It follows that if G is a cubic Hamiltonian graph with $v(G) \equiv 1 \bmod 3$, then $\gamma(G) \leq \lfloor v(G)/3 \rfloor$ which is stronger than Reed's conjecture suggests. The following natural question arises:

5.2 Problem. *Is it true that $\gamma(G) \leq \lfloor v(G)/3 \rfloor$ for every cubic 3-connected graph G with $v(G) \equiv 1 \bmod 3$?*

In [12] we gave a construction providing infinitely many cubic cyclically 4-connected graphs G with $v(G) \in \{0, 2\}$ mod 3 for which $\gamma(G) = \lceil v(G)/3 \rceil$, and so Reed's suggested bound is tight even in the class of cyclically 4-connected graphs. From this construction it also follows that the claim similar to **5.1** for graphs G with $v(G) \not\equiv 1 \mod 3$ is not true, namely, the graphs provided by the construction have no $\{\Lambda, Y\}$-factor.

No bipartite counterexamples to Reed's conjecture have been found. We can show that if claim (P) in Problem **3.7** is true for bipartite graphs, then Reed's conjecture is also true for bipartite 3-connected cubic graphs.

Let $\gamma_i(G)$ denote the size of a minimum independent domination set, and so $\gamma_i(G) \geq \gamma(G)$. It is easy to see that if G is a claw-free graph, then $\gamma_i(G) = \gamma(G)$.

From **4.13** and **4.14** we have the following upper bounds on the domination number of claw-free graphs:

5.3. *Let G be a 2-connected claw-free graph. Then $\gamma(G) \leq \lceil v(G)/3 \rceil$ and if, in addition, G is not a cycle and $v(G) \equiv 1 \mod 3$, then $\gamma(G) = \gamma_i(G) \leq \lfloor v(G)/3 \rfloor$.*

6. Further related results and questions

Given a family $\mathcal{F}$ of non-isomorphic graphs, an *edge disjoint $\mathcal{F}$-packing* $\mathcal{Q}$ of G is a set $\{Q_1, \ldots, Q_k\}$ such that each $Q_i \subseteq E(G)$, every two members of $\mathcal{Q}$ are disjoint, and the subgraph $\dot{Q}_i$ induced by Q_i in G is isomorphic to a member of $\mathcal{F}$. Let $E(\mathcal{Q}) = \cup\{E(Q_i) : i \in \{1, \ldots, k\}\}$ and $k(\mathcal{Q}) = k$. An edge disjoint $\mathcal{F}$-packing $\mathcal{Q}$ in G is called an *edge disjoint $\mathcal{F}$-factor* of G if $E(P) = E(G)$. The *edge disjoint $\mathcal{F}$-packing problem* is the problem of finding in G an edge $\mathcal{F}$-packing $\mathcal{Q}$ having the maximum number of edges $|E(\mathcal{Q})|$. If $\mathcal{F}$ consists of one graph F, then an edge disjoint $\mathcal{F}$-packing and an edge disjoint $\mathcal{F}$-factor are called simply an *edge disjoint F-packing* and an *edge disjoint F-factor*, respectively. Accordingly, the *edge disjoint F-packing problem* is the problem of finding in G an edge disjoint F-packing $\mathcal{Q}$ having the maximum number of edges $|E(\mathcal{Q})|$ or, equivalently, the maximum number of parts $k(\mathcal{Q})$. Let $\lambda_e(G)$ denote the number $k(\mathcal{Q})$ of parts in a maximum edge disjoint Λ-packing of G.

A graph D is called the *line graph of* a simple graph G if $V(D) = E(G)$ and $ab \in E(D)$ if and only if edges a and b in G have a common end-vertex. Let $L(G)$ denote the line graph of a graph G. A graph G is called a *line graph* if there exists a graph F such that $G = L(F)$. It is known (and easy to show) that if two non-isomorphic connected graphs A and B are such that $L(A)$ and $L(B)$ are isomorphic, then $\{A, B\} = \{Y, \Delta\}$ and both $L(A)$ and $L(B)$ are triangles, where Y is a claw and Δ is a triangle. Therefore if $H \not\cong \Delta$ and H is a connected line graph, then there is a unique graph F such that $H = L(F)$. A packing P in a graph G is called an *induced packing* in G if P is an induced subgraph of G.

We need the following simple observations. Let $\mathcal{F}$ be a family of non-isomorphic connected graphs and $L(\mathcal{F}) = \{L(F) : F \in \mathcal{F}\}$.

6.1. *Let G be a graph. If P is an $\mathcal{F}$-packing in G, then $L(P)$ is an induced $L(\mathcal{F})$-packing in $L(G)$. If Δ is not in $\mathcal{F}$ and $L(P)$ is an induced $L(\mathcal{F})$-packing in $L(G)$, then P is an $\mathcal{F}$-packing in G. In particular, if P is an $\mathcal{F}$-factor in G, then $L(P)$ is a vertex maximum induced $L(\mathcal{F})$-packing in $L(G)$.*

6.2. *Let G be a graph and $D = L(G)$. Then the following holds.*

(a1) *Let $Q = \{Q_i : i \in \{1, \ldots, k\}\}$ be an edge disjoint $\mathcal{F}$-packing in G. Then $L(Q)$ is an $\mathcal{F}'$-packing in D, where $Cmp(L(Q)) = \{L(Q_i) : i \in \{1, \ldots, k\}\}$ and $\mathcal{F}' = \{L(F) : F \in \mathcal{F}\}$.*
(a2) *Let P be a packing in D and $Cmp(P) = \{P_i : i \in \{1, \ldots, k\}\}$. Then $\{Q_i = V(P_i) : i \in \{1, \ldots, k\}\}$ is an edge disjoint packing in G with P_i being a spanning subgraph of $L(\dot{Q}_i)$, where $\dot{Q}_i$ is the subgraph in G induced by edge subset Q_i.*

These observations allow to deduce various byproducts from the packing results described before and obtain some facts on edge disjoint packings in a graph. Here are some of these results.

From **6.1** we have, in particular:

6.3. *Let G be a graph. Then P is a Λ-packing in G if and only if $L(P)$ is an induced matching in $L(G)$.*

Since the Λ-packing problem is NP-hard even for cubic graphs, we have:

6.4. *The induced matching problem is NP-hard for line graphs of cubic graphs.*

Using a procedure for connected graphs similar to Procedure $\mathcal{E}'$ for 2-connected graphs, it is easy to show the following:

6.5. *Let G be a connected claw-free graph. Then the following holds.*

(a1) *If M is a maximum matching in G, then $e(M) = \lfloor v(G)/2 \rfloor$.*
(a2) *If $v(G) \equiv 1 \mod 2$, then for every edge e in G there exist a maximum matching M in G that avoids e.*

Since $L(G)$ is a claw-free graph, we have from **6.2** and **6.5**:

6.6. *Let G be a connected graph. Then*

$(a1)$ $\lambda_e(G) = \lfloor e(G)/2 \rfloor$ *and*

$(a2)$ *if* $e(G) \equiv 1 \bmod 2$, *then for every 3-vertex path L in G there exists a maximum edge disjoint* Λ*-packing* $\mathcal{Q}$ *such that L is not a member of* $\mathcal{Q}$.

An edge disjoint factor $\mathcal{Q}$ of G is said to be an *edge k-factor* if every member of $\mathcal{Q}$ induces in G a connected graph having k edges.

6.7. *Suppose that G is a graph such that* $L(G)$ *is connected and has at most two end-blocks. If* $e(G) \equiv 0 \bmod 3$, *then G has an edge 3-factor.*

Proof (Uses **4.1** and **6.2**). Since $V(L(G)) = E(G)$, we have: $e(G) \equiv 0 \bmod 3 \Rightarrow v(L(G)) \equiv 0 \bmod 3$. Since $L(G)$ is claw-free, by **4.1**, $L(G)$ has a Λ-factor. Therefore we are done by **6.2**. □

We call a graph G an *edge-chain* if $G - Lv(G) = (\cup\{B_i : i \in \{1, \ldots, k\}\}) \cup \{e_i : i \in \{1, \ldots, k-1\}\}$, where each B_i is an edge 2-connected graph, all B_i' are disjoint, and each e_i is an edge with one end-vertex in B_i and the other end-vertex in B_{i+1}.

It is easy to see the following.

6.8. *If G is an edge-chain, then* $L(G)$ *has at most two end-blocks.*

From **4.13**$(a1)$, **4.20**, **6.7** and **6.8** we have:

6.9. *If G is an edge-chain and* $e(G) \equiv 0 \bmod 3$, *then G has a 3-edge factor. Moreover,*

$(a1)$ *if* $G - Lv(G)$ *is edge 2-connected and* $e(G) \equiv 0 \bmod 3$, *then for every 3-vertex path L in G there exists an edge 3-factor* $\mathcal{Q}$ *with no member containing L,*

$(a2)$ *if G is edge 3-connected* $e(G) \equiv 0 \bmod 3$, *then for every 3-vertex path L in G there exists an edge 3-factor* $\mathcal{Q}$ *with a member containing L.*

In [6] we put forward the following conjecture.

6.10 Conjecture. *Every 3-connected claw-free graph with* $v(G) \equiv 0 \bmod 4$ *has a* Π*-factor.*

By **6.1**, Conjecture **6.10** is equivalent to the following conjecture on induced Λ-packings.

6.11 Conjecture. *If G is a 3-connected claw-free graph with* $v(G) \equiv 0 \bmod 4$ *and P is a maximum induced* Λ*-packing in* $L(G)$, *then* $\lambda(P) = v(G)/4$.

As we mentioned in the introduction, the problem of packing induced 3-vertex paths in a claw-free graph, interesting in itself, is also related to the Hadwiger conjecture.

Let $h(G)$ be the maximum integer r such that G has K_r as a minor. In 1943 Hadwiger conjectured that if a graph G has no proper vertex coloring with $s-1$ colors, then $h(G) \geq s$ (see [2]). Now consider a graph F with $\alpha(F) = 2$, where $\alpha(F)$ in the size of a maximum vertex subset in F with no two adjacent vertices, and so F is claw-free. Then obviously the vertices of F cannot be colored properly with $s-1$ colors, where $s = \lceil v(G)/2 \rceil$. Thus, a natural (open) question is whether $h(F) \geq s$ as the Hadwiger conjecture claims. If P is a Λ-packing in F such that every component (3-vertex path) of P is an induced subgraph in F, then every two components of P are connected by an edge in G. Therefore contracting each component L of P to a new vertex $c(L)$ results in a graph G' having the complete subgraph K with $V(K) = \{c(L) : L \in Cmp(P)\}$, and so $v(K) = \lambda(P)$. Thus, the maximum packing of induced 3-vertex paths in F provides a maximum complete minor K of F in which every vertex corresponds to an induced 3-vertex path in F.

Let $h'(F)$ be the maximum integer r such that F has a minor K_r in which every vertex corresponds to either a vertex or an edge in F. Obviously, $h(F) \geq h'(F)$. In 1999 [15] we proved that if F is not s-connected, then the Hadwiger conjecture is true, moreover, $h'(F) \geq s$.

From **6.2** we have in particular:

6.12. *Let G be a graph and P a subgraph of G. Then P is an edge disjoint* Π*-packing in G if and only if* $L(P)$ *is a packing of induced 3-vertex paths in* $L(G)$.

It is known [3] that the maximum edge disjoint Π-packing problem is *NP*-hard. Therefore by **6.12**, the maximum packing of induced 3-vertex paths is also *NP*-hard. However, probably the following is true.

6.13 Conjecture. *Let* α *be a positive integer. Then there exists a polynomial-time algorithm* A_α *for finding a maximum packing of disjoint induced 3-vertex paths in a claw-free graph G with* $\alpha(G) \leq \alpha$.

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