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A Reaction–Diffusion System of a Competitor–Competitor–Mutualist Model

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We investigate the homogeneous Dirichlet problem and Neumann problem to a reaction–diffusion system of a competitor–competitor-mutualist model. The existence, uniqueness, and boundedness of the solutions are established by means of the comparison principle and the monotonicity method. For the Dirichlet problem, we study the existence of trivial and nontrivial nonnegative equilibrium solutions and their stabilities. For the Neumann problem, we analyze the contant equilibrium solutions and their stabilities. The main method used in studying of the stabilities is the spectral analysis to the linearized operators. The O.D.E. problem to the same model was proposed and studied by B. Rai, H. I. Freedman, and J. F. Addicott (*Math. Biosci.* **65** (1983), 13–50).

1. INTRODUCTION

We consider the following reaction–diffusion system of competitor–competitor-mutualist model

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 + \alpha u_1\left(1 - \frac{u_1}{K_1}\right) - \frac{\alpha\beta u_1 u_2}{1 + m u_3}, \\
\frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + \delta u_2\left(1 - \frac{u_2}{K_2}\right) - \eta u_1 u_2, \\
\frac{\partial u_3}{\partial t} &= d_3 \Delta u_3 + \gamma u_3\left(1 - \frac{u_3}{L_0 + l u_1}\right), \qquad \text{in} \quad \Omega \times \mathbb{R}^+.
\end{aligned} \tag{1.1}$$

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with initial condition

$$u_i(x, 0) = u_{i0}(x), \qquad i = 1, 2, 3, \quad \text{on} \quad \Omega \tag{1.2}$$

and Dirichlet boundary condition

$$u_i(x, t) = 0, \qquad i = 1, 2, 3, \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \tag{1.3}$$

or Neumann boundary condition

$$\frac{\partial u_i}{\partial n}(x, t) = 0, \qquad i = 1, 2, 3, \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+. \tag{1.4}$$

Here, $u_1(x, t)$, $u_2(x, t)$, and $u_3(x, t)$ represent the populations of two competitors and a mutualist with diffusion constants d_1, d_2 and d_3, respectively; Ω is a bounded domain in $\mathbb{R}^n$, $\partial\Omega$ is its boundary, $\partial/\partial n$ is the outward normal derivatives on $\partial\Omega$; Δ is Laplace operator. The all parameters in (1.1) are positive constants, m and l being the mutualist constants. It can be seen that in this model, the mutualist u_3 tends to reduce the competition effect of the second competitor u_2 on the first one u_1 but has no direct effect on u_2 or vice versa.

The homogeneous Neumann boundary condition (1.4) is to be interpreted as "no flux" condition; i.e., there is no migration of all species across the boundary of their habitat. While the homogeneous Dirichlet boundary condition (1.3) can be considered as such a condition that under which neither of the three species can exist on the boundary.

We establish the existence, uniqueness, and boundedness by means of the comparison principle and the monotonicity method. For the Dirichlet problem, we study the existence of trivial and nontrivial nonnegative equilibrium solutions and their stabilities. For the Neumann problem, we analyze the constant equilibrium solutions and their stabilities. The main method used in studying of the stabilities is the spectral analysis to the linearized operators.

The corresponding competitor–competitor-mutualist O.D.E. model was proposed and studied by Rai, Freedman, and Addicott in [9], where the explanations of the ecological background of this model can be found as well. Another reaction–diffusion system corresponding to a predator–prey-mutualist O.D.E. model of [9] was studied by us in [13]. As for the studies on three species reaction–diffusion systems of predator–prey model, the readers can see [4, 5].

2. Preliminaries

First, we consider the more general semilinear parabolic system with more general boundary condition and initial condition

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - L_1 u_1 = f_1(u_1, u_2, u_3), \\ &\frac{\partial u_2}{\partial t} - L_2 u_2 = f_2(u_1, u_2), \\ &\frac{\partial u_3}{\partial t} - L_3 u_3 = f_3(u_1, u_3) \qquad \text{in} \quad \Omega \times (0, T], \end{aligned} \tag{2.1}$$

$$u_i(x, 0) = u_{i0}(x), \qquad i = 1, 2, 3 \quad \text{on} \quad \Omega, \tag{2.2}$$

$$B_i[u_i] = \alpha_i(x)\, u_i + \beta_i(x) \frac{\partial u_i}{\partial n} = h_i(x), \qquad i = 1, 2, 3 \quad \text{on} \quad \partial\Omega \times (0, T] \tag{2.3}$$

as well as the corresponding elliptic system

$$\begin{aligned} &-L_1 u_1 = f_1(u_1, u_2, u_3), \\ &-L_2 u_2 = f_2(u_1, u_2), \\ &-L_3 u_3 = f_3(u_1, u_3) \qquad \text{in} \quad \Omega, \end{aligned} \tag{2.4}$$

$$B_i[u_i] = h_i(x), \qquad i = 1, 2, 3 \quad \text{on} \quad \partial\Omega, \tag{2.5}$$

where L_i is a uniformly elliptic operator in Ω; $\alpha_i(x)$, $\beta_i(x)$, and $u_{i0}(x)$ are smooth functions with $u_{i0} \not\equiv 0$ and $\alpha_i + \beta_i > 0$; f_i is continuously differentiable with respect to its variables for $u_k \geqslant 0$, i, $k = 1, 2, 3$. In addition, we assume

$$\begin{aligned} &\frac{\partial f_1}{\partial u_2} \leqslant 0, \qquad \frac{\partial f_1}{\partial u_3} \geqslant 0, \\ &\frac{\partial f_2}{\partial u_1} \leqslant 0, \\ &\frac{\partial f_3}{\partial u_1} \geqslant 0 \qquad \text{for} \quad u_i \geqslant 0,\ i = 1, 2, 3, \end{aligned} \tag{2.6}$$

which are obviously satisfied by the reaction terms of (1.1) as well as by those of a more general system corresponding to (2.2) of [9].

Denote $Q_T = \Omega \times (0, T]$, $S_T = \partial\Omega \times (0, T]$, where T is an arbitrary positive constant.

DEFINITION 2.1. Ordered smooth functions $\bar{U}(x,t)=(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ and $\underline{U}(x,t)=(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ in Q_T are called upper and lower solutions of (2.1)–(2.3), if they satisfy

$$\begin{aligned}
&(\bar{u}_1)_t - L_1\bar{u}_1 \geqslant f_1(\bar{u}_1, \underline{u}_2, \bar{u}_3),\\
&(\bar{u}_2)_t - L_2\bar{u}_2 \geqslant f_2(\underline{u}_1, \bar{u}_2),\\
&(\bar{u}_3)_t - L_3\bar{u}_3 \geqslant f_3(\bar{u}_1, \bar{u}_3),\\
&(\underline{u}_1)_t - L_1\underline{u}_1 \leqslant f_1(\underline{u}_1, \bar{u}_2, \underline{u}_3),\\
&(\underline{u}_i)_t - L_2\underline{u}_2 \leqslant f_2(\bar{u}_1, \underline{u}_2),\\
&(\underline{u}_3)_t - L_3\underline{u}_3 \leqslant f_3(\underline{u}_1, \underline{u}_3) \qquad \text{in} \quad Q_T,
\end{aligned} \tag{2.7}$$

$$B_i[\bar{u}_i] \geqslant h_i(x) \geqslant B_i[\underline{u}_i], \qquad i=1,2,3 \quad \text{on} \quad S_T \tag{2.8}$$

and

$$\bar{u}_i(x,0) \geqslant u_{i0}(x) \geqslant \underline{u}_i(x,0), \qquad i=1,2,3 \quad \text{on} \quad \Omega. \tag{2.9}$$

Suppose such $\bar{U}$ and $\underline{U}$ exist. Denote

$$S=\{(u_1,u_2,u_3)\in\mathbb{R}^3 : \underline{\rho}_i \leqslant u_i \leqslant \bar{\rho}_i,\ i=1,2,3\}, \tag{2.10}$$

where $\underline{\rho}_i = \inf_{Q_T} \underline{u}_i(x,t)$, $\bar{\rho}_i = \sup_{Q_T} \bar{u}_i(x,t)$, $i=1, 2, 3$. Define

$$N_i = \sup_S \left\{ \left| \frac{\partial f_i}{\partial u_i} \right| \right\}, \qquad i=1,2,3. \tag{2.11}$$

Construct the sequences $\{\bar{U}^{(k)}\}$ and $\{\underline{U}^{(k)}\}$ with $\bar{U}^{(0)}=\bar{U}$ and $\underline{U}^{(0)}=\underline{U}$ as follows:

$$\begin{aligned}
&(\bar{u}_1^{(k)})_t - L_1\bar{u}_1^{(k)} + N_1\bar{u}_i^{(k)} = N_1\bar{u}_1^{(k-1)} + f_1(\bar{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}, \bar{u}_3^{(k-1)}),\\
&(\bar{u}_2^{(k)})_t - L_2\bar{u}_2^{(k)} + N_2\bar{u}_2^{(k)} = N_2\bar{u}_2^{(k-1)} + f_2(\underline{u}_1^{(k-1)}, \bar{u}_2^{(k-1)}),\\
&(\bar{u}_3^{(k)})_t - L_3\bar{u}_3^{(k)} + N_3\bar{u}_3^{(k)} = N_3\bar{u}_3^{(k-1)} + f_3(\bar{u}_1^{(k-1)}, \bar{u}_3^{(k-1)}),\\
&(\underline{u}_1^{(k)})_t - L_1\underline{u}_1^{(k)} + N_1\underline{u}_1^{(k)} = N_1\underline{u}_1^{(k-1)} + f_1(\underline{u}_1^{(k-1)}, \bar{u}_2^{(k-1)}, \underline{u}_3^{(k-1)}),\\
&(\underline{u}_2^{(k)})_t - L_2\underline{u}_2^{(k)} + N_2\underline{u}_2^{(k)} = N_2\underline{u}_2^{(k-1)} + f_2(\bar{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}),\\
&(\underline{u}_3^{(k)})_t - L_3\underline{u}_3^{(k)} + N_3\underline{u}_3^{(k)} = N_3\underline{u}_3^{(k-1)} + f_3(\underline{u}_1^{(k-1)}, \underline{u}_3^{(k-1)}),
\end{aligned} \tag{2.12}$$

$$B_i[\bar{u}_i^{(k)}] = h_i(x) = B_i[\underline{u}_i^{(k)}], \tag{2.13}$$

$$\bar{u}_i^{(k)}(x,0) = u_{i0}(x) = \underline{u}_i^{(k)}(x,0), \tag{2.14}$$

where $i=1,2,3$, $k=1,2,3,\ldots$.

As for the existence of solution of (2.1)–(2.3) we have

THEOREM 2.1. *Suppose that there exists a pair of upper and lower solutions* $\bar{U}=(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ *and* $\underline{U}=(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ *satisfying* $\underline{u}_i(x,t)\leqslant \bar{u}_i(x,t)$, $i=1,2,3$. *Then the sequences* $\{\bar{U}^{(k)}\}$ *and* $\{\underline{U}^{(k)}\}$ *obtained by solving* (2.12)–(2.14) *monotonically from above and below, respectively, to a unique solution* $U=(u_1, u_2, u_3)$ *of* (2.1)–(2.3) *such that*

$$\underline{u}_i(x,t)\leqslant u_i(x,t)\leqslant \bar{u}_i(x,t), \qquad i=1,2,3,\ (x,t)\in Q_T.$$

The proof of Theorem 2.1 is standard. We omit it here.

Next consider the corresponding elliptic system (2.4) and (2.5).

DEFINITION 2.2. Ordered smooth functions $\bar{U}(x)=(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ and $\underline{U}(x)=(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ in Ω are called upper and lower solutions of (2.4) and (2.5) if they satisfy

$$\begin{aligned}
&-L_1\bar{u}_1\geqslant f_1(\bar{u}_1, \underline{u}_2, \bar{u}_3),\\
&-L_2\bar{u}_2\geqslant f_2(\underline{u}_1, \bar{u}_2),\\
&-L_3\bar{u}_3\geqslant f_3(\bar{u}_1, \bar{u}_3),\\
&-L_1\underline{u}_1\leqslant f_1(\underline{u}_1, \bar{u}_2, \underline{u}_3),\\
&-L_2, \underline{u}_2\leqslant f_2(\bar{u}_1, \underline{u}_2),\\
&-L_3\underline{u}_3\leqslant f_3(\underline{u}_1, \underline{u}_3), \qquad x\in\Omega
\end{aligned} \tag{2.15}$$

and

$$B_i[\bar{u}_i]\geqslant h_i(x)\geqslant B_i[\underline{u}_i], \qquad i=1,2,3,\ x\in\partial\Omega. \tag{2.16}$$

THEOREM 2.2. *Suppose* $\bar{U}$ *and* $\underline{U}$ *are a pair of upper and lower solutions of* (2.4) *and* (2.5) *with* $\bar{u}_i\geqslant\underline{u}_i$ $i=1,2,3$, *on* Ω, *then there exists at least one solution* $U(x)=(u_1, u_2, u_3)$ *of* (2.4) *and* (2.5), *such that*

$$\underline{u}_i(x)\leqslant u_i(x)\leqslant \bar{u}_i(x), \qquad i=1,2,3,\ x\in\Omega.$$

The proof of Theorem 2.2 is substantially the same as for the scalar equation case [10, 12]. See, e.g., [7, 10]. Note that the theorem does not guarantee the uniqueness of solutions of (2.4) and (2.5) even restricted between $\underline{U}$ and $\bar{U}$. In fact, as will be seen in the next section, multiple nontrivial nonnegative equilibrium solutions of (1.1)–(1.3) do exist.

3. Dirichlet Problem

In this section we consider the system (1.1) with initial condition (1.2) and homogeneous Dirichlet boundary condition (1.3).

We first establish the boundedness and nonnegativity to the solution.

Theorem 3.1. *Let $U(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ be a solution of (1.1)–(1.3). Then for $(x, t) \in \Omega \times \mathbb{R}^+$* we have

$$0 \leqslant u_1(x, t) \leqslant \max\{K_1, \sup_\Omega u_{10}(x)\},$$

$$0 \leqslant u_2(x, t) \leqslant \max\{K_2, \sup_\Omega u_{20}(x)\},$$

$$0 \leqslant u_3(x, t) \leqslant \max\{L_0 + lK_1, \sup_\Omega u_{30}(x)\}.$$

Proof. The case where u_i takes its maximum on superplane $t = 0$ is trivial.

Now, let us prove that (1.1) has the following invariant region [12]

$$\Sigma = \{(u_1, u_2, u_3) \in \mathbb{R}^3, 0 \leqslant u_1 \leqslant K_1, 0 \leqslant u_2 \leqslant K_2, 0 \leqslant u_3 \leqslant L_0 + lK_1\}.$$

Set

$$V = \left[\alpha u_1\left(1 - \frac{u_1}{K_1}\right) - \frac{\alpha\beta u_1 u_2}{1 + mu_3}, \delta u_2\left(1 - \frac{u_2}{K_2}\right) - \eta u_1 u_2, \gamma u_3\left(1 - \frac{u_3}{L_0 + lu_1}\right)\right],$$

$$G_j = -u_j \quad j = 1, 2, 3, \qquad G_4 = u_1 - K_1, \qquad G_5 = u_2 - K_2,$$

$$G_6 = u_3 - (L_0 + lK_1).$$

Then, according to Theorem 14.13 of [12] as well as the fact that the intersection of invariant regions is an invariant region (see Sect. B, Chapt. 14 of [12]), we have

$$\nabla G_j \cdot V|_{u_j = 0} = 0 \quad \text{in} \quad \Sigma, \qquad \text{so} \quad u_i \geqslant 0, j = 1, 2, 3,$$

$$\nabla G_4 \cdot V|_{u_1 = K_1} = -\frac{\alpha\beta K_1 u_2}{1 + mu_3} \leqslant 0$$

$$\text{in} \quad \Sigma, \qquad \text{so} \quad u_1 \leqslant K_1,$$

$$\nabla G_5 \cdot V|_{u_2 = K_2} = -\eta u_1 K_2 \leqslant 0$$

$$\text{in} \quad \Sigma, \qquad \text{so} \quad u_2 \leqslant K_2,$$

$$\nabla G_6 \cdot V|_{u_3 = L_0 + lK_1} = \gamma(L_0 + lK_1)\left(1 - \frac{L_0 + lK_1}{L_0 + lu_1}\right) \leqslant 0$$

$$\text{in} \quad \Sigma, \qquad \text{so} \quad u_3 \leqslant L_0 + lK_1.$$

The proof of the theorem is completed.

Due to Theorem 2.1, in order to establish the existence of solutions of (1.1)–(1.3), we need only to construct a pair of upper and lower solutions.

Let λ_0 be the principal eigenvalue of operator $-\Delta$ with homogeneous Dirichlet boundary condition. We can construct a function $\tilde{\varphi}_0(x)$ [1, 11], normalized by $\sup_\Omega \tilde{\varphi}_0(x) = 1$, such that

$$\begin{aligned} \Delta\tilde{\varphi}_0 + \lambda_0\tilde{\varphi}_0 \leqslant 0, &\qquad x \in \Omega, \\ \tilde{\varphi}_0 > 0, &\qquad x \in \bar{\Omega}. \end{aligned} \tag{3.1}$$

Choose positive constants M_i, $i = 1, 2, 3$, such that

$$u_{i0}(x) \leqslant M_i \tilde{\varphi}_0(x), \qquad i = 1, 2, 3,\ x \in \bar{\Omega}.$$

Set

$$\begin{aligned} \bar{u}_1(x, t) &= M_1 \tilde{\varphi}_0(x) \exp\{(-d_1\lambda_0 + \alpha)\, t\}, \\ \bar{u}_2(x, t) &= M_2 \tilde{\varphi}_0(x) \exp\{(-d_2\lambda_0 + \delta)\, t\}, \\ \bar{u}_3(x, t) &= M_3 \tilde{\varphi}_0(x) \exp\{(-d_3\lambda_0 + \gamma)\, t\}. \end{aligned} \tag{3.2}$$

It is easy to check that, $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ and $(0, 0, 0)$ define a pair of upper and lower solutions of (1.1)–(1.3). According to Theorem 2.1 and the arbitrariness of T, we get

THEOREM 3.2. *There exists the unique solution* $(u_1(x, t), u_2(x, t), u_3(x, t))$ *of problem* (1.1)–(1.3) *satisfying*

$$0 \leqslant u_i(x, t) \leqslant \bar{u}_i(x, t), \qquad i = 1, 2, 3,\ (x, t) \in \Omega \times \mathbb{R}^+,$$

where $\bar{u}_i(x, t)$ $(i = 1, 2, 3)$ *are defined by* (3.2).

Now study the existence and the stabilities of equilibrium solutions of (1.1)–(1.3).

We first discuss the trivial equilibrium solution $(0, 0, 0)$.

THEOREM 3.3. *Let* λ_0 *be the principal eigenvalue of* $-\Delta$ *with homogeneous Dirichlet boundary condition. Assume*

$$\Lambda = \max\{-d_1\lambda_0 + \alpha,\ -d_2\lambda_0 + \delta,\ -d_3\lambda_0 + \gamma\} < 0.$$

Then $(0, 0, 0)$ *is the only nonnegative equilibrium solution of* (1.1)–(1.3). *Moreover, it is globally asymptotically stable.*

Proof. Let $(u_1^*(x), u_2^*(x), u_3^*(x))$ be an arbitrary nonnegative equilibrium solution, i.e.,

$$
\begin{aligned}
&d_1 \Delta u_1^* + \alpha u_1^* \left(1 - \frac{u_1^*}{K_1}\right) - \frac{\alpha\beta u_1^* u_2^*}{1 + m u_3^*} = 0, \\
&d_2 \Delta u_2^* + \delta u_2^* \left(1 - \frac{u_1^*}{K_2}\right) - \eta u_1^* u_2^* = 0, \\
&d_3 \Delta u_3^* + \gamma u_3^* \left(1 - \frac{u_3^*}{L_0 + l u_1^*}\right) = 0, \qquad x \in \Omega, \\
&u_i^* |_{\partial\Omega} = 0, \qquad i = 1, 2, 3,
\end{aligned}
\tag{3.3}
$$

which implies that $u_1^*(x)$ is a nonnegative solution of the following linear elliptic equation

$$
\begin{aligned}
&d_1 \Delta u_1 + a(x)\, u_1 = 0, \qquad x \in \Omega, \\
&u_1 |_{\partial\Omega} = 0,
\end{aligned}
\tag{3.4}
$$

where $a(x) = \alpha(1 - u_1^*/K_1 - \beta u_2^*/(1 - m u_3^*))$.

Due to $u_i^* \geqslant 0$, $i = 1, 2, 3$ and $-d_1 \lambda_0 + \alpha < 0$, we know that

$$a(x) \leqslant \alpha < d_1 \lambda_0.$$

Thus, we deduce

$$\left\{\frac{a(x)}{d_1} : x \in \Omega\right\} \cap \sigma(-\Delta) = \varnothing, \tag{3.5}$$

where $\sigma(-\Delta)$ denotes the point spectrum of $-\Delta$ with homogeneous Dirichlet boundary condition. (3.5) implies

$$\Delta u_1^* + \frac{a(x)}{d_1} u_1^* \neq 0 \qquad \text{for all} \quad x \in \Omega$$

whenever

$$u_1^* \neq \text{constant zero.}$$

So,

$$u_1^*(x) \equiv 0, \qquad x \in \bar{\Omega}.$$

It can be shown in the same way that

$$u_2^*(x) = u_3^*(x) \equiv 0, \qquad x \in \bar{\Omega},$$

and hence $(0, 0, 0)$ is the only equilibrium solution of (1.1)–(1.3).

The conclusion on globally asymptotic stability follows from the assumption $\Lambda < 0$ and the upper solution formula (3.2). This completes the proof.

If the assumption $\Lambda < 0$ is violated, then there may exist some nontrivial nonnegative equilibrium solutions to (1.1)–(1.3).

THEOREM 3.4. *Assume* $-d_1\lambda_0 + d > 0$, $-d_2\lambda_0 + \delta$, $-d_3\lambda_0 + \gamma < 0$, *then there exists a nontrivial nonnegative equilibrium solution* $(u_1^*(x), 0, 0)$ *of* (1.1)–(1.3), *which is linearly stable.*

Proof. Consider the following Dirichlet problem of semilinear elliptic equation

$$\begin{aligned} d_1 \Delta u_1 + \alpha u_1 \left(1 - \frac{u_1}{K_1}\right) &= 0, \qquad x \in \Omega, \\ u_1 |_{\partial\Omega} &= 0. \end{aligned} \tag{3.6}$$

Take

$$\begin{aligned} \bar{u}_1(x) &= M\tilde{\varphi}_0(\chi), \\ \underline{u}_1(x) &= \varepsilon\varphi_0(x), \qquad x \in \bar{\Omega}, \end{aligned} \tag{3.7}$$

where $\tilde{\varphi}_0(x)$ satisfies (3.1), $\varphi_0(x)$ is the principal eigenfunction of $-\Delta$ with eigenvalue λ_0; both of them are normalized by $\sup_\Omega \tilde{\varphi}_0(x) = 1$, $\sup_\Omega \varphi_0(x) = 1$; constants $M \geqslant (K_1(-d_1\lambda_0 + d)/\alpha \inf_\Omega \tilde{\varphi}_0(x))$, $\varepsilon \leqslant K_1(-d_1\lambda_0 + \alpha)/\alpha$. We can check that $\bar{u}_1(x)$ and $\underline{u}_1(x)$ defined by (3.7) form a pair of upper and lower solutions of (3.6). Due to Theorem 2.2, we know that there exists a nontrivial solution of (3.6), such that

$$0 < \underline{u}_1(x) \leqslant u_1^*(x) \leqslant \bar{u}_1(x), \qquad x \in \Omega.$$

So, $(u_1^*(x), 0, 0)$ is a nontrivial nonnegative equilibrium solution of (1.1)–(1.3).

Let us linearize the reaction terms of (1.1) at $U^* = (u_1^*(x),\ 0,\ 0)$ and analyze the spectrum of the linearized operators. Rewrite system (1.1) into an evolution equation in Banach space $X = \oplus_{i=1}^3 X_i = \oplus_1^3 L^2(\Omega) \cap C^2(\Omega)$:

$$\frac{dU}{dt} = AU + F(U), \tag{3.8}$$

where

$$A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix} = \begin{bmatrix} d_1\Delta + \alpha & & \\ & d_2\Delta + \delta & \\ & & d_3\Delta + \gamma \end{bmatrix}, \tag{3.9}$$

$$U(t) = (u_1(t), u_2(t), u_3(t)) \in D(A) \subset X,$$

$$D(A) = \{U \in X\colon U|_{\partial\Omega} = 0\},$$

$$F(U) = \left(-\frac{\alpha u_1^2}{K_1} - \frac{\alpha\beta u_1 u_2}{1 + m u_3}, -\frac{\delta u_2^2}{K_2} - \eta u_1 u_2, -\frac{\gamma u_3^2}{L_0 + l u_1} \right)^T. \tag{3.10}$$

Linearizing $F(U)$ at U^*, we get

$$F(U + U^*) = F(U^*) + BU + g(U), \tag{3.11}$$

where

$$g(U) = o(\| U \|_X),$$

$$B = B_1 = \begin{bmatrix} -\dfrac{2\alpha u_1^*}{K_1} & -\alpha\beta u_1^* & 0 \\ 0 & -\eta u_1^* & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.12}$$

We analyze the spectrum of operator $\mathscr{U} \equiv A + B_1$ [2, 3, 6]. The resolvent equation for $\mathscr{U}$ is

$$(A + B_1 - \mu I)\, U = V, \qquad \mu \in \mathbb{C},\ V = (v_1, v_2, v_3)^T \in X,$$

i.e.,

$$\begin{bmatrix} A_1 - \dfrac{2\alpha u_1^*}{K_1} - \mu & -\alpha\beta u_1^* & 0 \\ 0 & A_2 - \eta u_1^* - \mu & 0 \\ 0 & 0 & A_3 - \mu \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

for $\mu \in \rho(A_2 - \eta u_1^*) \cap \rho(A_3)$, where $\rho(A_3)$ denotes the resolvent set of A_3, etc. Setting $\bar{A}_1(\mu) = A_1 - (2\alpha u_1^*/K_1) - \mu$, we have

$$\begin{bmatrix} \bar{A}_1(\mu)\, u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} I & \alpha\beta u_1^* R(\mu, A_2 - \eta u_1^*) & 0 \\ 0 & R(\mu, A_2 - \eta u_1^*) & 0 \\ 0 & 0 & R(\mu, A_3) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},$$

where $R(\mu, A_3)$ is the resolvent operator of A_3, etc.

We have the following four lemmas:

LEMMA 3.1. *Denote*

$$\mathscr{R} = \{\mu\colon \mu \in \rho(A_2 - \eta u_1^*) \cap \rho(A_3),\ \bar{A}_1(\mu) \text{ is invertible in } L^2(\Omega)\},$$

then

$$\mathscr{R} \subset \rho(\mathscr{U}).$$

LEMMA 3.2. *Let $\mu \in \rho(A_2 - \eta u_1^*) \cap (A_3)$, then $\bar{A}_1(\mu)$ is invertible in $L^2(\Omega)$ if and only if zero is not an eigenvalue of $\bar{A}_1(\mu)$.*

LEMMA 3.3. *There exist $\theta^* \in (0, \pi/2)$ and $\gamma^* > 0$, such that*

$$S^* \equiv \left\{\mu \in \mathbb{C} : |\arg(\mu + \gamma^*)| \leqslant \frac{\pi}{2} + \theta^*\right\} \subset \mathscr{R}.$$

LEMMA 3.4.

$$\sigma(\mathscr{U}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < -\gamma^*\}.$$

Lemmas 3.1 and 3.2 are obvious [6]. Lemma 3.4 follows from Lemmas 3.1–3.3 while the linear stability of $(u_1^*(x), 0, 0)$ results from Lemma 3.4 [12]. So, we need only to prove Lemma 3.3.

Proof of Lemma 3.3. Due to $d_2\lambda_0 + \delta$, $d_3\lambda_0 + \gamma < 0$, it is clear that

$$K^* = \max\{\operatorname{Re} \lambda : \lambda \in \sigma(A_2 - \eta u_1^*(x)) \cup \sigma(A_3)\} < 0. \tag{3.13}$$

In view of Lemma 3.2, it suffices to show that there exist $\theta^* \in (0, \pi/2)$ and $\gamma^* > 0$, such that $\bar{A}_1(\mu)$ does not have zero eigenvalue whenever $\mu \in S^*$.

Let $\eta(\mu)$ be an arbitrary eigenvalue of $\bar{A}_1(\mu)$, $\varphi(\mu)$ be the corresponding eigenfunction, $\varphi(\mu) \geqslant 0$, normalized by $\|\varphi\|_{L^2(\Omega)} = 1$. Put

$$\mu_1 = \operatorname{Re} \mu, \qquad \mu_2 = \operatorname{Im} \mu,$$
$$\eta_1(\mu) = \operatorname{Re} \eta(\mu), \qquad \eta_2(\mu) = \operatorname{Im} \eta(\mu).$$

Then

$$\begin{aligned}
\bar{A}_1(\mu)\,\varphi(\mu) &= \eta(\mu)\,\varphi(\mu),\\
\eta(\mu) &= \langle \bar{A}_1(\mu)\,\varphi(\mu), \varphi(\mu)\rangle\\
&= \left\langle \left(A_1 - \frac{\alpha u_1^*}{K_1}\right)\varphi(\mu), \varphi(\mu)\right\rangle + \left\langle -\frac{\alpha u_1^*}{K_1}\varphi(\mu), \varphi(\mu)\right\rangle - \mu,
\end{aligned}$$

and hence

$$\eta_1(\mu) = \left\langle \left(A_1 - \frac{\alpha u_1^*}{K_1}\right)\varphi(\mu), \varphi(\mu)\right\rangle + \left\langle -\frac{\alpha u_1^*}{K_1}\varphi(\mu), \varphi(\mu)\right\rangle - \mu_1,$$

$$\eta_2(\mu) = -\mu_2.$$

We recall that $u_1^*(x)$ satisfies (3.5), i.e.,

$$\left(A_1 - \frac{\alpha u_1^*}{K_1}\right) u_1^* = 0, \qquad x \in \Omega,$$
$$u_1^* |_{\partial\Omega} = 0. \tag{3.14}$$

This means that the second order elliptic operator $A_1 - (\alpha u_1^*/K_1)$ has positive function $u_1^*(x)$ as its eigenfunction with zero eigenvalue. Therefore, we deduce that [6]

$$\left\langle \left(A_1 - \frac{\alpha u_1^*}{K_1}\right) \psi, \psi \right\rangle \leqslant 0, \qquad \forall \psi \in H_0^1(\Omega)$$

and hence

$$\eta_1(0, \mu_2) \leqslant \left\langle -\frac{\alpha u_1^*}{K_1} \varphi(\mu), \varphi(\mu) \right\rangle < 0. \tag{3.15}$$

In view of the continuous dependence of $\eta(\mu)$ on μ, we know from (3.13) that there exist $c^* > 0$ and $\alpha^* > 0$, $-\alpha^* \in (K^*, 0)$, such that $\eta_1(\mu_1, \mu_2) < 0$ whenever

$$\mu \in \{\mu \in \mathbb{C} : \mu \in \rho(A_2 - \eta u_1^*) \cap \rho(A_3), \operatorname{Re} \mu \in (-\alpha^*, 0), |\operatorname{Im} \mu| < c^*\}.$$

Observing $|\eta_2(\mu_1, \mu_2)| = |\mu_2| \geqslant c^* > 0$ when $|\mu_2| \geqslant c^*$, we claim that $\bar{A}_1(\mu)$ does not have zero as an eigenvalue if

$$\mu \notin \{\mu \in \mathbb{C} : \mu \in \rho(A_2 - \eta u_1^*) \cap \rho(A_3), \operatorname{Re} \mu < -\alpha^*, |\operatorname{Im} \mu| < c^*\}.$$

Taking

$$\theta^* \in \left(0, \arctan \frac{\alpha^* - \gamma^*}{c^*}\right) \qquad \text{and} \qquad \gamma^* \in (0, \alpha^*),$$

we complete the proof of Lemma 3.3 and hence get the conclusion of the theorem.

Similarly, we have the following two theorems:

THEOREM 3.5. *Assume* $-d_2\lambda_0 + \delta > 0$, $-d_1\lambda_0 + \alpha$, $-d_3\lambda_0 + \gamma < 0$, *then there exists a nontrivial nonnegative equilibrium solution* $(0, u_2^*(x), 0)$ *of* (1.1)–(1.3), *which is linearly stable.*

THEOREM 3.6. *Assume* $-d_3\lambda_0 + \gamma > 0$, $-d_1\lambda_0 + \alpha$, $-d_2\lambda_0 + \delta < 0$, *then there exists a nontrivial nonnegative equilibrium solution* $(0, 0, u_3^*(x))$ *of* (1.1)–(1.3), *which is linearly stable.*

The proofs of Theorems 3.5 and 3.6 are almost the same as that of Theorem 3.4. We observe that the linearizations at $(0, u_2^*, 0)$ and $(0, 0, u_3^*)$ are

$$B_2 = \begin{bmatrix} -\alpha\beta u_2^* & 0 & 0 \\ -\eta u_2^* & -\dfrac{2\delta u_2^*}{K_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.16}$$

and

$$B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{l\gamma u_3^{*2}}{L_0^2} & 0 & -\dfrac{2\gamma u_3^*}{L_0} \end{bmatrix}, \tag{3.17}$$

respectively, and that

$$\max\{\operatorname{Re}\lambda : \lambda \in \sigma(A_1 - \alpha\beta u_2^*) \cup \sigma(A_3)\} < 0. \tag{3.18}$$

and

$$\max\{\operatorname{Re}\lambda : \lambda \in \sigma(A_1) \cup \sigma(A_2)\} < 0 \tag{3.19}$$

hold, respectively. Inequality (3.18) ((3.19)) comes from the assumption of Theorem 3.5 (Theorem 3.6) that $-d_1\lambda_0 + \alpha$, $-d_3\lambda_0 + \gamma < 0$ $(-d_1\lambda_0 + \alpha$, $-d_2\lambda_0 + \delta < 0)$.

The following simple theorem is the complement of Theorem 3.3.

THEOREM 3.7. *If*

$$\Lambda = \max\{-d_1\lambda_0 + \alpha,\ -d_2\lambda_0 + \delta,\ -d_3\lambda_0 + y\} > 0,$$

then the trivial equilibrium solution $(0, 0, 0)$ *is unstable.*

Proof. Linearizing $F(U)$ at $(0, 0, 0)$ we get

$$\mathscr{U} = A + B = \begin{bmatrix} d_1\Delta + \alpha & & \\ & d_2\Delta + \delta & \\ & & d_3\Delta + \gamma \end{bmatrix} = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix}$$

and hence

$$\sigma(\mathscr{U}) = \sigma(A_1) \cup \sigma(A_2) \cup \sigma(A_3).$$

Since $\Lambda > 0$, it is clear that

$$\sigma(\mathscr{U}) \cap \{\mu \in \mathbb{C} : \operatorname{Re} \mu > 0\} \neq \varnothing.$$

This completes the proof of the theorem.

The last three theorems of this section will deal with the cases where the habitat Ω is even "larger" (hence λ_0 is smaller), or the diffusion mechanism of the species is even weaker, or their growth rates are even greater, such that two of $-d_1\lambda_0 + \alpha$, $-d_2\lambda_0 + \delta$, and $-d_3\lambda_0 + \gamma$ are positive, but the left one is negative.

THEOREM 3.8. *Assume* $-d_1\lambda_0 + \alpha$, $-d_2\lambda_0 + \delta > 0$, $-d_3\lambda_0 + \gamma < 0$, *then*

(i) *There exists a nontrivial nonnegative equilibrium solution* $(u_1^*(x), 0, 0)$, *which is linearly stable if*

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma(A_2 - \eta u_1^*)\} < 0 \tag{3.20}$$

and is unstable if

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma(A_2 - \eta u_1^*)\} > 0. \tag{3.21}$$

(ii) *There exists a nontrivial nonnegative equilibrium solution* $(0, u_2^*(x), 0)$, *which is linearly stable if*

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma(A_1 - \alpha\beta u_2^*)\} < 0 \tag{3.22}$$

and is unstable if

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma(A_1 - \alpha\beta u_2^*)\} > 0. \tag{3.23}$$

(iii) *There exists a nontrivial nonnegative equilibrium solution* $(\tilde{u}_1(x), \tilde{u}_2(x), 0)$ *if*

$$\frac{K_1(-d_1\lambda_0 + \alpha)}{\alpha \inf_\Omega \tilde{\varphi}_0(x)} < \frac{-d_2\lambda_0 + \delta}{\eta}, \tag{3.24}$$

$$\frac{K_2(-d_2\lambda_0 + \delta)}{\delta \inf_\Omega \tilde{\varphi}_0(x)} < \frac{-d_1\lambda_0 + \alpha}{\alpha\beta}. \tag{3.25}$$

Proof. (i) Note that $\max\{\lambda : \lambda \in \sigma(d_3\Delta + \gamma)\} < 0$ since $-d_3\lambda_0 + \gamma < 0$. So, (3.13) holds if (3.20) is true. According to the proof of Theorem 3.4, we know that $(u_1^*(x), 0, 0)$ is linearly stable under the assumption (3.17).

Observe that

$$\sigma(A_2 - \eta u_1^*(x)) \subset \sigma(\mathscr{U}),$$

where $\mathscr{U} = A + B_1$, B_1 is the linearization at $(u_1^*(x), 0, 0)$. Therefore, $(u_1^*(x), 0, 0)$ is unstable under (3.21).

The proof of (ii) is the same as that of (i).

(iii) Consider

$$\begin{aligned} &d_1 \Delta u_1 + \alpha u_1 \left(1 - \frac{u_1}{K_1}\right) - \alpha\beta u_1 u_2 = 0, \\ &d_2 \Delta u_2 + \delta u_2 \left(1 - \frac{u_2}{K_2}\right) - \eta u_1 u_2 = 0. \qquad x \in \Omega, \\ &u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = 0. \end{aligned} \tag{3.26}$$

Put

$$\begin{aligned} \bar{u}_1(x) &= M_1 \tilde{\varphi}_0(x), \qquad \underline{u}_1(x) = \varepsilon_1 \varphi_0(x), \\ \bar{u}_2(x) &= M_2 \tilde{\varphi}_0(x), \qquad \underline{u}_2(x) = \varepsilon_2 \varphi_0(x), \end{aligned} \tag{3.27}$$

where $\tilde{\varphi}_0(x)$ and $\varphi_0(x)$ are described as before, M_i and ε_i are to be chosen, $i = 1, 2$. We hope that (3.27) defines a pair of upper and lower solutions of (3.26).

Observe that

$$-d_1 M_1 \lambda_0 \tilde{\varphi}_0 + \alpha M_1 \tilde{\varphi}_0 \left(1 - \frac{M_1 \tilde{\varphi}_0}{K_1}\right) - \alpha\beta M_1 \varepsilon_2 \tilde{\varphi}_0 \varphi_0 \leqslant 0$$

if

$$M_1 \geqslant \frac{K_1(-d_1\lambda_0 + \alpha)}{\alpha \inf_\Omega \tilde{\varphi}_0(x)}, \tag{3.28}$$

$$-d_2 M_2 \lambda_0 \tilde{\varphi}_0 + \delta M_2 \tilde{\varphi}_0 \left(1 - \frac{M_2 \tilde{\varphi}_0}{K_2}\right) - \eta M_2 \varepsilon_1 \tilde{\varphi}_0 \varphi_0 \leqslant 0$$

if

$$M_2 \geqslant \frac{K_2(-d_2\lambda_0 + \delta)}{\delta \inf_\Omega \tilde{\varphi}_0(x)}, \tag{3.29}$$

$$-d_1 \varepsilon_1 \lambda_0 \varphi_0 + \alpha \varepsilon_1 \varphi_0 \left(1 - \frac{\varepsilon_1 \varphi_0}{K_1}\right) - \alpha\beta\varepsilon_1 M_2 \varphi_0 \tilde{\varphi}_0 \geqslant 0$$

if

$$M_2 < \frac{-d_1\lambda_0 + \alpha}{\alpha\beta} \tag{3.30}$$

and take

$$\varepsilon_1 \leqslant \frac{\beta K_1}{2}\left(\frac{-d_1\lambda_0+\alpha}{\alpha\beta}-M_2\right), \tag{3.31}$$

$$-d_2\varepsilon_2\lambda_0\varphi_0+\delta\varepsilon_2\varphi_0\left(1-\frac{\varepsilon_2\varphi_0}{K_2}\right)-\eta\varepsilon_2 M_1\varphi_0\tilde{\varphi}_0 \geqslant 0$$

if

$$M_1 < \frac{-d_2\lambda_0+\delta}{\eta} \tag{3.32}$$

and take

$$\varepsilon_2 \leqslant \frac{\eta K_2}{2\delta}\left(\frac{-d_2\lambda_0+\delta}{\eta}-M_1\right). \tag{3.33}$$

Obviously, under conditions (3.24) and (3.25), we can choose M_i, ε_i, $i=1,2$, such that (3.28)–(3.33) hold. So, $(\bar{u}_1(x), \bar{u}_2(x))$ and $(\underline{u}_1(x), \underline{u}_2(x))$ defined by (3.27) form a pair of upper and lower solutions of (3.26). Due to Theorem 2.2, there exists a solution $(\tilde{u}_1(x), \tilde{u}_2(x))$ of (3.26) satisfying

$$0<\underline{u}_i(x)\leqslant\tilde{u}_i(x)\leqslant\bar{u}_i(x), \qquad i=1,2,\ x\in\Omega.$$

This completes the proof of (iii).

THEOREM 3.9. *Assume* $-d_1\lambda_0+\alpha$, $-d_3\lambda_0+\gamma>0$, $-d_2\lambda_0+\delta<0$, *then*

(i) *There exist nontrivial nonnegative equilibrium solutions of the forms* $(u_1^*(x),0,0)$ *and* $(0,0,u_3^*(x))$, *both of which are unstable.*

(ii) *There exists a nontrivial nonnegative equilibrium solution of the form* $(\tilde{u}_1(x), o, \tilde{u}_3(x))$, *which is always linearly stable.*

Proof. (i) We will only prove the instabilities. Recall from (3.12) and (3.17) that the linearization at $(u_1^*(x),0,0)$ and $(0,0,u_3^*(x))$ are

$$B_1=\begin{bmatrix} -\dfrac{2du_1^*}{K_1} & -\alpha\beta u_1^* & 0 \\ 0 & -\eta u_1^* & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B_3=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{l\gamma u_3^{*2}}{L_0^2} & 0 & -\dfrac{2\gamma u_3^*}{L_0} \end{bmatrix},$$

respectively. So,

$$\sigma(A_3) \subset \sigma(A + B_1),$$
$$\sigma(A_1) \subset \sigma(A + B_3).$$

We know that

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma(A_3)\} > 0$$

since $-d_3\lambda_0 + \gamma > 0$ as well as that

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma(A_1)\} > 0$$

since $-d_1\lambda_0 + \alpha > 0$. Therefore, both of $(u_1^*(x), 0, 0)$ and $(0, 0, u_3^*(x))$ are unstable.

(ii) We know that $\tilde{u}_1(x)$ and $\tilde{u}_3(x)$ satisfy (3.6) and

$$\begin{aligned} &d_3 \Delta u_3 + \gamma u_3 \left(1 - \frac{u_3}{L_0 + l u_1}\right) = 0, \qquad x \in \Omega, \\ &u_3|_{\partial\Omega} = 0. \end{aligned} \tag{3.34}$$

Due to Theorem 3.4, there exists a positive solution $\tilde{u}_1(x)$ of (3.6). Substituting $\tilde{u}_1(x)$ for u_1 in (3.34), we obtain in the same way that there exists a positive solution $\tilde{u}_3(x)$ of (3.34).

As to the linear stability, consider the linearization at this point

$$B_4 = \begin{bmatrix} -\dfrac{2\alpha\tilde{u}_1}{K_1} & -\dfrac{\alpha\beta\tilde{u}_1}{1 + m\tilde{u}_3} & 0 \\ 0 & -\eta\tilde{u}_1 & 0 \\ -\dfrac{\gamma l \tilde{u}_3}{(L_0 + l\tilde{u}_1)^2} & 0 & -\dfrac{2\gamma\tilde{u}_3}{L_0 + l\tilde{u}_1} \end{bmatrix}.$$

The resolvent equation for $\mathscr{U} = A + B_4$ is

$$\begin{bmatrix} A_1 - \dfrac{2\alpha\tilde{u}_1}{K_1} - \mu & -\dfrac{\alpha\beta\tilde{u}_1}{1 + m\tilde{u}_3} & 0 \\ 0 & A_2 - \eta\tilde{u}_1 - \mu & 0 \\ -\dfrac{\gamma l \tilde{u}_3}{(L_0 + l\tilde{u}_1)^2} & 0 & A_3 - \dfrac{2\gamma\tilde{u}_3}{L_0 + l\tilde{u}_1} - \mu \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

for $\mu \in \rho(A_2 - \eta\tilde{u}_1)$, $(v_1, v_2, v_3) \in X$. Setting

$$\bar{A}_1(\mu) = A_1 - \frac{2\alpha\tilde{u}_1}{K_1} - \mu, \qquad \bar{A}_3(\mu) = A_3 - \frac{2\gamma\tilde{u}_3}{L_0 + l\tilde{u}_1} - \mu,$$

then

$$\bar{A}_1(\mu)\, u_1 = v_1 + \frac{\alpha\beta\tilde{u}_1}{1+m\tilde{u}_3} R(\mu, A_2 - \eta\tilde{u}_1)\, v_2,$$

$$u_2 = R(\mu, A_2 - \eta\tilde{u}_1)\, v_2,$$

$$-\frac{\gamma l \tilde{u}_3}{(L_0 + l\tilde{u}_1)^2} u_1 + \bar{A}_3(\mu)\, u_3 = v_3.$$

Denote

$$\mathscr{R} = \{\mu : \mu \in \rho(A_2 - \eta\tilde{u}_1), \text{ both } \bar{A}_1(\mu) \text{ and } \bar{A}_3(\mu) \text{ are invertible in } L^2(\Omega)\}.$$

As did in the proof of Theorem 3.4, we can get four lemmas similar to Lemmas 3.1–3.4. Observe that $K^* = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A_2 - \eta\tilde{u}_1)\} < 0$ and that $\tilde{u}_1(x)$ and $\tilde{u}_3(x)$ satisfy (3.6) and (3.34), and hence they are positive eigenfunctions of second order elliptic operators $A_1 - \alpha\tilde{u}_1/K_1$ and $A_3 - (\gamma\tilde{u}_3/(L_0 + l\tilde{u}_1))$, respectively, with zero eigenvalue. We omit the details.

THEOREM 3.10. *Assume* $-d_2\lambda_0 + \delta$, $-d_3\lambda_0 + \gamma > 0$, $-d_1\lambda_0 + \alpha < 0$, *then*

(i) *There exist nontrivial nonnegative equilibrium solutions of the forms* $(0, u_2^*(x), 0)$ *and* $(0, 0, u_3^*(x))$, *both of them are unstable.*

(ii) *There exists a nontrivial nonnegative equilibrium solution of the form* $(0, \tilde{u}_2(x), \tilde{u}_3(x))$, *which is always linearly stable.*

We omit the proof of Theorem 3.10, which is somewhat similar to those of Theorems 3.8 and 3.9.

4. NEUMANN PROBLEM

In this section we consider system (1.1) with initial condition (1.2) and homogeneous Neumann boundary condition (1.4).

First, we can prove in the same way as in Section 3 that the solution is nonnegative and bounded:

THEOREM 4.1. *Let* $U(x, t) = (u_1(x, t)\, u_2(x, t)\, u_3(x, t))$ *be a solution of* (1.1), (1.2) *and* (1.4), *then*

$$0 \leqslant u_1(x, t) \leqslant \max\{K_1, \sup_\Omega u_{10}(x)\},$$

$$0 \leqslant u_2(x, t) \leqslant \max\{K_2, \sup_\Omega u_{20}(x)\},$$

$$0 \leqslant u_3(x, t) \leqslant \max\{L_0 + lK_1, \sup_\Omega u_{30}(x)\}.$$

Now establish the existence of solutions.
Set

$$\hat{u}_i = \sup_{\Omega} u_{i0}(x) > 0, \qquad i = 1, 2, 3.$$

Consider the following initial problem of ODE system

$$\begin{aligned}
&u_1'(t) = \alpha u_1(t)\left(1 - \frac{u_1(t)}{K_1}\right), \\
&u_2'(t) = \delta u_2(t)\left(1 - \frac{u_2(t)}{K_2}\right), \\
&u_3'(t) = \gamma u_3(t)\left(1 - \frac{u_3(t)}{L_0 + lM_1}\right), \qquad \text{in} \quad Q_T, \\
&u_i(0) = \hat{u}_i, \qquad i = 1, 2, 3,
\end{aligned} \tag{4.1}$$

where constant M_1 is to be taken later. Obviously,

$$\left.\frac{\partial u_i}{\partial n}\right|_{\partial\Omega} = 0 \qquad i = 1, 2, 3.$$

It is easy to get

$$u_1(t) = K_1\left[1 + \frac{K_1 - \hat{u}_1}{\hat{u}_1} e^{-\alpha t}\right]^{-1},$$

$$u_2(t) = K_2\left[1 + \frac{K_2 - \hat{u}_2}{\hat{u}_2} e^{-\delta t}\right]^{-1}.$$

Take

$$M_1 = \sup_{t \geq 0} u_2(t) < \infty.$$

Then

$$u_3(t) = (L_0 + lM_1)\left[1 + \frac{L_0 + lM_1 - \hat{u}_3}{\hat{u}_3}\right]^{-1}.$$

Clearly, $(0, 0, 0)$ and $(u_1(t), u_2(t), u_3(t))$ define a pair of upper and lower solutions of (1.1), (1.2), and (1.4). According to Theorem 2.1 and arbitrariness of T, we get immediately

THEOREM 4.2. *There exists a solution* $(u_1(x, t), u_2(x, t), u_3(x, t))$ *of* (1.1), (1.2), *and* (1.4), *such that*

$$0 \leqslant u_i(x, t) \leqslant u_i(t) \leqslant M_i < \infty, \qquad i = 1, 2, 3, \ (x, t) \in \Omega \times \mathbb{R}^+,$$

where $M_i = \sup_{t \geqslant 0} u_i(t)$, $u_i(t)$ *is the solution of* (4.1), $i = 1, 2, 3$.

Next study the constant equilibrium solutions of (1.1), (1.2), and (1.4), i.e., solutions of

$$\begin{aligned} \alpha u_1 \left[1 - \frac{u_1}{K_1} - \frac{\beta u_2}{1 + m u_3}\right] &= 0, \\ u_2 \left[\delta\left(1 - \frac{u_2}{K_2}\right) - \eta u_1\right] &= 0, \\ \gamma u_3 \left[1 - \frac{u_3}{L_0 + l u_1}\right] &= 0. \end{aligned} \tag{4.2}$$

The all possible solutions of (4.2) (see [9]) have the forms of

$$\begin{aligned} &E_0 = (0, 0, 0), && E_1 = (K_1, 0, 0), \\ &E_2 = (0, K_2, 0), && E_3 = (0, 0, L_0), \\ &E_4 = (0, K_2, L_0), && E_5 = \left(\frac{\delta K_1(\beta K_2 - 1)}{K_1 K_2 \beta\eta - \delta}, \frac{K_2(\eta K_1 - \delta)}{K_1 K_2 \beta\eta - \delta}, 0\right), \\ &E_6 = (K_1, 0, L_0 + l K_1), && E_7 = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3), \end{aligned}$$

$$\tilde{u}_i > 0, \ i = 1, 2, 3,$$

where $E_0, \ldots, E_4$, E_6 always exist. E_5 exists if

$$(\beta K_2 - 1)(\eta K_1 - \delta) > 0, \tag{4.3}$$

$$(\beta K_2 - 1)(K_1 K_2 \beta\eta - \delta) > 0. \tag{4.4}$$

While the condition under which E_7 exists will be given later.

Now analyze the stabilities of these equilibrium solutions. Let us rewrite (1.1), (1.2), and (1.4) into an envolution equation in Banach space $Y = \oplus_1^3 C^2(\Omega) \cap L^2(\Omega)$;

$$\frac{dU}{dt} = AU + F(U), \tag{4.5}$$

where A and F are defined by (3.9) and (3.10),

$$D(A) = \left\{U \in Y : \left.\frac{\partial U}{\partial n}\right|_{\partial\Omega} = 0\right\}.$$

Linearizing the right side of (4.5) at E_i, $i=0, 1, \ldots, 6$, respectively, we get

$$M_0(E_0)=\begin{bmatrix} d_1\Delta+\alpha & 0 & 0 \\ 0 & d_2\Delta+\delta & 0 \\ 0 & 0 & d_3\Delta+\gamma \end{bmatrix},$$

$$M_1(E_1)=\begin{bmatrix} d_1\Delta-\alpha & -\alpha\beta K_1 & 0 \\ 0 & d_2\Delta+\delta-\eta K_1 & 0 \\ 0 & 0 & d_3\Delta+\gamma \end{bmatrix},$$

$$M_2(E_2)=\begin{bmatrix} d_1\Delta+\alpha-\alpha\beta K_2 & 0 & 0 \\ -\eta K_2 & d_2\Delta-\delta & 0 \\ 0 & 0 & d_3\Delta+\gamma \end{bmatrix},$$

$$M_3(E_3)=\begin{bmatrix} d_1\Delta+\alpha & 0 & 0 \\ 0 & d_2\Delta+\delta & 0 \\ \gamma l & 0 & d_3\Delta-\gamma \end{bmatrix},$$

$$M_4(E_4)=\begin{bmatrix} d_1\Delta+\alpha-\dfrac{\alpha\beta K_2}{1+mL_0} & 0 & 0 \\ -\eta K_2 & d_2\Delta-\delta & 0 \\ \gamma l & 0 & d_3\Delta-\gamma \end{bmatrix},$$

$$M_5(E_5)=\begin{bmatrix} d_1\Delta-\dfrac{\alpha u_1^*}{K_1} & -\alpha\beta u_1^* & \alpha\beta m u_1^* u_2^* \\ -\eta u_2^* & d_2\Delta-\dfrac{\delta}{K_2}u_2^* & 0 \\ 0 & 0 & d_3\Delta+\gamma \end{bmatrix},$$

$$u_1^*=\frac{\delta K_1(\beta K_2-1)}{K_1K_2\beta\eta-\delta}, \qquad u_2^*=\frac{K_2(\eta K_1-\delta)}{K_1K_2\beta\eta-\delta},$$

$$M_6(E_6)=\begin{bmatrix} d_1\Delta-\alpha & -\dfrac{\alpha\beta K_1}{1+m(L_0+lK_1)} & 0 \\ 0 & d_2\Delta+\delta-\eta K_1 & 0 \\ \gamma l & 0 & d_3\Delta-\gamma \end{bmatrix}.$$

Denote

$$P_i(\lambda, \Delta)=\det(\lambda I-M_i),$$

$$\Lambda_i=\{\lambda : P_i(\lambda, \mu)=0, \text{ for some } \mu\in\sigma(\Delta)\}, \qquad i=0, 1, \ldots, 7,$$

where $\sigma(\Delta)$ is the point spectrum of Δ with homogeneous Neumann boundary condition. It can be shown that [2]

$$\sigma(M_i) \subset \Lambda_i, \qquad i = 0, 1, ..., 7.$$

Recall that $\sigma(\Delta)$ is an infinite but discrete set of simple real eigenvalues bounded from above, i.e.,

$$0 = \mu_0 > \mu_1 > \mu_2 > \cdots > \mu_n > \cdots.$$

Clearly, E_0, E_1, E_2, E_3, and E_5 are unstable since the corresponding $P_i(\eta, \mu)$, $i = 0, 1, 2, 3, 5$, has at least one positive root for $\mu_0 = 0 \in \sigma(\Delta)$.

E_4 is linearly stable if

$$1 - \frac{\beta K_2}{1 + mL_0} < 0 \tag{4.6}$$

and is unstable if

$$1 - \frac{\beta K_2}{1 + mL_0} > 0. \tag{4.7}$$

E_6 is linearly stable if

$$\delta - \eta K_1 < 0 \tag{4.8}$$

and is unstable if

$$\delta - \eta K_1 > 0. \tag{4.9}$$

We have proved

THEOREM 4.3. (i) *Nonnegative equilibrium solution E_5 exists if* (4.3) *and* (4.4) *hold.*

(ii) *Neither of E_0, E_1, E_2, E_3, and E_5 is stable.*

(iii) *E_4 is linearly stable under* (4.6) *and is unstable under* (4.7).

(iv) *E_6 is linearly stable under* (4.8) *and is unstable under* (4.9).

Finally, discuss the conditions of the existence and the stability for $E_7 = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, $\tilde{u}_i > 0$, $i = 1, 2, 3$.

THEOREM 4.4. (*Rai et al.* [9]).

(i) *If $1 + mL_0 = \beta K_2$ and $\eta K_1 \geq \delta$, then E_7 does not exist.*

(ii) *If $1 + mL_0 = \beta K_2$ and $\eta K_1 < \delta$, then E_7 exists uniquely and is given by*

$$\tilde{u}_1 = K_1 - (\delta - \eta K_1)\,\beta K_2/ml\delta,\ \tilde{u}_2 = K_2(\delta - \eta \tilde{u}_1)/\delta,\ \tilde{u}_3 = L_0 + l\tilde{u}_1.$$

(iii) *If* $1+mL_0-\beta K_2>0$, *then* E_7 *is given uniquely by the positive value of* $\tilde{u}_1=\{\tau\pm[\tau^2\pm 4ml\delta^2K_1(1+mL_0-\beta K_2)]^{1/2}\}/2ml\delta$, $\tilde{u}_2=K_2\delta^{-1}(\delta-\eta\tilde{u}_1)$, $\tilde{u}_3=L_0+l\tilde{u}_1$, *where*

$$\tau=ml\delta K_1+\beta\eta K_1K_2-\delta(1+mL_0) \tag{4.10}$$

provided $\tilde{u}_1<\delta/\eta$.

(iv) *If* $1+mL_0-\beta K_2<0$ *and* $\tau\leqslant 0$, *then* E_7 *does not exist.*

(v) *If* $1+mL_0-\beta K_2<0$ *and* $\tau>0$, *then* E_7 *does not exist, exists uniquely, or has two possible values according as* $\tau^2+4ml\delta^2K_1(1+mL_0-\beta K_2)$ *is negative, zero, or positive, and in the latter two cases* $\tilde{u}_1<\delta/\eta$, *where* $\tilde{u}$, $\tilde{u}_2$, $\tilde{u}_3$ *are given as in* (iii).

Suppose there exists $E_7=(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)$ with $\tilde{u}_i>0$, $i=1,2,3$. Linearizing at E_7 we get

$$M_7(E_7)=\begin{bmatrix} d_1\Delta-\dfrac{\alpha\tilde{u}_1}{K_1} & -\dfrac{\alpha\beta\tilde{u}_1}{1+m\tilde{u}_3} & \dfrac{\alpha\beta m\tilde{u}_1\tilde{u}_2}{(1+m\tilde{u}_3)^2} \\ -\eta\tilde{u}_2 & d_2\Delta-\dfrac{d\tilde{u}_2}{K_2} & 0 \\ \gamma l & 0 & d_3\Delta-\gamma \end{bmatrix}.$$

Now, let us analyze the spectrum of $M_7(E_7)$. Put

$$P_7=\lambda^3+a_1\lambda^2+a_2\lambda+a_3 \qquad \text{for} \quad \text{some } \mu\in\sigma(\Delta).$$

Then

$$a_1=\frac{\alpha\tilde{u}_1}{K_1}+\frac{\delta\tilde{u}_2}{K_2}+\gamma-\mu(d_1+d_2+d_3)>0,$$

$$\begin{aligned} a_2={}&\frac{\tilde{u}_1\tilde{u}_2}{K_1K_2}\left(\alpha\delta+\frac{\alpha\gamma K_2}{\tilde{u}_2}+\frac{\gamma\delta K_1}{\tilde{u}_1}\right)-\frac{\alpha\beta l\gamma m\tilde{u}_1\tilde{u}_2}{(1+m\tilde{u}_3)^2}-\frac{\alpha\beta\eta\tilde{u}_1\tilde{u}_2}{1+m\tilde{u}_3} \\ &+\mu^2(d_1d_2+d_1d_3+d_2d_3)-\mu\left[\gamma(d_1+d_2)+\frac{\alpha\tilde{u}_1}{K_1}(d_2+d_3)\right. \\ &\left.+\frac{\delta\tilde{u}_2}{K_2}(d_1+d_3)\right], \end{aligned}$$

$$a_3=b_1+b_2,$$

where

$$b_1 = \frac{\alpha\gamma\tilde{u}_1\tilde{u}_2}{K_1K_2(1+m\tilde{u}_3)^2}\{\delta m^2l^2\tilde{u}_1^2 + (1+mL_0)[2ml\delta\tilde{u}_1 + \delta(1+mL_0)$$

$$-\beta\eta K_1K_2] - ml\beta\delta K_1K_2\}, \tag{B1}$$

$$b_2 = -\mu^3 d_1d_2d_3 + \mu^2\left(\gamma d_1d_2 + \frac{\alpha\tilde{u}_1}{K_1}d_2d_3 + \frac{\delta\tilde{u}_2}{K_2}d_1d_3\right)$$

$$-\mu\left[d_1\gamma\frac{\delta\tilde{u}_2}{K_2} + d_2\gamma\frac{\alpha\tilde{u}_1}{K_1} + d_3\frac{\alpha\delta\tilde{u}_1\tilde{u}_2}{K_1K_2} - d_3\frac{\alpha\beta\eta\tilde{u}_1\tilde{u}_2}{1+m\tilde{u}_3}\right.$$

$$\left.-d_2\frac{l\gamma\alpha\beta m\tilde{u}_1\tilde{u}_2}{(1+m\tilde{u}_3)^2}\right]. \tag{B2}$$

Here, the fact that $\mu \leqslant 0$ for $\mu \in \sigma(\Delta)$ and that

$$\left(1-\frac{\tilde{u}_1}{K_1}\right) - \frac{\beta\tilde{u}_2}{1+m\tilde{u}_3} = 0,$$

$$\delta\left(1-\frac{\tilde{u}_2}{K_2}\right) - \eta\tilde{u}_1 = 0, \tag{4.11}$$

$$1 - \frac{\tilde{u}_3}{L_0 + l\tilde{u}_1} = 0, \qquad x \in \Omega$$

are used.

Assume

$$\beta K_2 \leqslant 1 + mL_0, \tag{4.12}$$

$$\eta K_1 < \delta. \tag{4.13}$$

Then

$$\tau^2 + 4ml\delta^2K_1(1+mL_0-\beta K_2) > 0, \qquad \tilde{u}_1 < K_1 < \delta/\eta,$$

$$ml\delta\tilde{u}_1 \geqslant \tau,$$

thus

$$\begin{aligned}(1+mL_0)&[2ml\delta\tilde{u}_1 + \delta(1+mL_0) - \beta\eta K_1K_2] - ml\beta K_1K_2\\ &\geqslant (1+mL_0)[\tau + \delta(1+mL_0) - \beta\eta K_1K_2] - ml\beta K_1K_2\\ &= (1+mL_0)\,ml\delta K_1 - ml\beta\delta K_1K_2 \geqslant m^2l\delta K_1L_0 > 0\end{aligned}$$

and hence $b_1 > 0$.

On the other hand, due to (4.11) we have that the last term of (B2) equals

$$-\mu\left\{d_1\gamma(\delta-\eta\tilde{u}_1)+\frac{d_2\gamma d\tilde{u}_1(1+mL_0+2ml\tilde{u}_1-lmK_1)}{K_1(1+mL_0+ml\tilde{u}_1)}\right.$$
$$\left.+d_3\alpha\tilde{u}_1\left(\frac{\delta}{K_1}-\eta\right)\right\}.$$

It can be shown that $b_2\geqslant 0$ if

$$1+mL_0+2ml\tilde{u}_1-lmK_1\geqslant 0. \tag{4.14}$$

Note

$$1+mL_0+2ml\tilde{u}_1-lmK_1\geqslant 1+mL_0+\frac{\tau}{\delta}-lmK_1$$
$$=1+mL_0+mlK_1+\frac{\beta\eta}{\delta}K_1K_2-(1+mL_0)-lmK_1$$
$$=\frac{\beta\eta}{\delta}K_1K_2>0.$$

This means (4.14) is true. So, $a_3>0$ under assumptions (4.12) and (4.13).

By a computation we know [9]

$$a_1a_2-a_3=c_1+c_2,$$

where

$$c_1=\frac{m^2l^2\alpha\gamma\tilde{u}_1^3}{K_1^2+(1+m\tilde{u}_3)^2}(\alpha u_1^2+\gamma K_2)+\frac{\alpha\gamma\tilde{u}_1}{\delta K_1(1+m\tilde{u}_3)}$$
$$\times\left\{\left[(1+mL_0)(2ml\delta\tilde{u}_1+\delta(1+mL_0))-ml\beta\delta K_1K_2\left(\frac{\alpha\tilde{u}_1}{K_1}+\gamma\right)\right.\right.$$
$$\left.\left.+ml\beta\eta K_1K_2\tilde{u}_1\left(\frac{\alpha\tilde{u}_1}{K_1}+\frac{\delta\tilde{u}_2}{K_2}+\gamma\right)+\beta\delta\eta K_1\tilde{u}_2(1+mL_0)\right]\right\},$$

$$c_2=-\mu^3(d_1+d_2+d_3)(d_1d_2+d_1d_3+d_2d_3)$$
$$+\mu^2\left\{(d_1+d_2+d_3)\left[\gamma(d_1+d_2)+\frac{\delta\tilde{u}_2}{K_2}(d_1+d_3)+\frac{\alpha\tilde{u}_1}{K_1}(d_2+d_3)\right]\right.$$

$$+\left(\frac{\alpha\tilde{u}_1}{K_1}+\frac{\delta\tilde{u}_2}{K_2}+\gamma\right)(d_1d_2+d_1d_3+d_2d_3)-\mu\left\{\left(\frac{\alpha\tilde{u}_1}{K_1}+\frac{\delta\tilde{u}_2}{K_2}+\gamma\right)\right.$$

$$\times\left[\gamma(d_1+d_2)+\frac{\alpha\tilde{u}_1}{K_1}(d_2+d_3)+\frac{\delta\tilde{u}_2}{K_2}(d_1+d_3)\right]+(d_1+d_2+d_3)$$

$$\left.\times\left[\frac{\tilde{u}_1\tilde{u}_2}{K_1K_2}\left(\alpha\delta+\frac{\alpha\gamma K_2}{\tilde{u}_2}+\frac{\delta\gamma K_1}{\tilde{u}_1}\right)-\frac{\alpha\beta\gamma l\tilde{u}_1\tilde{u}_2}{(1+m\tilde{u}_3)^2}-\frac{\alpha\beta\eta\tilde{u}_1\tilde{u}_2}{1+m\tilde{u}_3}\right]\right\}$$

$$+\mu^3d_1d_2d_3-\mu^2\left(\gamma d_1d_2+\frac{\alpha\tilde{u}_1}{K_1}d_2d_3+\frac{\delta\tilde{u}_2}{K_2}d_1d_3\right)$$

$$+\mu\left\{d_1\gamma\frac{\delta\tilde{u}_2}{K_2}+d_3\frac{\alpha\delta\tilde{u}_1\tilde{u}_2}{K_1K_2}-d_3\frac{\eta\alpha\beta\tilde{u}_1\tilde{u}_2}{1+m\tilde{u}_3}+d_2\gamma\frac{\alpha\tilde{u}_1}{K_1}-d_2\frac{l\gamma\alpha\beta m\tilde{u}_1\tilde{u}_2}{(1+m\tilde{u}_3)^2}\right\}.$$

In [9], it had been shown that $c_1>0$. Due to $\mu\leqslant 0$ for $\mu\in\sigma(\Delta)$, it is not difficult to check that $c_2\geqslant 0$. Hence

$$a_1a_2-a_3=c_1+c_2>0.$$

By using the Routh–Hurwitz criteria and Theorem 4.4, we obtain our last theorem

THEOREM 4.5. *If* (4.12) *and* (4.13) *hold, then* E_7 *exists uniquely. Moreover, it is linearly stable. Particularly, if* $\beta K_2\leqslant 1$, *then* E_7 *is linearly stable for all mutualism constant* $m\geqslant 0$.

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