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Asymptotic Duality over Closed Convex Sets*†

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The asymptotic duality theory of linear programming over closed convex cones [4] is extended to closed convex sets, by embedding such sets in appropriate cones. Applications to convex programming and to approximation theory are given.

INTRODUCTION

For any set K in R^n and any real α, let $K_\alpha^* = \{y \in R^n : x \in K \Rightarrow (y, x) \geq \alpha\}$. Only the two values $\alpha = 0, -1$ are used in this paper, and for convenience the subscript $\alpha = 0$ will be deleted. The following pair of problems,

(I.C) $\quad \sup(c, x) \quad$ s.t. $\quad Ax = b, \quad x \in C,$

(II.C*) $\quad \inf(b, y) \quad$ s.t. $\quad A^T y - c \in C^*,$

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with given $A \in R^{m \times n}$, $b \in R^m$, $c \in R^n$, and a closed convex cone $C \subset R^n$, were given an asymptotic duality theory in [4, 9], generalizing the classical duality theorem of linear programming.

This asymptotic duality theory is extended here to problem pairs of the type

$$\text{(I)} \qquad \sup(c, x) \quad \text{s.t.} \quad Ax = b, \quad x \in K,$$

$$\text{(II)} \qquad \inf\{(b, y) + \eta\} \quad \text{s.t.} \quad A^T y - c \in \eta K_{-1}^*, \quad \eta \geqslant 0,$$

where K is a closed convex set in R^n containing 0, and A, b, c are as above.

This is done by transforming (I) to an equivalent problem of type (I.C) (see Theorem 1) whose dual problem is equivalent to (II) (see Theorem 2). The duality theorem of Section 3 reduces to that of [4] if K is a cone. Applications to convex programming and to approximation problems are given in Section 4. The results of this paper and of [4] which are stated for R^n can be extended to complex spaces using the solvability theorem of [2] and to locally convex spaces using the solvability theorem of [3]. Indeed the pair of problems (I) and (II) in a different but equivalent form were studied by Fan [12] in locally convex spaces concentrating on consistent problems and duality situations with equal optimal functional values, which correspond to duality state 1 in our classification. Our results have been applied in [13, 14] to classes of generalized moment and approximation problems.

Notations and Definitions

K — *a* closed convex set in R^n, containing the origin.[1] $A \in R^{m \times n}$, $b \in R^m$, $c \in R^n$.

The following definitions are analogous to [4, p. 318].

$F(I) = \{x \in K : Ax = b\}$, the set of feasible solutions of problem (I). (1)

$AF(I) = \{\{x_k : k = 1, 2, \ldots\} \subset K : \lim_k Ax_k = b\}$, the set of asymptotically feasible solutions of problem (I). (2)

$AF(I)$ is a set of sequences. If $F(I) \neq \varnothing$, then it can be embedded naturally in $AF(I)$.

$$F(II) = \{[y, z] \in R^m \times R^n : z \in \eta K_{-1}^*, \eta \geqslant 0, A^T y - z = c\} \tag{3}$$

$$AF(II) = \left\{\left[\begin{pmatrix} y_k \\ \eta_k \end{pmatrix}, \begin{pmatrix} z_k \\ \zeta_k \end{pmatrix}\right] \in R^{m+1} \times R^{n+1} : z_k \in \zeta_k K_{-1}^*, \eta_k \geqslant 0, \right.$$
$$\zeta_k \geqslant 0 \text{ for } k = 1, 2, \ldots, \tag{4}$$

[1] No generality is lost by assuming $0 \in K$, since this is guaranteed by a translation which preserves the form (I).

and

$$\lim_k (A^T y_k - z_k) = c, \lim_k(\eta_k - \zeta_k) = 0\Big\}.$$

DEFINITIONS. The following terminology is drawn from [4] and defines the possible states for problems (I) and (II) in terms of consistency and boundedness:

(i) The problem (I) [(II)] is

CONS (consistent) if $F(I) \neq \varnothing$ [$F(II) \neq \varnothing$],
INC (inconsistent) if $F(I) = \varnothing$ [$F(II) = \varnothing$],
AC (asymptotically consistent) if $AF(I) \neq \varnothing$ [$AF(II) \neq \varnothing$],
SINC (strongly inconsistent) if $AF(I) = \varnothing$ [$AF(II) = \varnothing$].

(ii) Let (I) [(II)] be AC. Then (I) [(II)] is PAC (properly AC) if

$$\exists\{x_k\} \in AF(I) \ni \lim_k \sup(c, x_k) > -\infty$$

$$\left[\exists \left\{\begin{pmatrix} y_k \\ \eta_k \end{pmatrix}, \begin{pmatrix} z_k \\ \zeta_k \end{pmatrix}\right\} \in AF(II) \ni \lim_k \inf(b, y_k) + \eta_k < \infty\right].$$

Otherwise, it is IAC (improperly AC)

(iii) Let (I) [(II)] be CONS. Then it is BD (bounded) if

$$\sup\{(c, x) : x \in F(I)\} < \infty \qquad [\inf\{(b, y) + \eta : [y, z] \in F(II)\} > -\infty].$$

Otherwise it is UBD (unbounded).

(iv) Let (I) [(II)] be PAC. Then it is ABD (asymptotically BD) if

$$\sup\{\lim_k \sup(c, x_k) : \{x_k\} \in AF(I)\} < \infty$$

$$\left[\inf \left\{\lim_k \inf(b, y_k) + \eta_k : \left[\begin{pmatrix} y_k \\ \eta_k \end{pmatrix}, \begin{pmatrix} z_k \\ \zeta_k \end{pmatrix}\right] \in AF(II)\right\} > -\infty\right].$$

Otherwise it is AUBD (asymptotically UBD).

(v) A *duality state* of the pair of problems (I) and (II) is a pair of states, one of problem (I) and the second of problem (II).

Using the states in (i)–(iv), there are 49 duality states of which at most 11 are possible (see Theorem 3 and the remarks following it).

1. The Associated Cone $C(K)$

For a closed convex set $K \subset R^n$ with $0 \in K$, let the associated cone $C(K)$ in R^{n+1} be defined by

$$C(K) = \left\{ \begin{pmatrix} \lambda x \\ \lambda \end{pmatrix} : \lambda \geqslant 0, x \in K \right\}.$$

$C(K)$ is a convex cone, but not necessarily closed.

Let $r(K)$ denote the union of all rays $\{rx : r \geqslant 0,\ x \in K\}$ lying in K. $r(K) \neq \varnothing$ since $0 \in K$. $r(K)$ is the maximal closed convex cone contained in K.

The closure of $C(K)$, $cl\{C(K)\}$, is now given:

Lemma 1.

$$cl\{C(K)\} = C(K) \cup \begin{pmatrix} r(K) \\ 0 \end{pmatrix}.$$

Proof.

(i) $$C(K) \cup \begin{pmatrix} r(K) \\ 0 \end{pmatrix} \subset cl\{C(K)\}.$$

Since $C(K) \subset cl\{C(K)\}$, it suffices to show that

$$\begin{pmatrix} r(K) \\ 0 \end{pmatrix} \subset cl\{C(K)\}. \tag{5}$$

Let $x \in r(K)$, and define

$$x_k = kx, \qquad \lambda_k = \frac{1}{k} \qquad (k = 1, 2,...).$$

Then

$$\begin{pmatrix} \lambda_k x_k \\ \lambda_k \end{pmatrix} \to \begin{pmatrix} x \\ 0 \end{pmatrix}, \qquad \text{proving (5).}$$

(ii) $$cl\{C(K)\} \subset C(K) \cup \begin{pmatrix} r(K) \\ 0 \end{pmatrix}.$$

Let $x_k \in K$, $\lambda_k \geqslant 0$ $(k = 1, 2,...)$, and let

$$\begin{pmatrix} \lambda_k x_k \\ \lambda_k \end{pmatrix} \to \begin{pmatrix} a \\ \alpha \end{pmatrix}.$$

Now there are two cases:

(a) $\alpha = 0$. In this case $a \in r(K)$, i.e., $ra \in K$ for all $r \geq 0$. Indeed, for any $r > 0$, $r\lambda_k$ is eventually < 1 (since $\lambda_k \to 0$) and $(1 - r\lambda_k)0 + r\lambda_k x_k \in K$ since $0 \in K$.

$$\therefore \quad ra = \lim_k r\lambda_k x_k \in K, \quad \text{since } K \text{ is closed.}$$

$$\therefore \quad \lim \begin{pmatrix} \lambda_k x_k \\ \lambda_k \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \in \begin{pmatrix} r(K) \\ 0 \end{pmatrix}$$

(b) $\alpha > 0$. Here $x_k \to 1/\alpha(a) \in K$.

$$\therefore \quad \lim \begin{pmatrix} \lambda_k x_k \\ \lambda_k \end{pmatrix} = \begin{pmatrix} a \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha((1/\alpha)a) \\ \alpha \end{pmatrix} \in C(K) \qquad \text{Q.E.D.}$$

Remark. The set $cl\{C(K)\}$ may, alternatively, be given by

$$cl\{C(K)\} = C(K) + \begin{pmatrix} r(K) \\ 0 \end{pmatrix}.$$

The set $(cl\{C(K)\})^*$ will now be given in a decomposed form in the following Lemma.

LEMMA 2.

$$(cl\{C(K)\})^* = C(K_{-1}^*) \cup \begin{pmatrix} K^* \\ 0 \end{pmatrix}$$

Proof. It is a standard result in the theory of convex sets that $(cl\{S\})^* = S^*$ for any $S \subset R^n$. Therefore, it suffices to show that

$$\{C(K)\}^* = C(K_{-1}^*) \cup \begin{pmatrix} K^* \\ 0 \end{pmatrix}.$$

We do this in two parts.

(i) $$\{C(K)\}^* \subset C(K_{-1}^*) \cup \begin{pmatrix} K^* \\ 0 \end{pmatrix}.$$

Let $\binom{y}{\eta} \in \{C(K)\}^*$. Then for any $\lambda \geq 0$, $x \in K$

$$\left(\begin{pmatrix} y \\ \eta \end{pmatrix}, \begin{pmatrix} \lambda x \\ \lambda \end{pmatrix} \right) = \lambda(x, y) + \lambda\eta \geq 0, \quad \text{since} \quad \begin{pmatrix} \lambda x \\ \lambda \end{pmatrix} \in C(K).$$

For $\lambda = 1$, we now distinguish the two cases:

$\eta = 0$. $\quad \therefore \; (x, y) \geqslant 0,$

$\therefore \; y \in K^*,$

$\therefore \; \begin{pmatrix} y \\ \eta \end{pmatrix} \in \begin{pmatrix} K^* \\ 0 \end{pmatrix}.$

$\eta > 0$. $\quad \therefore \; (x, y) \geqslant -\eta,$

$\therefore \; \left(x, \frac{y}{\eta}\right) \geqslant -1,$

$\therefore \; \frac{y}{\eta} \in K^*_{-1},$

$\therefore \; \begin{pmatrix} y \\ \eta \end{pmatrix} = \begin{pmatrix} \eta((1/\eta)y) \\ \eta \end{pmatrix} \in C(K^*_{-1}).$

(ii) $$C(K^*_{-1}) \cup \begin{pmatrix} K^* \\ 0 \end{pmatrix} \subset \{C(K)\}^*.$$

Let $\binom{\eta y}{\eta} \in C(K^*_{-1})$, i.e., $\eta \geqslant 0$, $y \in K^*_{-1}$, and $\binom{\lambda x}{\lambda} \in C(K)$, i.e., $\lambda \geqslant 0$, $x \in K$.

$$\therefore \; \left(\begin{pmatrix} \lambda x \\ \lambda \end{pmatrix}, \begin{pmatrix} \eta y \\ \eta \end{pmatrix}\right) = \lambda\eta[(x, y) + 1] \geqslant 0,$$

$$\therefore \; C(K^*_{-1}) \subset \{C(K)\}^*.$$

Finally, let $\binom{y}{0} \in \binom{K^*}{0}$.

$$\therefore \; \left(\begin{pmatrix} \lambda x \\ \lambda \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}\right) = \lambda(x, y) \geqslant 0 \qquad \text{for any} \quad \begin{pmatrix} \lambda x \\ \lambda \end{pmatrix} \in C(K)$$

$$\therefore \; \begin{pmatrix} K^* \\ 0 \end{pmatrix} \subset \{C(K)\}^*.$$ Q.E.D.

Alternatively, $(cl\{C(K)\})^*$ may be given by

$$(cl\{C(K)\})^* = C(K^*_{-1}) + \begin{pmatrix} K^* \\ 0 \end{pmatrix}.$$

2. The Equivalent Linear Programs over Closed Convex Cones

With problem (I) we associate the following linear program over a closed convex cone:

$$(I.\, cl\{C(K)\}) \sup(c, x) \quad \text{s.t.} \quad \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} b \\ 1 \end{pmatrix}, \; \begin{pmatrix} x \\ \xi \end{pmatrix} \in cl\{C(K)\}.$$

Analogous to (1) and (2), we define, as in [4], the feasibility sets of (I. $cl\{C(K)\}$):

$$F(I.\, cl\{C(K)\}) = \left\{\begin{pmatrix} x \\ 1 \end{pmatrix} \in cl\{C(K)\} : Ax = b\right\}. \tag{6}$$

$$AF(I.\, cl\{C(K)\}) = \left\{\begin{pmatrix} x_k \\ \xi_k \end{pmatrix} \in cl\{C(K)\} : \lim_k \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_k \\ \xi_k \end{pmatrix} = \begin{pmatrix} b \\ 1 \end{pmatrix}\right\}. \tag{7}$$

The following special symbols are used for the optimal values and limiting values of the functionals:

$$s(I) = \sup\{(c, x) : x \in F(I)\}. \tag{8}$$

$$s(I.\, cl\{C(K)\}) = \sup \left\{(c, x) : \begin{pmatrix} x \\ 1 \end{pmatrix} \in F(I.\, cl\{C(K)\})\right\}. \tag{9}$$

$$sls(I) = \sup \left\{\lim_k \sup (c, x_k) : \{x_k\} \subset AF(I)\right\}. \tag{10}$$

$$sls(I.\, cl\{C(K)\}) = \sup \left\{\lim_k \sup (c, x_k) : \begin{pmatrix} x_k \\ \xi_k \end{pmatrix} \in AF(I.\, cl\{C(K)\})\right\}. \tag{11}$$

The relations between problems (I) and ($I.\, cl\{C(K)\}$) are given in the following theorem:

THEOREM 1. (a) *(I) is* CONS iff *(I. cl{C(K)}) is* CONS, i.e. $F(I.\, cl\{C(K)\}) \neq \emptyset$ *in which case*

$$s(I) = s(I.\, cl\{C(K)\}). \tag{12}$$

(b) *(I) is* AC iff *(I. cl{C(K)}) is* AC, i.e., $AF(I.\, cl\{C(K)\}) \neq \emptyset$, *in which case*

$$sls(I) = sls(I.\, cl\{C(K)\}). \tag{13}$$

Proof. (a) Follows from

$$x \in F(I) \quad \text{iff} \quad \begin{pmatrix} x \\ 1 \end{pmatrix} \in F(I.\, cl\{C(K)\}),$$

which is proved as follows:

Only if: Obvious from (1) and (6).

If: $\binom{x}{1} \in cl\{C(K)\}$ implies, by Lemma 1, that $\binom{x}{1} \in C(K)$, i.e., $x \in K$.

(b) Follows from the fact that, for any sequence $\xi_k \to 1$,

$$\left\{\left(\frac{x_k}{\xi_k}\right)\right\} \in AF(I) \quad \text{iff} \quad \left\{\begin{pmatrix} x_k \\ \xi_k \end{pmatrix}\right\} \in AF(I.\, cl\{C(K)\}).$$

Only if: Obvious.

If: $\{\binom{x_k}{\xi_k}\} \in AF(I.\, cl\{C(K)\})$ implies that $\xi_k \to 1$, and, eventually,

$$\left\{\binom{\xi_k(x_k/\xi_k)}{\xi_k}\right\} \in C(K), \qquad \text{by Lemma 1,}$$

which proves that, eventually, $\{(x_k/\xi_k)\} \subset K$. Also from (7) it follows that $\lim_k Ax_k = b$, proving

$$\left\{\left(\frac{x_k}{\xi_k}\right)\right\} \subset AF(I). \qquad \text{Q.E.D.}$$

Theorem 1 justifies calling problems (I) and $(I.\, cl\{C(K)\}$ *equivalent*. This means that one problem is in a given state iff the other problem is precisely in the same state, with equal values and limiting values of the functionals.

Similarly, with problem (II) we associate a linear program over a closed convex cone, which is shown in Theorem 2 below to be equivalent to (II). This problem is the dual problem of $(I.\, cl\{C(K)\}$, in the sense of [4], denoted by

$$(II.\, (cl\{C(K)\})^*) \ \inf\left(\binom{b}{1}, \binom{y}{\eta}\right) \quad \text{s.t.} \quad \begin{pmatrix} A^T & 0 \\ 0 & 1 \end{pmatrix}\binom{y}{\eta} - \binom{c}{0} \in (cl\{C(K)\})^*.$$

The feasibility sets of $(II.\, (cl\{C(K)\})^*)$ are defined analogously to (3) and (4):

$$F(II.\, (cl\{C(K)\})^*) = \left\{[y, z] \in R^m \times R^n : \binom{z}{\zeta} \in (cl\{C(K)\})^* \text{ for some } \zeta, \right.$$
$$\left. \text{and } A^Ty - z = c\right\}. \qquad (14)$$

$$AF(II.\, (cl\{C(K)\})^*) = \left\{\left[\binom{y_k}{\eta_k}, \binom{z_k}{\zeta_k}\right] \in R^{m+1} \times R^{n+1} : \quad k = 1, 2,...; \right.$$
$$\left. \binom{z_k}{\zeta_k} \in (cl\{C(K)\})^*; \quad \lim_k [A^Ty_k - z_k] = c \text{ and } \lim_k [\eta_k - \zeta_k] = 0\right\}. \qquad (15)$$

We also need the following problem

$$(II') \qquad \inf(b, y) \quad \text{s.t.} \quad A^Ty - c \in K^*$$

and the corresponding feasibility sets

$$F(II') = \{[y, z] \in R^m \times R^n : z \in K^* \quad \text{and} \quad A^Ty - z = c\}. \qquad (16)$$

$$AF(II') = \left\{\left[\binom{y_k}{\eta_k}, \binom{z_k}{0}\right] \in R^{m+1} \times R^{n+1} : \quad k = 1, 2,..., \right.$$
$$\left. \binom{z_k}{0} \in \binom{K^*}{0}, \quad \lim_k [A^Ty_k - z_k] = c \text{ and } \lim_k \eta_k = 0\right\}. \qquad (17)$$

The feasibility sets defined above satisfy, by Lemma 2,

$$F(II.(cl\{C(K)\})^*) = F(II) \cup F(II'), \tag{18}$$

and

$$AF(II.\,(cl\{C(K)\})^*) = AF(II) \cup AF(II'). \tag{19}$$

Analogously to (8)–(11), we denote

$$i(II) = \inf\{(b, y) : [y, z] \in F(II)\}. \tag{20}$$

$$i(II') = \inf\{(b, y) : [y, z] \in F(II')\}. \tag{21}$$

$$i(II.\,(cl\{C(K)\})^*) = \inf\left\{(b, y) + \eta : \left[\begin{pmatrix} y \\ \eta \end{pmatrix}, \begin{pmatrix} z \\ \eta \end{pmatrix}\right] \in F(II.\,(cl\{C(K)\})^*)\right\}. \tag{22}$$

$$ili(II) = \inf\left\{\liminf_k (b, y_k) + \eta_k : \left[\begin{pmatrix} y_k \\ \eta_k \end{pmatrix}, \begin{pmatrix} z_k \\ \zeta_k \end{pmatrix}\right] \in AF(II)\right\}. \tag{23}$$

$$ili(II') = \inf\left\{\liminf_k (b, y_k) : \left[\begin{pmatrix} y_k \\ \eta_k \end{pmatrix}, \begin{pmatrix} z_k \\ 0 \end{pmatrix}\right] \in AF(II')\right\}. \tag{24}$$

$$ili(II.\,(cl\{C(K)\})^*) = \inf\left\{\liminf_k (b, y_k) + \eta_k : \left[\begin{pmatrix} y_k \\ \eta_k \end{pmatrix}, \begin{pmatrix} z_k \\ \zeta_k \end{pmatrix}\right] \in AF(II.\,(cl\{C(K)\})^*)\right\}. \tag{25}$$

Problem (II') is introduced because, by Lemma 2, solving $(II.\,(cl\{C(K)\})^*)$ requires solving problems (II) and (II') and taking the infimum of their infima.

In proving that problems (II) and $(II.\,(cl\{C(K)\})^*)$ are equivalent, in the sense of Theorem 2 below, we show that in fact problem (II') need not be considered, i.e., (II) is CONS if (II') is CONS, in which case

$$i(II) \leqslant i(II') \tag{26}$$

and (II) is AC if (II') is AC, in which case

$$ili(II) \leqslant ili(II'). \tag{27}$$

THEOREM 2. *Problems* (II) *and* $(II.\,(cl\{C(K)\})^*)$ *satisfy the following statements*:

(a) (II) *is* CONS *iff* $(II.\,(cl\{C(K)\})^*)$ *is* CONS, *in which case*

$$i(II) = i(II.\,(cl\{C(K)\})^*). \tag{28}$$

(b) (II) *is* AC *iff* $(II.\,(cl\{C(K)\})^*)$ *is* AC, *in which case*

$$ili(II) = ili(II.\,(cl\{C(K)\})^*) \tag{29}$$

Proof. (a) From (3), (16), and the fact that $K^* \subset K^*_{-1}$, it follows that

$$F(II') \subset F(II). \tag{30}$$

(30) proves (26) which together with (18) proves (a).

(b) We show now that

$$(II')\text{AC} \Rightarrow (II)\text{AC}, \quad \text{i.e.,} \tag{31}$$
$$AF(II') \neq \varnothing \Rightarrow AF(II) \neq \varnothing$$

and also (27).

Let $\{[\binom{y_k}{\eta_k}, \binom{z_k}{0}]\} \in AF(II')$ and let $\lim_k \inf(b, y_k) = \beta$. (31) follows, since, for any sequence $\{\zeta_k > 0 : k = 1, 2,...\}$, the sequence

$$\left\{\left[\begin{pmatrix} y_k \\ \zeta_k \end{pmatrix}, \begin{pmatrix} z_k \\ \zeta_k \end{pmatrix}\right]\right\} \tag{32}$$

is in $AF(II)$. To prove this, we rewrite (32) as

$$\left\{\left[\begin{pmatrix} y_k \\ \zeta_k \end{pmatrix}, \begin{pmatrix} \zeta_k(z_k/\zeta_k) \\ \zeta_k \end{pmatrix}\right]\right\},$$

which is in AF(II) because $(z_k/\zeta_k) \in K^*_{-1}$ since $z_k \in K^* \subset K^*_{-1}$ and K^* is a cone.

A sequence (32) will be constructed such that $\lim_k \inf(b, y_k) + \zeta_k = \beta$, proving (27). Such a sequence is (32) with

$$\zeta_k = \frac{1}{k} \text{ if } \beta \text{ is finite.}$$

$$\zeta_k = 1 \text{ if } \beta \text{ is infinite.}$$

From (19) and (31) it follows that $AF(II. (cl\{C(K)\})^*) \neq \varnothing$ iff $AF(II) \neq \varnothing$, proving the first part of (b). (29) follows then from (19) and (27). Q.E.D.

3. The Duality States

THEOREM 3. *Let $A \in R^{m \times n}$, $b \in R^m$, $c \in R^n$, and let $K \subset R^n$ be a closed convex set, $0 \in K$. Of the 49 mutually exclusive and collectively exhaustive states for the pair of problems*

$$\sup(c, x) \quad \text{s.t.} \quad Ax = b, \qquad x \in K \tag{I}$$
$$\inf(b, y) + \eta \quad \text{s.t.} \quad A^T y - c \in \eta K^*_{-1}, \qquad \eta \geqslant 0, \tag{II}$$

only 11 *are possible, and are those denoted in the table below by positive integers. A zero in the table means that the corresponding state is impossible. Furhtermore,*

(a) $$sls(I) \leqslant i(II) \qquad \text{in states } 1, 2, 3,$$

and there is a sequence $\{x_k\} \in AF(I)$ *such that*

$$\lim_k \sup(c, x_k) = i(II) \qquad \text{in states } 1, 2.$$

(b) $$s(I) \leqslant ili(II) \qquad \text{in states } 1, 5, 7$$

and there is a sequence $\{[(\begin{smallmatrix} y_k \\ \eta_k \end{smallmatrix}), (\begin{smallmatrix} z_k \\ \zeta_k \end{smallmatrix})]\} \in \mathrm{AF}(II)$ *such that*

$$s(I) = \lim_k \inf\{(b, y_k) + \eta_k\} \qquad \text{in states } 1, 5.$$

(c) $$sls(I) = i(II) \geqslant ili(II) = s(I) \qquad \text{in state } 1.$$

TABLE A

II \ I	AC / PAC / CONS / ABD	AC / PAC / CONS / AUBD / BD	AC / PAC / CONS / AUBD / UBD	AC / PAC / INC / ABD	AC / PAC / INC / AUBD	AC / IAC / INC / IAC	SINC / INC / SINC
AC / PAC / CONS / ABD / BD	1	0	0	0	0	0	0
AC / PAC / CONS / AUBD / BD	0	0	0	2	0	0	0
AC / PAC / CONS / AUBD / UBD	0	0	0	0	0	3	4
AC / PAC / INC / ABD	0	5	0	0	0	0	0
AC / PAC / INC / AUBD	0	0	0	0	11	0	6
AC / IAC / INC / IAC	0	0	7	0	0	0	0
SINC / INC / SINC	0	0	8	0	9	0	10

Proof. Follows from the results of [4] and [5] applied to the pair $(I.\ cl\{C(K)\}$ and $(II.\ (cl\{C(K)\})^*)$ which are equivalent to (I) and (II), respectively, by Theorems 1 and 2.

In particular, the possibility of duality states 1–11 is demonstrated by the examples in [4] and [5], since when K is a closed convex cone, problems (I) and (II) are linear programs over closed convex cones. To see this for (II) we observe that K closed convex cone $\Rightarrow K_{-1}^* = K^*$ and $\eta K_{-1}^* = K_{-1}^*$ for all $\eta > 0$.

Parts (a), (b), and (c) follow from Lemmas 1, 2 and 5, respectively, of [4]. Q.E.D.

Remarks. Theorem 3 can be sharpened, i.e., more states can be excluded, when further conditions are imposed on the closed convex set K.

Remark 1. *K bounded* $\Rightarrow$ only states 1 and 4 are possible.

Proof. The boundedness of K implies that int K_{-1}^*, i.e., the interior of K_{-1}^*, is nonempty and $0 \in \text{int}\, K_{-1}^*$[2]. This clearly guarantees that the problem (II) is CONS for any A, c, which excludes all but states 1, 2, 3 and 4 in the Table A. Finally, states 2 and 3 are impossible since K is bounded.

Remark 2. *K bounded and* $0 \in \text{int}\, K$ and, if $a \in R^n$, then

$$\inf\{\eta \geqslant 0 : a \in \eta K\} = \sup\{(a, x) : x \in -K_{-1}^*\}. \tag{33}$$

Proof. Let $A = 0$, $b = 0$, $c = -a$ and let the roles of K, K_{-1}^* be interchanged[3] in the theorem, for which Remark 1 allows only states 1 and 4. By the choice of A, b, the problem (I) is CONS so that state 4 is also excluded. Moreover $F(I) = AF(I)$, resulting in an equality in part (c) of the theorem, thus proving (33).

Recognizing that $-K_{-1}^*$ is the *polar* (see [6, 16]) or the *dual* [15] set of K, (33) expresses the duality between the "distance" and "support" functions of K (see [11, p. 55]).

Remark 3. *K polyhedral* $\Rightarrow$ only states 1, 4, 8, and 10 are possible.

Proof. Since K is polyhedral, it follows that (I) is either CONS or SINC. By a theorem of Klee [16, Theorem 2.11], K_{-1}^* is also polyhedral, implying that problem (II) is also either CONS or SINC.

4. Applications

Two applications of the above duality theorem, to convex programming and to approximation problems, will now be given.

Convex programming

Consider the following problem:

$$(CP) \qquad \inf f(x) \quad \text{s.t.} \quad Ax = b, \qquad x \in K,$$

[2] For example, [6] where this is actually proved for the "polar set" $-K_{-1}^*$.

[3] This interchange is possible since $K = (K_{-1}^*)_{-1}^*$ for any closed convex set K containing 0.

where $f: R^n \to R$ is a convex function, $A \in R^{m\times n}$, $b \in R^m$, and K is a closed convex set, with $0 \in K$. For the set $\tilde{K}$ defined below to contain 0, we also assume that $f(0) \leqslant 0$[4]. The problem (CP) is equivalent to

$$(CP1) \qquad \sup -\xi \quad \text{s.t.} \quad (A\ \ 0)\begin{pmatrix} x \\ \xi \end{pmatrix} = b, \qquad \begin{pmatrix} x \\ \xi \end{pmatrix} \in \tilde{K},$$

where $\tilde{K} = \binom{K}{R} \cap \{\binom{x}{\xi} : f(x) \leqslant \xi\}$ is a closed convex set in R^{n+1}, containing 0. The problem $(CP1)$ is of type (I) and our duality theory assigns to it the following dual problem

$$(DP) \qquad \inf(b, y) + \eta \quad \text{s.t.} \quad \begin{pmatrix} A^T \\ 0 \end{pmatrix} y - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \eta \tilde{K}^*_{-1}, \qquad \eta \geqslant 0.$$

Identifying the problems (CP) and (DP) with (I) and (II), respectively, of the Theorem 3 gives a new duality theorem for convex programming and a complete classification of the 11 possible duality states. The duality of (CP) and (DP) holds if the convexity assumption on f in (CP) is weakened to quasiconvexity; see e.g., [18].

Approximation problems

Let the set K of the theorem be an equilibrated convex body (see e.g., [15, p. 39], and let

$$\| x \|_K = \inf\{\nu \geqslant 0 : x \in \nu K\}$$

be the *associated norm*. Then K^*_{-1} is also equilibrated, and hence it coincides with the *polar* $K' = -K^*_{-1}$ of K, and $\| \ \|_{K^*_{-1}}$ is the dual norm (see, e.g., [15, p. 43]).

By Remark 1, only states 1 and 4 are possible here, and the resulting duality theorem gives the following well-known results in approximation theory:

COROLLARY 1. *Let K be an equilibrated convex body in R^n, $\| \ \|_K$ the associated norm, and let $K' = -K^*_{-1}$ be the polar of K. Then, for any $a \in R^n$,*

$$\| a \|_K = \sup\{(a, x) : x \in K'\}.$$

Proof. Follows from Remark 2. Q.E.D.

COROLLARY 2. *Let K, $\| \ \|_K$ and K' be as in Corollary 1. Then for any $A \in R^{m\times n}$, $b \in R^m$*

$$\inf \| Ax - b \|_K = \sup\{(b, y) : A^T y = 0, y \in K'\}.$$

[4] Like the assumption that $0 \in K$, this assumption can also be satisfied by a translation.

Proof. Follows from Theorem 3, by taking $b = 0$ and then interchanging the roles of A and A^T, of b and c, and of K and K'. Q.E.D.

Corollary 2 is a duality theorem for the best approximate solution, in the sense of $\| \ \|_K$, of the equation $Ax = b$. The analogous result for the distance $d(b, L)$ of a point $b \in R^n$ from a subspace L of R^n is

COROLLARY 3. *Let K, $\| \ \|_K$, and K' be as above, and let L be any subspace of R^n, $L^\perp$ its orthogonal complement. Then for any $b \in R^n$,*

$$d(b, L) = \inf\{\| x - b \|_K : x \in L\} = \sup\{(b, y) : y \in L^\perp \cap K'\}.$$

Proof. Similarly proved. Q.E.D.

The role of duality in approximation theory has been extensively studied (see [1, pp. 124–125; [7; 8; 10; 17, Section 4.2] and their references, for generalizations and extensions of Corollaries 1, 2, and 3, chosen here to demonstrate applications of our duality theorem).

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