Explicit Construction of an Inertial Manifold for a Reaction Diffusion Equation

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Received February 24, 1988; revised May 24, 1988

An inertial manifold is constructed for the scalar reaction–diffusion equation

\[ u_t = vu_{xx} + f(u) \quad \text{with Dirichlet boundary conditions,} \]

where \( f \) in \( C^3(\mathbb{R}) \) satisfies

\[
\begin{align*}
  f(0) &= 0, \quad f'(0) = 1 \\
  \limsup_{|u| \to \infty} u \cdot f(u) &< 0 \\
  f''(u) \cdot u &
\end{align*}
\]

When restricted to this positively invariant, finite-dimensional manifold the flow becomes that of a finite set of ordinary differential equations, termed an inertial form, which captures not only the behavior at infinity, but also, due to the invariance of the manifold, some of the transients as well.


1. INTRODUCTION

In this paper we explicitly construct an inertial manifold for the reaction–diffusion equation

\[ u_t = vu_{xx} + f(u), \quad u(x, 0) = u_0(x), \quad 0 < x < \pi, \]

with Dirichlet boundary conditions, where \( f \) in \( C^3(\mathbb{R}) \) satisfies

\[
\begin{align*}
  f(0) &= 0, \quad f'(0) = 1 \\
  \limsup_{|u| \to \infty} u \cdot f(u) &< 0 \\
  f''(u) \cdot u &
\end{align*}
\]
Properties (1.1a), (1.1b), and (1.1c) ensure that for each $f$ there are unique numbers $r_{\pm}$, with $r_- < 0 < r_+$ and $f(r_{\pm}) = 0$. The cubic function $f(u) = -u(u - r_+)(u - r_-)$ satisfies these properties, for example.

There are several abstract theories for inertial manifolds. In [8, 6] general existence theories are developed which can be applied to a variety of problems. This has been done in [6, 9] for the Kuramoto–Sivashinsky equation, in [6, 19] for the Cahn–Hilliard equation, in [5, 7, 24] for the complex Ginzburg–Landau equation, and in [6] for the nonlocal Burger’s equation all in once space dimension. Essential in each case is the verification of a certain gap condition on the spectrum of an appropriately chosen linear operator, or equivalently a more geometric cone condition on the global attractor. Augmenting this approach with a new idea, the principle of spatial averaging, Mallet-Paret and Sell [15] have estimated the dimension of an inertial manifold for a reaction–diffusion equation in space dimensions two and three with restricted domains. Though general enough to apply to a variety of problems, many of which have complicated attractors, these techniques yield only an estimate, sometimes rather crude, for the dimension of the manifold.

The approach taken here is tailor made for (1.1) in that it uses the specific structure of the global attractor to construct a series of invariant manifolds culminating in the inertial manifold. The advantage is that the dimension of the manifold is the lowest possible as it coincides with that of the attractor. Since similar global attractors correspond to other odd degree polynomial choices for the nonlinearity $f$, a similar construction should also be possible. While such a construction may be possible for reaction–diffusion equations in higher space dimensions, the technique of proof used here will not apply.

The manifold constructed here is also significant for its orientation. Since current estimates on the dimension are obtained while characterizing the manifold as the graph of a function over a finite-dimensional space, they could ignore the possibility of a well-defined inertial manifold of lower dimension, without this property. We show here that the global attractor of which the manifold is a small extension is indeed the graph of a function defined in a linear space of the same dimension.

Equation (1.1) has a rich history. What is perhaps the first stability analysis of equilibrium solutions can be found in [17] under somewhat different assumptions on the nonlinearity $f$. A complete description of the bifurcation of equilibria is given by Chafee and Infante in [3]. The existence of orbits connecting different equilibria has been investigated in [12, 23]. Brunovsky and Fiedler in [2] obtained results regarding the number of zeros of solutions on the stable and unstable manifolds of equilibria. Both Angement [1] and Henry [12] have shown that (1.1) is an example of an infinite-dimensional Morse–Smale system. This last result,
which means that the stable and unstable manifolds of these points meet transversally, is what motivates our geometric approach.

This paper is structured as follows. In Section 2 we introduce notation and describe the bifurcation of the global attractor. We provide a precise statement of the main result in Section 3 along with a brief heuristic argument outlining the global extensions that result in the manifold. The global attractor is shown to be the graph of a function in Section 4 using a technical result which is proved in Section 5 and later used to prove the main result. The proof of the main result, presented in Section 6, uses a combination of global and local extensions linked through a zero number condition. It is the proof of the local extension that is more involved, requiring the next three sections, of considerable more detail. These sections describe the modification of a locally Lipschitz vector field in a Hilbert space of functions outside a neighborhood of an equilibrium. The objective is to connect a given invariant manifold with a linear subspace by creating a new globally Lipschitz vector field. Most of the effort is involved in controlling the size of the Lipschitz constant of the nonlinear term.

2. Preliminaries

Equation (1.1) is often viewed as an ordinary differential equation

\[
\frac{du}{dt} + Au = F(u),
\]

(2.1)

where \( A \), defined by \( Au = -v u_{xx} \) for \( u \in C^2([0, \pi]) \) is extended to \( \mathcal{D}(A) = H_0^1([0, \pi]) \cap H^2([0, \pi]) \), and \( F : L^2([0, \pi]) \to L^2([0, \pi]) \) is defined pointwise by \( F(u)(x) = f(u(x)) \). The solution at time \( t \) denoted \( u(x, t) \) defines a semiflow \( S : \mathcal{D}(A) \times \mathbb{R}^+ \to \mathcal{D}(A) \) by \( S_t u_0(x) = u(x, t) \) [11].

The sign condition on the nonlinearity in (1.1) guarantees that the system is dissipative which we will take to mean that not only are all solutions of (2.1) bounded but they all enter a sufficiently large ball \( \mathcal{B} \) in \( \mathcal{D}(A) \). Under these circumstances the global attractor \( \mathcal{A} \) is defined to be the \( \omega \)-limit set of \( \mathcal{B} \), that is,

\[
\mathcal{A} = \omega(\mathcal{B}) = \bigcap_{\tau \geq 0} \{ S_t(\mathcal{B}) : \tau \geq \tau \}.
\]

This is not to be confused with \( \bigcup_{u_0 \in \mathcal{B}} \omega(u_0) \) which consists solely of the equilibria by virtue of a Lyapunov function

\[
V(u) = \int_0^\pi \left\{ \frac{v}{2} u_x^2 - \int_0^u f(s) \, ds \right\} \, dx.
\]
The advantage of studying (1.1) with this particular choice of $f$ is that the global attractor can be described in great detail. The following theorem summarizes the main results regarding the dynamics of Eq. (1.1).

**Theorem 2.1.** For $(k+1)^{-2} < \nu < k^{-2}$, $k = 1, 2, 3, \ldots$, there exists a $k$ pair(s) of equilibrium solutions $\varphi_0^\pm$, $\varphi_1^\pm$, $\ldots$, $\varphi_k^\pm$ to (1.1). These solutions are the only nontrivial equilibria for (1.1) and they satisfy the following properties:

1. Each $\varphi_j^\pm$ has exactly $j$ simple zeros in $(0, \pi)$, for $0 < j < k$.
2. Each $\varphi_j^\pm$ is hyperbolic, with $\dim W^u(\varphi_j^\pm) = j$ for $0 < j < k$.
3. There exist connecting orbits from the zero solution to $\varphi_j^\pm$, $0 < j < k$, and from $\varphi_i^\pm$ to $\varphi_j^\pm$ if $0 < j < i < k$.
4. The global attractor $\mathcal{A} = \bigcup \{ W^u(\varphi) : \varphi \text{ is an equilibrium} \}$.

A proof of (1) can be found in [3] while the rest appear in [12]. By drawing one of each of the saddle connections one obtains a qualitative picture of the global attractor. Such a picture is provided in Fig. 2.1.

![Fig. 2.1. The bifurcation of the global attractor for (1.1).](image-url)
3. STATEMENT OF THE MAIN RESULT

We begin by making several notions more precise. The basic space is taken as $H = L^2(0, \pi)$ and its norm denoted by simply $\| \cdot \|$. All other norms will have a special designation.

DEFINITION. A finite-dimensional manifold $\mathcal{M} \subset H$ is an inertial manifold for (1.1) if it is positively invariant and exponentially attracting, that is:

1. $S_t \mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$,
2. there exists $\lambda \geq 0$ such that for every $u_0 \in \mathcal{D}(A)$ there exists a $t_0$ and a constant $c \geq 0$ (uniform for $u_0$ in bounded sets) such that, for $t \geq t_0$,

$$\text{dist}(S_t u_0, \mathcal{M}) \leq c \exp(\lambda t).$$

Recall that a set $\mathcal{N} \subset H$ is an $m$-dimensional manifold with boundary if each point of $\mathcal{N}$ possesses a neighborhood homeomorphic to an open set in the half-space $\{(p_1, \ldots, p_m) \in \mathbb{R}^m : p_m \geq 0\}$. The boundary of $\mathcal{N}$, denoted $\partial \mathcal{N}$, consists of those points that belong to the image of $\{(p_1, \ldots, p_m) \in \mathbb{R}^m : p_m = 0\}$ under some local parametrization. The interior of $\mathcal{N}$, denoted $\text{Int}(\mathcal{N})$, is defined to be the complement of the boundary relative to $\mathcal{N}$, that is, $\text{Int}(\mathcal{N}) = \mathcal{N} \setminus \partial \mathcal{N}$. It is easy to confuse the boundary and interior of $\mathcal{N}$ as above with the topological boundary and interior of $\mathcal{N}$ as a subset of $H$. We shall, however, use the terms only in the former sense here. Observe that the global attractor for (1.1) by itself falls short of being an inertial manifold in that it is a manifold with boundary but not a manifold in the usual sense. A subset $\mathcal{N} \subset H$ is said to be locally invariant under $S$ if for each $u_0 \in \mathcal{N}$ there is an $\epsilon > 0$ such that $S_t u_0 \in \mathcal{N}$ for $0 \leq t < \epsilon$. We shall build an inertial manifold by making a series of locally invariant extensions of the attractor beyond its boundary.

A local extension theorem used for the main result requires a gap, however small, in the spectra of the linearizations at each equilibria. Though it is sufficient that this gap occur between the $m$th and $(m + 1)$st eigenvalues, where $m$ is the dimension of the attractor, the following assumption holds for almost all $v$ and greatly simplifies the discussion.

STANDING ASSUMPTION. The parameter $v$ is chosen so that the eigenvalues of the linearization at each equilibria of Eq. (1.1) are simple and nonzero.

We now state precisely the content of the main result.
**Theorem 3.1.** Under the standing assumption there exists an inertial manifold $\mathcal{M}$ for Eq. (1.1) with dimension equal to that of the global attractor.

We first illustrate the construction procedure by outlining it for the case where the global attractor $\mathcal{A}$ is two dimensional. The first task is to extend $\mathcal{A}$ in neighborhoods of $\varphi^+_1$ to locally invariant two-dimensional manifolds $\mathcal{N}^+_1$. The collection of all forward trajectories starting on $\mathcal{N}^+_1$ together with $\mathcal{A}$ forms a single positively invariant two-dimensional manifold with boundary, $\mathcal{M}_1$. All of $\mathcal{A}$ is contained in the interior of $\mathcal{M}_1$ with the exception of the stable equilibria $\varphi^+_0$.

To complete an inertial manifold, local extensions $\mathcal{N}^+_0$ of $\mathcal{M}_1$ are made in small enough neighborhoods of $\varphi^+_0$ so as to be positively invariant. We then take as our inertial manifold $\mathcal{M}_1$ the set $\text{Int}(\mathcal{M}_1 \cup \mathcal{N}^+_0 \cup \mathcal{N}^-_0)$. This sequence of extensions is depicted in Fig. 3.1.

![Diagram](image_url)

*Fig. 3.1.* The construction of $\mathcal{M}$ when $\dim \mathcal{A} = 2.$
4. Global Parametrization

In this section we show that the global attractor is the graph of a function defined within a finite-dimensional linear subspace. In the process we also derive a technical result to be used in the construction of an inertial manifold. Since, for Eq. (1.1) $\mathcal{A}$ is merely the closure of $\mathcal{W}^u(0)$ in $\mathcal{D}(A)$ we need only parametrize the unstable manifold at 0. Since the same method works for the unstable manifolds of the nontrivial equilibria we prove the result in general.

The results in this and the next section hold for the more general class of equations

$$u_t = vu_{xx} + f(x, u), \quad v > 0, \quad (4.1)$$

with boundary conditions

$$u(0, t) \cos \theta - u_x(0, t) \sin \theta = 0$$

$$u(\pi, t) \cos \theta - u_x(\pi, t) \sin \theta = 0,$$

where $\theta \in [0, \pi/2], f: [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is $C^k$, $k \geq 2$. The only other assumption on $f$ is that $uf(xu) < 0$ for large $|u|$ and $x \in [0, \pi]$. Note that for $\theta = 0$, BC reduces to the Dirichlet condition.

We actually have a great deal of freedom in choosing the parametrization. Let $\phi$ be an equilibrium where $\mathcal{W}^u(\phi)$ has dimension, say $m$. It is natural to try to express $\mathcal{W}^u(\phi)$ as the graph of a function of the first $m$ eigenmodes of the linearization at $\phi$ as is well known to be the case for local manifolds. What is somewhat surprising is that this can be done for the unstable manifold of a given equilibrium using the first $m$ eigenmodes associated with any (other) equilibrium.

Our methods include the study of the number of spatial zeroes of differences of solutions, which needs to be defined carefully for an arbitrary continuous function.

**Definition.** For any continuous function $\psi: [0, \pi] \to \mathbb{R}$ the zero number $Z(\psi)$ is defined as the maximal element $n \geq 0$ of $\mathbb{N} \cup \{\infty\}$ such that there is a strictly increasing sequence $0 \leq x_1 < x_2 < \cdots < x_n \leq \pi$ with $\psi(x_j)$ of alternating signs, that is, $\psi(x_j) \cdot \psi(x_{j+1}) < 0$ for $0 \leq j < n$.

In other words, $Z(\psi)$ is the number of times the graph of $\psi(x)$ versus $x$ crosses the $x$-axis between $x = 0$ and $x = \pi$. We will make repeated use of the following direct consequences of the definition:

1. The function $Z: C^0[0, \pi] \to \mathbb{N} \cup \{\infty\}$ is lower semicontinuous.
2. The function $Z$ is constant in a $C^1$-neighborhood of any $C^1$-function with only simple zeros.
A deeper result stemming from the maximum principle states that the zero number is nonincreasing along solutions of a certain class of semilinear equations. This asserts that $Z$ is actually an integer-valued Lyapunov function. Results of this nature can be found in [17, 19, 21]. We now state a version which is not the most general but quite appropriate for our purposes.

**Theorem 4.1.** Let $u(x, t)$ be a solution to
\[
u_t = v u_{xx} + g(x, t, u)
\]
satisfying the boundary condition $BC(\theta)$ and $g: [0, \pi] \times \mathbb{R}^+ \times \mathbb{R}$ is $C^k$, $k > 1$. If $g(x, t, 0) = 0$ for all $(x, t) \in [0, \pi] \times \mathbb{R}^+$ then $Z(u(\cdot, t))$ is a nonincreasing function of $t$.

A proof of Theorem 4.1 can be found in [2].

The zero number plays an essential role in this approach to global parametrization. Since there is no analogue of the zero number for higher space dimension this is inherently a dimension one result. The key is to show that the zero number of the difference of any two elements on $\mathcal{W}^u(\varphi)$ is strictly less than the dimension of the manifold. In other words we prove the following.

**Theorem 4.2.** Let $\varphi$ be a hyperbolic equilibrium of (4.1) with only simple eigenvalues. If $v_0, w_0 \in \mathcal{W}^u(\varphi)$, then
\[
Z(v_0 - w_0) < \dim \mathcal{W}^u(\varphi).
\]

We will need several preliminary results for the proof of Theorem 4.2. The first gives bounds for $Z$ on the span of Sturm-Liouville eigenfunctions. Let $A$ be an arbitrary Sturm-Liouville operator of the form
\[
A: \mathcal{D}(A) \subset L^2(0, \pi) \to L^2(0, \pi)
\]
\[
u \mapsto -u_{xx} - a(x) u
\]
with continuous potential $a$, and boundary condition $BC(\theta)$. The following result can be found in [2].

**Theorem 4.3.** Let $\psi_1, \psi_2, \psi_3, \ldots$ be the eigenfunctions of a Sturm-Liouville operator $A$ such as above. Suppose $A$ has simple eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$. For any $0 \leq j < i < \infty$ and $\alpha_j, \alpha_{j+1}, \ldots, \alpha_i$ in $\mathbb{R}$, we have
\[
j - 1 \leq Z\left(\sum_{k=j}^{i} \alpha_k \psi_k\right) \leq i - 1.
\]
Moreover, if \( \sum_{k=j}^{\infty} \alpha_k^2 < \infty \), then

\[
j - 1 \leq Z \left( \sum_{k=j}^{\infty} \alpha_k \psi_k \right).
\]

We now state a general global parametrization result to be used again in the construction of an inertial manifold.

**Theorem 4.4.** Let \( \psi_1, \psi_2, \psi_3, \ldots \), be the eigenfunctions for a Sturm-Liouville operator \( A \) as in Theorem 4.3. Any set \( \mathcal{N} \in \mathcal{H} \) such that \( Z(v - w) < m \) for all \( v, w \in \mathcal{N} \) can be expressed as the graph of a function,

\[
\Phi: P\mathcal{N} \rightarrow \mathcal{QH},
\]

where \( P \) is the projection onto the first \( m \) eigenfunctions of \( A \) and \( \mathcal{Q} \) is that onto the orthogonal complement.

**Proof.** Suppose there exist elements \( v, w \in \mathcal{N} \) with \( v \neq w \) and \( P_v = P_w \).

It follows that \( v - w \in \mathcal{QH} \) so that

\[
v - w = \sum_{k=m+1}^{\infty} \langle v - w, \psi_k \rangle \psi_k.
\]

Yet by Theorem 4.2 one has \( Z(v - w) \geq m \), a contradiction. Thus \( P \) is one-to-one when restricted to \( \mathcal{N} \), and \( \Phi \) may be defined to be its inverse. \( \square \)

The global parametrization of \( \mathcal{W}^u(\varphi) \) is now an immediate corollary of Theorem 4.2 and Theorem 4.4.

**Corollary 4.5.** If the unstable manifold \( \mathcal{W}^u(\varphi) \) has dimension \( m \), then it can be expressed as the graph of a function

\[
\Phi: P\mathcal{W}^u(\varphi) \rightarrow \mathcal{QH},
\]

where \( P \) is the projection onto the first \( m \) eigenfunctions of \( A \) and \( \mathcal{Q} \) is that onto the orthogonal complement for a Sturm-Liouville operator \( A \) as in Theorem 4.3.

Before giving the formal proof of Theorem 4.2 we argue heuristically. Consider two initial conditions, \( v_0, w_0 \) in \( \mathcal{W}^u(\varphi) \). Since \( S_t v_0 \) and \( S_t w_0 \) are close to \( \varphi \) \( t \ll 0 \), by the tangency of \( \mathcal{W}^u(\varphi) \), the difference, \( S_t v_0 - S_t w_0 \) is close to the unstable manifold of the linearized equation \( \eta_t = \eta_{xx} + f_a(x, \varphi) \eta \) where \( \eta(\cdot, t) \) satisfies BC(\( \theta \)). Let \( m \) be the dimension of \( \mathcal{W}^u(\varphi) \). The linear unstable manifold is spanned by the first \( m \) Sturm-Liouville eigenfunctions \( \psi_1, \ldots, \psi_m \), for the potential \( a(x) = f_a(x, \varphi(x)) \). It follows from Theorem 4.3 that \( Z(p) < m \) for all \( p \) on the linear manifold.
By its proximity to the linear unstable manifold, the difference $S_{t}v_{0} - S_{t}w_{0}$ should also have less than $m$ zeros on $(0, \pi)$ for $t \neq 0$. Following the flow forward in time on $W^{-}\omega(\phi)$, we have that the zero number cannot increase so that $Z(v_{0} - w_{0}) \leq Z(S_{t}v_{0} - S_{t}w_{0}) < m$. To make the argument rigorous we will use the following lemma which is proved in the next section.

**Lemma 4.6.** For all $v_{0}, w_{0} \in W^{-}\omega(\phi)$,

$$
\frac{S_{t}v_{0} - S_{t}w_{0}}{\|S_{t}v_{0} - S_{t}w_{0}\|} \rightarrow \psi_{j}
$$

as $t \rightarrow -\infty$ for some $j \leq \dim W^{-}\omega(\phi)$.

Using this result, we now prove Theorem 4.2.

**Proof of Theorem 4.2.** Let $m = \dim W^{-}\omega(\phi)$ and let $\psi_{1}, ..., \psi_{m}$ be the first $m$ eigenfunctions of the linearization at $\phi$. By virtue of the $C^{1}$-convergence in Lemma 4.6 there exists a time $t_{0} < 0$ such that

$$
Z(S_{t}v_{0} - S_{t}w_{0}) = Z\left(\frac{S_{t}v_{0} - S_{t}w_{0}}{\|S_{t}v_{0} - S_{t}w_{0}\|}\right) = Z(\psi_{j}) = j - 1
$$

for all $t \leq t_{0}$.

Applying Theorem 4.1 we have that the function $Z(S_{t}v_{0} - S_{t}w_{0})$ is non-increasing in $t$. In particular, for $t = 0$ we have

$$
Z(v_{0} - w_{0}) \leq j - 1 \leq m - 1.
$$

5. **Asymptotic Behavior of Solutions**

In this section we determine the asymptotic behavior of differences of solutions to Eq. (4.1). To do this we rely on results in Henry [12] regarding

$$
\eta_{t} = v\eta_{xx} + a(x, t) \eta,
$$

where $\eta(\cdot, t)$ satisfies $BC(\theta)$. The statement requires some special notation. We define a family of Sturm–Liouville operators $\{A(t), t \in \mathbb{R}\}$ on the domains

$$
\mathcal{D}(A(t)) = \{\eta \in H^{2}(0, \pi): \eta \text{ satisfies } BC(\theta)\}
$$

by

$$
A(t): \mathcal{D}(A(t)) \subset L^{2}(0, \pi) \rightarrow L^{2}(0, \pi)
$$

$$
\eta \mapsto -v\eta_{xx} - a(x, t) \eta.
$$
Let us denote
\[
\lim_{t \to -\infty} A(t) = A_-, \quad \lim_{t \to +\infty} A(t) = A_+,
\]
\[
\lim_{t \to -\infty} a(x, t) = a_-(x), \quad \lim_{t \to +\infty} a(x, t) = a_+(x)
\]
whenever the limits exist.

In our applications these limits correspond to the linearization at equilibria. Let \( \lambda_k(t) \) be the kth eigenvalue of \( A(t) \), and \( \lambda_k^\pm \) that of \( A_\pm \). The eigenvalues of \( A_\pm \) are assumed to be simple. Let \( \psi_k^\pm \) denote the kth eigenfunction of \( A_\pm \). We are now in a position to state two results of Henry [12].

**Theorem 5.1.** Suppose the real-valued functions \( a(x, t) \) and \( a_i(x, t) \) are bounded and continuous on \( 0 \leq t < \infty, \ 0 \leq x \leq \pi \). In addition suppose
\[
\int_0^\infty \sup_{x \in [0, \pi]} |a_i(x, t)| \, dt < \infty. \tag{5.2}
\]
There exist classical solutions \( \eta_{k}^+ (x, t) \) of Eq. (5.1) defined for \( t \) greater than some \( t_k \) such that
\[
\eta_{k}^+ (x, t) = \exp \left( -\int_0^t \lambda_k(s) \, ds \right) [\psi_k^+(x) + o(1)],
\]
with convergence in the norm of \( C^1[0, \pi] \). For any solution \( \eta(x, t) \) of (5.1) not identically zero, there exists an integer \( k \geq 1 \) and a constant \( c \neq 0 \) such that
\[
\eta(x, t) = \exp \left( -\int_0^t \lambda_k(s) \, ds \right) [c\psi_k^+(x) + o(1)]
\]
as \( t \to +\infty \).

**Theorem 5.2.** Suppose the real-valued functions \( a(x, t) \) and \( a_i(x, t) \) are bounded and continuous on \( -\infty < t \leq 0, \ 0 \leq x \leq \pi \), and that
\[
\int_{-\infty}^0 \sup_{x \in [0, \pi]} |a_i(x, t)| \, dt < \infty. \tag{5.3}
\]
There exist classical solutions \( \eta_{k}^- \) of Eq. (5.1) defined for \( -\infty < t \leq 0 \) such that
\[
\eta_{k}^- (x, t) = \exp \left( \int_{t}^0 \lambda_k(s) \, ds \right) [\psi_k^-(x) + o(1)],
\]
with convergence in the norm of \( C^1[0, \pi] \). Also if \( \alpha < \lambda_{m+1} \) and \( \| \eta(\cdot, t) \| = o(e^{-\alpha t}) \) as \( t \to -\infty \), there exist \( c_1^-, \ldots, c_m^- \) such that

\[
\eta(x, t) = \sum_{k=1}^{m} c_k^- \eta_k^-(x, t).
\]

We should mention that the results contained in Theorems 5.1 and 5.2 are combined under the stronger assumption in [12] where

\[
\int_{-\infty}^{\infty} \sup_{0 \leq x \leq \pi} |a_i(x, t)| \, dt < \infty. \tag{5.4}
\]

This is natural when considering connecting orbits. We will show that a family of potential functions \( a(x, t) \) can be defined in terms of the differences of solutions to Eq. (4.1). If the solutions approach equilibria as \( t \to \pm \infty \), it will follow that condition (5.4) holds. This is true of solutions on the global attractor.

When constructing an inertial manifold, however, we will also be concerned with solutions off the attractor which need not even be defined for all negative time. This is why we state Theorem 5.1 with the weaker, forward in time hypothesis. The proof in [12] of the forward in time results in Theorem 5.1 uses only the forward in time hypothesis. We omit the details and refer the reader to that paper.

We now show that Theorem 5.2 applies to the equation satisfied by the difference \( S_t u_0 - S_t w_0 \) when \( u_0, w_0 \in W^u(\phi) \).

**Proof of Lemma 4.6.** Let \( \eta(x, t) = v(x, t) - w(x, t) \). Differentiating we find that \( \eta \) satisfies the equation

\[
\eta_t = v \eta_{xx} + f(x, v) - f(x, w).
\]

Now let \( \hat{f}(x, \alpha) = f(x, w - \alpha(w - v)) \). It follows that

\[
f(x, v) - f(x, w) = \int_0^1 \hat{f}_\alpha(x, \alpha) \, d\alpha
\]

\[
= \int_0^1 f(w, w - \alpha(w - v)) \, d\alpha [v - w].
\]

Thus \( \eta \) satisfies the linear equation

\[
\eta_t = v \eta_{xx} + a(x, t) \eta \tag{5.5}
\]

where

\[
a(x, t) = \int_0^1 f_u(x, w(x, t) + \alpha(w(x, t) - v(x, t))) \, d\alpha. \tag{5.6}
\]
Note that

$$\lim_{t \to -\infty} \alpha(x, t) = f_\alpha(x, \varphi(x)).$$

We now verify that $\alpha$ defined by (5.6) satisfies condition (5.3). Note that since $v_0$ and $w_0$ lie on the unstable manifold $\mathcal{W}^u(\varphi)$ their trajectories $S_t v_0$ and $S_t w_0$ are both bounded in $H^1$ and by the Sobolev imbedding theorem, also in $C^0[0, \pi]$. It follows that $f_\alpha$ is bounded in $C^0[0, \pi]$ along a linear combination of these trajectories.

We can express this bound as

$$\sup_{x \in [0, \pi]} | f_\alpha(x, 1 - \alpha) w(x, t) + \alpha v(x, t) | < M$$

(5.7)

for some $M < \infty$. By (5.7) and Fubini's theorem we obtain

$$\int_{-\infty}^{\infty} \sup_{x \in [0, \pi]} | f_\alpha((1 - \alpha) w + \alpha v) [ (1 - \alpha) w_t + \alpha v_t ] | dt$$

$$\leq M \int_{-\infty}^{\infty} \left( \int_{0}^{1} \sup_{x \in [0, \pi]} | (1 - \alpha) w_t + \alpha v_t | dx \right) \right) dt$$

By the hyperbolicity of the equilibria, $v_t$ and $w_t$ decay exponentially as $t \to \pm \infty$ in $C^0[0, \pi]$. It follows that

$$\int_{-\infty}^{\infty} \sup_{x \in [0, \pi]} | (1 - \alpha) w_t(x, t) + \alpha v_t(x, t) | dt < \infty$$

and thus condition (5.3) is satisfied.

Note that $\lambda_k(t) \to \lambda_k^-$ as $t \to -\infty$. By the simplicity of the eigenvalues of $A_-$ we have for $k < j$ that $\lambda_k(t) < \lambda_j(t)$ for all large and negative $t$, and hence

$$\lim_{t \to -\infty} \exp \left( - \int_{-t}^{0} \lambda_k(s) - \lambda_j(s) ds \right) = 0.$$

(5.8)

Since $S_t v_0, S_t w_0$ both tend to 0 as $t \to -\infty$ their difference $\eta(x, t)$ is in the unstable (linear) manifold of the origin for Eq. (5.5). Applying Theorem 5.2
with \( \alpha = 0 \), we find there exist constants \( c_1, \ldots, c_j \) with \( c_j \neq 0 \) and \( j \leq m \) such that

\[
\eta(x, t) = \sum_{k=1}^{j} c_k \eta_k(x, t).
\]

Factoring out the dominating decay term associated with \( \lambda_j \), we obtain

\[
\eta(x, t) - c_j \exp \left( \int_0^t \lambda_j(s) \, ds \right)
\]

\[
\times \left\{ \psi_j + o(1) + \sum_{k=1}^{j-1} c_k \left( \exp \int_0^t \lambda_k(s) - \lambda_j(s) \, ds \right) [\psi_k + o(1)] \right\}
\]

as \( t \to -\infty \) with convergence in the norm of \( C^1[0, \pi] \). Normalizing and noting the decay expressed in Eq. (5.8), we have

\[
\frac{\eta(x, t)}{||\eta(x, t)||} \xrightarrow{c^1} \psi_j
\]

as \( t \to -\infty \).

If the points are on the stable manifold of an equilibrium, and a zero number condition holds, then we obtain a similar result forward in time by the same methods only using Theorem 5.1.

**Lemma 5.3.** If \( v_0, w_0 \in W^s(\phi) \), and \( Z(v_0 - w_0) < m \), then

\[
S_t v_0 - S_t w_0 = \exp \left( - \int_0^t \lambda_k(\tau) \, d\tau \right) \{ c\psi_k + o(1) \}
\]

as \( t \to +\infty \) for some \( k \leq m \).

**Proof.** As in the proof of Lemma 4.6 one can verify that condition (5.2) holds. Thus we can apply Theorem 5.1 to obtain

\[
S_t v_0 - S_t w_0 = \exp \left( - \int_0^t \lambda_k(\tau) \, d\tau \right) \{ c\psi_k + o(1) \}, \tag{5.9}
\]

where \( k \) is some positive integer. Since the convergence in (5.9) is \( C^1 \) it follows that

\[
\frac{S_t v_0 - S_t w_0}{||S_t v_0 - S_t w_0||} \xrightarrow{c^1} \psi_k.
\]

By the fact that the zero number is nonincreasing and lower semicontinuous we have that \( k \leq m \).
6. PROOF OF THE MAIN RESULT

The proof of the main result involves a combination of local and global extensions of invariant manifolds. We begin by stating a local extension theorem which is general enough to be applied at each nontrivial equilibrium and then proceed to use it in the proof of Theorem 3.1. To be clear about the choice of basis as we move from equilibrium to equilibrium we let $P_\varphi$ denote the orthogonal projection onto the linear subspace spanned by the first $m$ eigenfunctions of the linearization at an equilibrium $\varphi$ and let $Q_\varphi = I - P_\varphi$.

Our methods require a tangency condition in the norm $\|A^2 \cdot \|$, where $A$ is the Sturm–Liouville operator defined by

$$u \mapsto -vu_{xx} - f'(\varphi) u$$

with Dirichlet boundary conditions. We denote the ball of radius $\delta$ in this norm by

$$B_\delta(\varphi) = \{u \in D(A) : \|A^2(u - \varphi)\| < \delta\}.$$

DEFINITION. An $m$-dimensional positively invariant manifold with boundary $\mathcal{N}$ has a stable orientation at $\varphi$ if $\mathcal{N} \cap \mathcal{W}^s(\varphi) \neq \emptyset$ and given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|A^2Q_\varphi(u_1 - u_2)\| \leq \varepsilon \|A^2P_\varphi(u_1 - u_2)\|$$

for all $u_1, u_2 \in \mathcal{N} \cap B_\delta(\varphi)$.

**Theorem 6.1 (local extension).** Let $\mathcal{N}$ be an $m$-dimensional positively invariant manifold for Eq. (1.1) with a stable orientation at an equilibrium $\varphi$. Under the standing assumption there exists a positive number $r$ and functions $\Phi: P_\varphi(\mathcal{N} \cap B_r^2(\varphi)) \to Q_\varphi \mathcal{H}$ and $\tilde{\Phi}: P_\varphi B_r^2(\varphi) \to Q_\varphi \mathcal{H}$ such that

1. $\mathcal{N} \cap B_r^2$ is the graph of $\Phi$,
2. $\tilde{\Phi}(p) = \{the\ graph\ of \ \Phi \in B_r^2(\varphi)\}$ is locally invariant,
3. $\tilde{\Phi}(p) = \Phi(p)$ for all $p \in P_\varphi(\mathcal{N} \cap B_r^2(\varphi))$,
4. $Z(u_1 - u_2) < m$ for all $u_1, u_2 \in \mathcal{N} \cap B_r^2(\varphi)$.

The proof of Theorem 6.1 is rather involved and, in fact, occupies the next two sections. We will apply this result repeatedly in the construction of a global manifold. To do so we need the following link in order to guarantee that the hypothesis of Theorem 6.1 is satisfied at each stage of the process.
LEMMA 6.2. Suppose dim \( \mathcal{A} = m \) and that \( \mathcal{N} \) is an \( m \)-dimensional positively invariant manifold with boundary such that \( Z(u_1 - u_2) < m \) for all \( u_1, u_2 \in \mathcal{N} \). Under the standing assumption \( \mathcal{N} \) has a stable orientation at each equilibrium \( \varphi \) such that \( \mathcal{N} \cap \mathcal{W}^s(\varphi) \neq \emptyset \).

The proof of Lemma 6.2 appears in the Appendix. We are now in a position to prove the main result.

Proof of Theorem 3.1. Let \( m = \text{dim} \mathcal{A} \). From Theorem 2.1, part 3, we have that \( \mathcal{A} \cap \mathcal{W}^s(\varphi) \neq \emptyset \) for all nonzero equilibria. Note that by part 4 of the same theorem, \( \mathcal{A} \) is the closure of \( \mathcal{W}'(\varphi) \) in \( H^1 \). Since \( Z(u_1 - u_2) < m \) for all \( u_1, u_2 \in \mathcal{W}^s(0) \) by Theorem 4.2 it follows from the lower semi-continuity of \( Z \) and Lemma 6.2 that \( \mathcal{A} \) has a stable orientation at each nonzero equilibrium.

In particular, the hypothesis of Theorem 6.1 is satisfied at the least stable nontrivial equilibria. Consequently there exist locally invariant extensions \( \mathcal{N}_{m-1}^{\pm} \) of \( \mathcal{A} \) near \( \varphi_{m-1}^{\pm} \). Let

\[
\mathcal{S}_{m-1}^{+} = \{ S, u_0 : u_0 \in \mathcal{N}_{m-1}^{+}, t \geq 0 \}
\]

with \( \mathcal{S}_{m-1}^{-} \) similarly defined. From Theorem 6.1, part 4, one has \( Z(u_1 - u_2) < m \) for all \( u_1, u_2 \in \mathcal{N}_{m-1}^{\pm} \). Yet the zero number is nonincreasing along solutions (Theorem 4.1) so that \( Z(u_1 - u_2) < m \) for all \( u_1, u_2 \in \mathcal{S}_{m-1}^{\pm} \) as well. Since \( \mathcal{W}_{\text{loc}}^u(\varphi_{m-1}^{+}) \subset \mathcal{N}_{m-1}^{+} \) it follows that \( \mathcal{W}^u(\varphi_{m-1}^{+}) \subset \mathcal{S}_{m-1}^{+} \). By Theorem 2.1 one has \( \mathcal{W}^u(\varphi_{m-1}^{+}) \cap \mathcal{W}^s(\varphi_{j}^{\pm}) \neq \emptyset \) and \( \mathcal{S}_{m-1}^{+} \) has a stable orientation at \( \varphi_{j}^{\pm} \) for \( 0 \leq j \leq m - 2 \). An equivalent argument holds for \( \mathcal{S}_{m-1}^{-} \). Thus \( \mathcal{A}, \mathcal{S}_{m-1}^{\pm} \) are the graphs of Lipschitz functions with small Lipschitz constants near the remaining equilibria. Due to the overlapping of \( \mathcal{S}_{m-1}^{\pm} \) with \( \mathcal{A} \) it follows that

\[
\mathcal{M}_{m-1} = \mathcal{A} \cup \mathcal{S}_{m-1}^{+} \cup \mathcal{S}_{m-1}^{-}
\]

has a stable orientation at \( \varphi_{j}^{\pm} \) for \( 0 \leq j \leq m - 2 \). Most of \( \mathcal{A} \) is contained in the interior of \( \mathcal{M}_{m-1} \) in that

\[
\mathcal{A} \cap \partial \mathcal{M}_{m-1} = \bigcup_{j=0}^{m-2} \mathcal{W}^u(\varphi_{j}^{\pm}).
\]

Applying Theorem 6.1 to \( \mathcal{M}_{m-1} \) one obtains locally invariant extensions \( \mathcal{N}_{m-2}^{\pm} \) near \( \varphi_{m-2}^{\pm} \). As above, this leads to a global extension \( \mathcal{M}_{m-2} = \mathcal{M}_{m-1} \cup \mathcal{S}_{m-2}^{+} \cup \mathcal{S}_{m-2}^{-} \) with a stable orientation at \( \varphi_{j}^{\pm}, 0 \leq j \leq m - 3 \), and

\[
\mathcal{A} \cap \partial \mathcal{M}_{m-2} = \bigcup_{j=0}^{m-3} \mathcal{W}^u(\varphi_{j}^{\pm}).
\]
This process may be repeated through the equilibria \( \varphi_{m-1}^\pm, \varphi_{m-2}^\pm, \ldots, \varphi_1^\pm \), producing nested manifolds with boundary \( \mathcal{M}_{m-1} \subset \mathcal{M}_{m-2} \subset \cdots \subset \mathcal{M}_1 \).

At this point the interior of \( \mathcal{M}_1 \) contains all of \( \mathcal{A} \) except \( \{ \varphi_0^\pm \} \). At the completely stable equilibria \( \varphi_0^\pm \), the local extensions \( \mathcal{N}_0^\pm \) can be chosen small enough to be positively invariant. The \( m \)-dimensional manifold \( \mathcal{M} = \text{Int}(\mathcal{M}_1 \cup \mathcal{N}_0^+ \cup \mathcal{N}_0^-) \) now contains the global attractor, and is positively invariant. Due to the existence of a Lyapunov function and hyperbolicity, all trajectories approach one of the finite number of equilibria at a minimal exponential rate. It follows that \( \mathcal{M} \) is exponentially attracting and therefore a well-defined inertial manifold.

7. PROOF OF THE LOCAL EXTENSION THEOREM

The proof of Theorem 6.1 involves several steps. Consider a nontrivial equilibrium \( \varphi \). The first step is to modify the semiflow in such a way that the given manifold is connected to \( P, \mathcal{H} \) outside a neighborhood of \( \varphi \). The second step amounts to shifting the spectrum with a change of variables so that the desired extension is the unstable manifold for the modified semiflow in the new variable.

To make this more precise we translate \( \varphi \) to the origin so that Eq. (1.1) becomes
\[
\frac{du}{dt} + Au = F(u),
\]
where \( A \) is the Sturm–Liouville operator defined by
\[
\varphi \mapsto -\nu u_{xx} - f'(\varphi) u
\]
with Dirichlet boundary conditions, and \( F: \mathcal{H} \rightarrow \mathcal{H} \) is defined pointwise by
\[
F(u)(x) = f(u(x) + \varphi(x)) - f(\varphi(x)) - f'(\varphi(x)).
\]

For \( \varphi = \varphi_j^\pm \) the operator \( A \) has simple eigenvalues
\[
\lambda_1 < \lambda_2 < \cdots < \lambda_m < \lambda_{m+1} < \cdots
\]
with corresponding eigenfunctions \( \psi_1, \psi_2, \psi_3, \ldots \). Let \( P \) be the projection onto the span of the first \( m \) eigenfunctions and \( P_w \) be that onto the span of the weakly stable eigenfunctions \( \psi_{j+1}, \ldots, \psi_m \). The projection onto the span of the unstable eigenfunctions, if any, is then \( P_u = P - P_w \). As before we let \( Q = I - P \).
The theory in what follows applies to equations of the form

\[
\frac{du}{dt} + A u = \hat{F}(u),
\]

(7.3)

where \( \hat{F} \) is globally Lipschitz continuous. Since \( F \) given by (7.2) is only locally Lipschitz, we will construct a globally Lipschitz function \( \hat{F} \) which is identical to \( F \) in a neighborhood of the origin. It is the freedom in choosing \( \hat{F} \) that enables us to extend the given invariant manifold \( \mathcal{N} \).

We now make the change of variables \( w = e^{\lambda t} u \) where \( \lambda_m < \lambda < \lambda_{m+1} \). This leads to an equation for \( w \) of the form

\[
\frac{dw}{dt} + [A - \lambda I] w = e^{\lambda t} \hat{F}(e^{-\lambda t} w).
\]

(7.4)

The unstable set \( U_2 \) is defined as the set of initial conditions for which the solution of Eq. (7.4) is bounded for \( t \leq 0 \). This can be expressed as

\[
U_2 = \{ u_0 \in H: \hat{s}_t u_0 \text{ is defined for all } t \leq 0, \text{ and } e^{\lambda t} \hat{s}_t u_0 \text{ is bounded for all } t \leq 0 \},
\]

where \( \hat{s} \) is the semiflow generated by (7.3). The key to proving Theorem 6.1 is that \( U_2 \) is invariant, both forward and backward in time, with respect to \( \hat{s} \). This is easily verified by letting \( u_0 \) be in \( U_2 \), \( v_0 = \hat{s}_t u_0 \), and noting that since \( s \) is a constant \( e^{\lambda t} \hat{s}_t v_0 = e^{-\lambda s} [e^{\lambda(t+s)} \hat{s}_{t+s} u_0] \) is bounded for all \( t \leq 0 \).

We return to the problem of choosing \( \hat{F} \). For \( \delta > 0 \), let \( \mathcal{R}_\delta \) denote the cube

\[
\mathcal{R}_\delta = \{ u \in \mathcal{D}(A^2): \| A^2 P_u u \| \leq \delta, \| A^2 P_w u \| \leq \delta, \| A^2 Q u \| < \delta \}.
\]

We take \( \delta_0 > 0 \) small enough so that for each \( \delta \) such that \( 0 < \delta < \delta_0 \) the solution \( S_t u_0 \) of (7.1) through \( u_0 \in \mathcal{N} \) with \( \| A^2 P_w u_0 \| = \delta \) \( \| A^2 P_w u \| = \delta \) immediately leaves \( \mathcal{R}_\delta \) in positive (negative) time. This is depicted in Fig. 7.1. Consider two cubes \( \mathcal{R}_3 \subset \mathcal{R}_{25} \). We will design \( \hat{F} \) so as to effectively bend the trajectories of Eq. (7.1) coming into \( \mathcal{N} \cap \mathcal{R}_3 \) so that for (7.3) they start on \( PH \) at the boundary of \( \mathcal{R}_{25} \). A complete description of \( \hat{F} \) is in the following statement.

**THEOREM 7.1.** Let \( \mathcal{N} \) be an \( m \)-dimensional manifold with a stable connection to \( \varphi \), with \( F \), \( \mathcal{R}_3 \), and \( \hat{s} \) as above. Given \( \varepsilon > 0 \) there exists \( \delta_\ast > 0 \), and a globally Lipschitz function \( \hat{F}: H \to H \) such that for \( \delta < \delta_\ast \)

1. \( \hat{F} = F \) in \( \mathcal{R}_3 \);
2. \( Q \hat{F} = 0 \) in \( \mathcal{D}(A^2) \setminus \mathcal{R}_{25} \);

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(3) For each $u_1 \in \mathcal{N}$ with $\| A^2 P_w u_1 \| = \delta$ there exist $p_0(u_1)$ and $t_1(u_1)$ such that $\hat{S}_t p_0 = u_1$, $p_0 \in PH$, and $\| A^2 P_w p_0 \| = 2\delta$;

(4) $\text{Lip}(\tilde{F}) < \varepsilon$.

The properties of $\hat{S}$ are illustrated in Fig. 7.2. The proof of Theorem 7.1 will be postponed until Section 8.

We next follow the approach taken in Sell [22] for finite-dimensional ordinary differential equations to realize the unstable set $U_d$ for (7.4) as the fixed point for a contraction mapping. The mapping is the used to show that $U_d$ is a Lipschitz manifold of dimension $m$ which coincides with $PH$.
outside $\mathcal{R}_\beta$. It will follow from the invariance of $U_\lambda$ and the construction of $\hat{F}$ that this will provide a local extension of $\mathcal{N}$ inside $\mathcal{R}_\delta$.

We now develop several tools for construction of the contraction mapping. Consider the linear differential equation

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0,$$

(7.5)

where $A$ is as above. The solution to (7.5) can be found directly by separation of variables to be of the form

$$u(x, t) = \sum_{k=1}^\infty e^{-\lambda_k t} \langle u_0, \psi_k \rangle_H \psi_k(x).$$

This is precisely the $C^0$-semigroup $e^{-At}$ operating on $u_0$.

For the remainder of this section we let

Let $A_\lambda$ denote the shifted operator $A - \lambda I$. We now derive an exponential dichotomy for the shifted linear equation.

**Lemma 7.2.** For $\alpha$ and $\lambda$ as above one has

$$\| e^{-A_{\lambda}^\tau} P \|_{L(H, H)} \leq e^{\alpha \tau}, \quad \tau \leq 0$$

(7.6)

$$\| e^{-A_{\lambda}^\tau} Q \|_{L(H, H)} \leq e^{\alpha \tau}, \quad \tau \geq 0,$$

(7.7)

where $L(H, H)$ denotes the space of linear operators on $H$.

**Proof.** Applying Parseval's identity we obtain for $\tau < 0$,

$$\| e^{-A_{\lambda}^\tau} P \|_{L(H, H)} = \sup_{u \in H \setminus \{0\}} \frac{\| e^{-A_{\lambda}^\tau} Pu \|}{\| u \|_H} \leq \sup_{p \in PH \setminus \{0\}} \frac{\| e^{-A_{\lambda}^\tau} p \|}{\| p \|}$$

$$= \sup_{p \in PH \setminus \{0\}} \left\{ \frac{\sum_{k=1}^m \frac{e^{2(\lambda_m - \lambda_k)\tau} p_k^2}{\sum_{k=1}^m p_k^2}} {e^{-\lambda_m \tau}} \right\}^{1/2},$$

where $p_k = \langle p, \psi_k \rangle_H$. Since the quantity in branches is bounded by 1, we have

$$\| e^{-A_{\lambda}^\tau} P \|_{L(H, H)} \leq e^{-\lambda_m \tau}.$$
It follows that

$$\| e^{-A_\tau P} \|_{L(H, H)} \leq e^{\tau} \| e^{-A_\tau P} \|_{L(H, H)} \leq e^{\alpha \tau}, \quad \tau \leq 0.$$ 

The proof of (7.7) is similar.

The following estimates are now easily obtained.

**Lemma 7.3.** For $\lambda$ and $\alpha$ as above, $a \in (-\infty, \infty]$, $b \in [-\infty, \infty)$ we have

$$\| \int_t^a e^{-A\lambda(s-r)} Pe^{\lambda s} \left[ \hat{F}(e^{-\lambda u(s)}) - \hat{F}(e^{-\lambda v(s)}) \right] ds \| \leq \alpha^{-1} \text{Lip}(\hat{F}) \sup_{t \leq s \leq a} \{ \| u(s) - v(s) \| \} \quad (7.8)$$

and

$$\| \int_b^t e^{-A\lambda(t-s)} Pe^{\lambda s} \left[ \hat{F}(e^{-\lambda u(s)}) - \hat{F}(e^{-\lambda v(s)}) \right] ds \| \leq \alpha^{-1} \text{Lip}(\hat{F}) \sup_{b \leq s \leq t} \{ \| u(s) - v(s) \| \}. \quad (7.9)$$

**Proof.** Using Lemma 7.2 we obtain

$$\left\| \int_t^a e^{-A\lambda(s-r)} Pe^{\lambda s} \left[ \hat{F}(e^{-\lambda u(s)}) - \hat{F}(e^{-\lambda v(s)}) \right] ds \right\| \leq \text{Lip}(\hat{F}) \int_t^a \| e^{-A\lambda(s-r)} P \|_{L(H, H)} \| u(s) - v(s) \| \ ds \leq \text{Lip}(\hat{F}) \sup_{a \leq s \leq t} \| u(s) - v(s) \| \int_t^a e^{\alpha(t-s)} \ ds.$$ 

Computing the integral, we obtain

$$\int_t^a e^{\alpha(t-s)} \ ds = \alpha^{-1} [1 - e^{\alpha(t-s)}] < \alpha^{-1},$$

completing the proof of (7.8). The proof of (7.9) is similar.

Let $\mathcal{F}$ be the collection of all continuous functions $w: \mathcal{PH} \times (-\infty, 0] \rightarrow H$ such that

(i) $\sup_{t \leq 0} \| w(p, t) \| < \infty$ for each $p \in P_mH$,

(ii) $\| w(p_1, t) - w(p_2, t) \| \leq 2 \| p_1 - p_2 \|. $
\( \mathcal{F} \) becomes a complete metric space when endowed with the topology generated by the family of pseudonorms

\[ \| w \|_k = \sup \{ \| w(p, t) \|_H : \| p \|_H \leq k, t \leq 0 \}. \]

We will continue to denote \( \| \cdot \|_H \) by simply \( \| \cdot \| \) unless special emphasis is needed.

The following version of the contradiction mapping theorem will be applied to the family \( \mathcal{F} \).

**Theorem 7.4.** Suppose \( \mathcal{F} \) is a mapping from \( \mathcal{F} \) to \( \mathcal{F} \) such that for some \( 0 < L < 1 \)

\[ \| \mathcal{F}w_1 - \mathcal{F}w_2 \|_k \leq L \| w_1 - w_2 \|_k \]

for all \( k = 1, 2, \ldots \), and all \( w_1, w_2 \in \mathcal{F} \). Then there is a unique fixed point \( w \in \mathcal{F} \), that is, \( \mathcal{F}w = w \).

A proof of Theorem 7.4 following a standard argument can be found in [10]. We formally define a mapping \( \mathcal{F} \) on \( \mathcal{F} \) by

\[
\mathcal{F}w(p, t) = e^{-A_\lambda t}p - \int_0^t Pe^{-A_\lambda(t-s)}e^{\lambda s}\hat{F}(e^{-\lambda s}w(p, s)) \, ds + \int_{-\infty}^t Qe^{-A_\lambda(t-s)}e^{\lambda s}\hat{F}(e^{-\lambda s}w(p, s)) \, ds.
\]

Note that by Lemma 7.3 with \( u(s) = w(p, s) \) and \( v(s) = 0 \), we have that the second integral exists so \( \mathcal{F} \) is well-defined.

**Theorem 7.5.** Let \( \mathcal{F} \) be defined as above. If \( 4\alpha^{-1}\text{Lip}(\hat{F}) < 1 \), the mapping \( \mathcal{F} \) is a contraction on \( \mathcal{F} \) with a unique fixed point \( w \) such that

\[ \{ w(p, 0) : p \in PH \} = U_\lambda. \]

Moreover, \( U_\lambda \) is the graph of a Lipschitz continuous function \( \Phi : PH \to QH. \)

**Proof.** We proceed as in the finite-dimensional case in [22].

We first verify that \( \mathcal{F} \) maps \( \mathcal{F} \) into itself. By Lemmas 7.2 and 7.3 we have

\[
\sup_{t \leq 0} \| \mathcal{F}w(p, t) \| \leq \| p \| + 2\alpha^{-1}\text{Lip}(\hat{F}) \sup_{s \leq 0} \| w(p, s) \| < \infty,
\]

or

\[ \| \mathcal{F}w \|_k \leq k + 2\alpha^{-1} \text{Lip}(\hat{F}) \| w \|_k. \]
Also by Lemmas 7.2 and 7.3 and the assumption that $4a^{-1}\text{Lip}(\hat{F}) < 1$ we obtain

\[
\| Fw(p_1, t) - Fw(p_2, t) \|
\leq \| p_1 - p_2 \| + 2a^{-1}\text{Lip}(\hat{F}) \sup_{s \leq 0} \| w(p_1, s) - w(p_2, s) \|
\leq \| p_1 - p_2 \| + 4a^{-1}\text{Lip}(\hat{F}) \| p_1 - p_2 \|
\leq 2 \| p_1 - p_2 \|.
\]

Thus $F$ maps $\mathcal{F}$ into $\mathcal{F}$.

Next, we show that $F$ is a contraction mapping. Another application of Lemma 7.3 yields

\[
\sup_{t \leq 0} \| Fw_1(p, t) - Fw_2(p, t) \|_{\mathcal{H}} \leq 2a^{-1}\text{Lip}(\hat{F}) \sup_{s \leq 0} \| w_1(p, s) - w_2(p, s) \|_{\mathcal{H}}.
\]

Thus in the pseudonorm, since $4a^{-1}\text{Lip}(\hat{F}) < 1$, we have

\[
\| Fw_1 - Fw_2 \|_k \leq \frac{1}{2} \| w_1 - w_2 \|_k
\]

for $k = 1, 2, \ldots$.

Now let $w = Fw$ be the fixed point of $F$ given by Theorem 7.4. Note that by the orthogonality of $P$ and $Q$ we have

\[
Pw(p, 0) = P\mathcal{F}w(p, 0) = Pp = p,
\]

for all $p \in \mathcal{P}$. The inverse $\hat{F}$ given by

\[
\hat{F}: \mathcal{P} \rightarrow \{ w(p, 0) \mid p \in \mathcal{P} \}
\]

is therefore a well-defined Lipschitz continuous function with $\text{Lip}(\hat{F}) \leq 2$. To complete the proof we show that

\[
U_\lambda = \{ w(p, 0) \mid p \in \mathcal{P} \}.
\]

Differentiating $w(p, t) = Fw(p, t)$, we obtain

\[
w_\lambda(p, t) = -A_\lambda w(p, t) + e^{\lambda t}\hat{F}(e^{-\lambda t}w(p, t)).
\]

Thus, $w(p, t)$ is a solution to (7.4). Since $w(p, t)$ is bounded for all $t \leq 0$ we have

\[
\{ w(p, 0) \mid p \in \mathcal{P} \} \subset U_\lambda.
\]
The reverse containment follows from the uniqueness of the fixed point of another mapping $\mathcal{F}_p$ defined by replacing $w(p, s)$ with $w(s)$ in the formula for $\mathcal{T}$. See [22] for details.

We are now ready to prove the local extension theorem.

**Proof of Theorem 6.1.** Take $\delta$ small enough in Theorem 7.1 so that the function $\hat{F}$ satisfies $4\lambda^{-1}\text{Lip}(\hat{F}) < 1$. By Theorem 7.5, $U_{\lambda}$ is a global manifold of dimension $m$ which is invariant under the semiflow $\hat{S}$. By the construction of $\hat{F}$, every element in $\mathcal{N} \cap \mathcal{R}_0 \backslash \mathcal{W}_w(\phi)$ is on a trajectory defined by $\hat{S}$ passing through the hyperplane in $PH$ defined by $\| A^2 P_w p \| = 2\delta$. We now show that this hyperplane is contained in $U_{\lambda}$.

More precisely, we show that $w(p, 0) = p$ for all $p \in PH$ with $\| A^2 P_w p \| = 2\delta$. Expanding the fixed point $w$ in terms of the mapping $\mathcal{F}$ and setting $t = 0$ we obtain

$$w(p, 0) = \mathcal{T} w(p, 0) = p + \int_{-\infty}^{0} e^{A\lambda} Q e^{\lambda} \hat{F}(e^{-\lambda} w(p, s)) \, ds.$$ 

Since $e^{-\lambda t} w = u$ is a solution to (7.3) its projection in the weakly stable direction explodes in negative time. In fact, one has $\| A^2 P_w e^{-\lambda s} w(p, s) \| \geq 2\delta$ for all $s \leq 0$. Yet by design, we have $Q\hat{F} = 0$ in $\mathcal{D}(A^2) \backslash \mathcal{R}_0$ so the integral vanishes. Thus $p \in U_{\lambda}$ and, by the invariance of $U_{\lambda}$, we have that $\mathcal{N}$ is contained in $U_{\lambda}$. Since $S = \hat{S}$ in $\mathcal{R}_0$ it follows that $\mathcal{N} = U_{\lambda} \cap \mathcal{R}_0$ is a locally invariant extension of $\mathcal{N}$.

For the proof of part (4), let $u_1, u_2 \in \mathcal{N} \cap \mathcal{R}_0$, $w_i(t) = e^{\lambda t} S u_i$, and $\hat{w}(t) = w_1(t) - w_2(t)$. The difference $\hat{w}$ satisfies

$$\begin{aligned} \frac{d\hat{w}}{dt} + A_{\lambda} \hat{w} &= e^{\lambda t} [\hat{F}(e^{-\lambda} w_1) - \hat{F}(e^{-\lambda} w_2)] \equiv g(t), \end{aligned}$$

so $g(t) = O(e^{\lambda t})$ as $t \to -\infty$ since $\hat{F}$ is bounded. The normalized difference $y = \hat{w} \parallel \hat{w} \parallel^{-1}$ satisfies

$$\begin{aligned} \frac{dy}{dt} &= [ - A_{\lambda} \hat{w} + g(t) ] \parallel \hat{w} \parallel^{-1} - \langle \hat{w}, \hat{w} \rangle^{-3/2} \langle - A_{\lambda} \hat{w} + g(t), \hat{w} \rangle \hat{w} \\ &= - A_{\lambda} y + \langle A_{\lambda} y, y \rangle y + G(t), \end{aligned}$$

where $G(t) = g(t) \parallel \hat{w} \parallel^{-1} + \langle g(t), \hat{w} \parallel^{-1}, y \rangle y$. Observe that since $\hat{w}(t)$ decays no faster than $e^{(\lambda - k)t}$ as $t \to -\infty$ it follows that $G(t)$ decays like $e^{\lambda t}$ and the differential equation

$$\begin{aligned} \frac{dy}{dt} &= - A_{\lambda} y + \langle A_{\lambda} y, y \rangle y + G(t) \tag{7.10} \end{aligned}$$
is asymptotically autonomous. It follows from the results in [21] that the $\alpha$-limit sets for (7.10) are also $\alpha$-limit sets for

$$\frac{dy}{dt} = -A_\psi y + \langle A_\psi y, y \rangle y.$$  \hspace{1cm} (7.11)

Let $\psi$ be a nontrivial equilibrium solution for (7.11), that is,

$$-A_\psi \psi + \langle A_\psi \psi, \psi \rangle \psi = 0.$$

Since $\langle A_\psi \psi, \psi \rangle$ is a scalar quantity, $\psi$ must be an eigenfunction of $A_\psi$, and hence $A$.

It follows that for each pair of initial conditions $u_1, u_2$ in $\mathcal{N}^2 \cap \mathcal{R}_\delta$ one has $y(t) \to \psi_k$ for some $k \geq 1$. Suppose $k > m$ for some $u_1, u_2$. Let $p_1(t) = P\psi_k(t)$. Since $\|P\psi_k\| = 0$ it follows that $\|Qy(t)\| \|Py(t)\|^{-1} \to \infty$ as $t \to -\infty$. Yet $\text{Lip}(\Phi) \leq 2$, so

$$\frac{\|Qy(t)\|}{\|Py(t)\|} \leq 2 \frac{\|p_1(t) - p_2(t)\|}{\|p_1(t) - p_2(t)\|}$$

for all $t$, a contradiction. Thus for each pair of initial conditions $u_1, u_2$ in $\mathcal{N}^2 \cap \mathcal{R}_\delta, y(t) \to \psi_k$ with $k \leq m$.

For each $k, 1 \leq k \leq m$, let $\mathcal{V}_k$ be a $C^1$-neighborhood of $\psi_k$ in which $Z$ is constant. Take $t_0$ so large and negative that for each $u_1, u_2 \in \mathcal{N}^2 \cap \mathcal{R}_\delta$ one has $y(t) \in \mathcal{V}_k$ for some $k \leq m$ and all $t \leq t_0$. Thus $Z(u_1 - u_2) < m$ for all $u_1, u_2 \in \mathcal{N}^2 \cap \mathcal{R}_r$, where

$$r = \min_{u \in \mathcal{X}} \{ \|A^2e^{i\alpha}S_0u\| \},$$

and $\mathcal{X}$ is the compact set given by

$$\mathcal{X} = \{ u \in \mathcal{N}^2 : \|A^2u\| = \delta \}.$$

8. **Designing a Nonlinearity**

This section is devoted to the proof of Theorem 7.1. We begin with an outline of the major steps.

Step (1) Show that $\text{Lip}(A^kF |_{\mathcal{R}_\delta}) \to 0$ as $\delta \to 0$ for $k = 0, 1, 2$.

Step (2) Construct the wing of trajectories that connect $\mathcal{N} \cap \mathcal{R}_3$ with $PH$ as in Fig. 7.2.

Step (3) Differentiate along the wing to derive a formula for $Q\hat{F}$ there.

Step (4) Show $\text{Lip}(Q\hat{F} |_{\text{wing}}) \to 0$ as $\delta \to 0$.
Step (5) Extend $Q\hat{F}$ to equal $F$ in $\mathcal{R}_\delta$, 0 in $\mathcal{D}(A^2)\setminus\mathcal{R}_\delta$; estimate $\text{Lip}(Q\hat{F})$ on the combined domains.

Step (6) Extend $Q\hat{F}$ to all of $H$, preserving the Lipschitz constant in the process.

Step (7) Perform a scalar truncation of $PF$ between $\mathcal{B}^2_{\delta}$ and $\mathcal{B}^2_{\infty}$.

For the first step we state and prove a general lemma for $k = 0, 1, 2, \ldots$. This is stated in greater generality than necessary. The proof, however, is no more burdensome than the special case $k = 2$ which we do require.

**Lemma 8.1.** Suppose $f$ is $C^{2k+1}$, and that $F$ is defined pointwise by (7.2). There exists $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that $L(\delta) \to 0$ as $\delta \to 0$ and

$$\|A^k[F(u) - F(v)]\| \leq L(\delta) \|A^k[u - v]\|$$

for all $u, v \in \mathcal{B}^k_{\delta}$.

**Proof.** By the equivalence of the norms $\|A^k\|$ and $\|\cdot\|_{\mathcal{H}^k}$ a proof by induction on $k$ reduces to estimating $\|[f(u)]^{(k)} - [f(v)]^{(k)}\|$ in terms of $\|u - v\|_{\mathcal{H}^k}$. The formula of Faà di Bruno for the derivatives of a composition (cf. [14]) may expressed as

$$[f(u)]^{(k)} = \sum f^{(j)}(u) \frac{k!}{i_1! \ldots i_j!} u_{i_1}^{j_1} \ldots u_{i_j}^{j_j},$$

where $u_{i_m} = (j_m!)^{-1} u^{(j_m)}$ and the sum is taken over all partitions

$$i_1 + \ldots + i_j = j$$

$$i_1 j_1 + \ldots + i_j j_j - k$$

with $1 \leq j \leq k$.

The only term involving $u^{(k)}$ has the trivial partition with $j = 1$. Applying the mean value theorem pointwise to the contribution of this term one obtains

$$\|f'(u) u^{(k)} - f'(v) v^{(k)}\| \leq \|f'(u) u^{(k)} - f'(v) u^{(k)}\| + \|f'(v) u^{(k)} - f'(v) v^{(k)}\|$$

$$\leq \|f'(u) - f'(v)\|_{\infty} \|u^{(k)}\| + \|f'(v)\|_{\infty} \|u^{(k)} - v^{(k)}\|$$

$$\leq \|f''(c)\|_{\infty} \|u - v\|_{\infty} \|u^{(k)}\| + \|f'(v)\|_{\infty} \|u - v\|_{\mathcal{H}^k},$$

where $c : [0, \pi] \to \mathbb{R}$ satisfies

$$u(x) \leq c(x) \leq v(x) \quad \text{if} \quad u(x) \leq v(x);$$

$$v(x) \leq c(x) \leq u(x) \quad \text{if} \quad v(x) \leq u(x).$$
By the Sobolev imbedding theorem there exists a constant $K$ such that
\[ \| u^{(n)} \|_{\infty} \leq K \| u^{(n)} \|_{H^k} \quad \text{for} \quad 1 \leq n < k. \quad (8.1) \]

Let
\[ M = \max_{2 \leq n \leq k + 1} \left\{ \sup_{-K \leq x \leq K} |f^{(n)}(x)| \right\} \]
so that
\[ \| f'(u) u^{(k)} - f'(v) v^{(k)} \| \leq L_1(\delta) \| u - v \|_{H^k}, \]
where, for $\delta \leq 1$
\[ L_1(\delta) = K \| f'''(c) \|_{\infty} \| u^{(k)} \| + \| f'(v) \|_{\infty} \leq MK\delta + \| f'(v) \|_{\infty}. \]

By (8.1) one has that $\| f'(v) \|_{\infty} \rightarrow 0$ as $\| v \|_{H^k} \rightarrow 0$ and hence $L_1(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

To complete the proof it is sufficient to estimate for an arbitrary partition with $j \geq 2$ the quantity
\[ A = \| f^{(j)}(u) u_{j_1}^{i_1} \cdots u_{j_l}^{i_l} - f^{(j)}(v) v_{j_1}^{i_1} \cdots u_{j_l}^{i_l} \| . \]

Let $U_m = u_{j_m}^{i_m} \cdots u_{j_n}^{i_n}$, $V_m = v_{j_m}^{i_m} \cdots v_{j_n}^{i_n}$, and $U_{l+1} \equiv 1 \equiv V_1$. Note that
\[ U'_1 - V'_1 = U'_1 - v_{j_1}^{i_1} U'_2 + v_{j_1}^{i_1} U'_2 + v_{j_1}^{i_1} U'_2 - V'_1 = (u_{j_1}^{i_1} - v_{j_1}^{i_1}) U'_2 + v_{j_1}^{i_1} (U'_2 - V'_2). \]

Repeating this computation for $U'_m - V'_m$, $2 \leq m \leq l$ one obtains
\[ U'_1 - V'_1 = \sum_{m=1}^{l} (u_{j_m}^{i_m} - v_{j_m}^{i_m}) U_{m+1}^{i_m} V_1^{i_m - 1} \]
\[ = \sum_{m=1}^{l} (u_{j_m}^{i_m} - v_{j_m}^{i_m}) U_{m+1}^{i_m} V_1^{i_m - 1} \sum_{y=0}^{i_m-1} u_{j_m}^{i_m-1-y} v_{j_m}^{y} \]
\[ = \sum_{m=1}^{l} \left[ u_{j_m}^{i_m} - v_{j_m}^{i_m} \right] W_m, \quad (8.2) \]
where
\[ W_m = \frac{1}{j_m} U_{m+1}^{i_m} V_1^{i_m - 1} \sum_{y=0}^{i_m-1} u_{j_m}^{i_m-1-y} v_{j_m}^{y}. \]
Since each of the products $U_m, V_m$, and $W_m$ involve derivatives of order at most $k - 1$, there exists one constant $K_1$, such that $\| U_m \|_\infty \leq K_1 \delta$ and $\| W_m \|_\infty \leq K_1 \delta$ for $1 \leq m \leq l$. It follows from another application of the mean value theorem and (8.2) that

$$
\Delta = \| f^{(j)}(u) U'_1 - f^{(j)}(v) V'_1 \| \\
\leq \| f^{(j)}(u) U'_1 - f^{(j)}(v) U'_1 \| + \| f^{(j)}(v) U'_1 - f^{(j)}(v) V'_1 \| \\
\leq \| f^{(j)}(u) - f^{(j)}(v) \| \| U'_1 \|_\infty + \| f^{(j)}(v) \|_\infty \| U'_1 - V'_1 \| \\
\leq \pi \| f^{(j+1)}(c_1) \|_\infty \| u - v \|_\infty \| U'_1 \|_\infty \\
+ \| f^{(j)}(v) \|_\infty \sum_{m=1}^l \| u^{(j_m)} - v^{(j_m)} \| \| W'_m \|_\infty.
$$

Thus, $\Delta \leq L_2(\delta) \| u - v \|_{H^k}$ where

$$
L_2(\delta) = \pi K \| f^{(j+1)}(c_1) \|_\infty \| U'_1 \|_\infty + l \| f^{(j)}(v) \|_\infty \| W'_m \|_\infty \\
\leq (\pi K + l) MK_1 \delta.
$$

We now begin Step (2). Since $Z(u_1 - u_2) < m$ for $u_1, u_2 \in N$, it follows from Theorem 4.4 that $N$ is the graph of a function $\Phi : P.N \rightarrow QH$. On $N$ we split Eq. (7.1) into

$$
\frac{dp}{dt} + Ap = PF(p + \Phi(p)) \quad (8.3p) \\
\frac{dq}{dt} + Aq = QF(p + \Phi(p)). \quad (8.3q)
$$

Let $\mathcal{D}$ be the projection onto $PH$ of the trajectory segments in $N \cap (R_2 \cup R_3)$ which enter $R_3$ as in Fig. 7.1. Fix an element $p_\ast$ in $\mathcal{D}$. Let $p(t)$ denote the solution of $(8.3p)$ through $p_\ast$ such that $\| A^2P_wp(0) \| = 2\delta$. Let $t_1 = t_1(p_\ast)$ be the time the solution takes to traverse $\mathcal{D}$ so that $\| A^2P_wp(t_1) \| = \delta$. Let $q(t)$ denote the $Q$-component of the trajectory in $N$ over $p(t)$, that is, $q(t) = \Phi(p(t))$.

We seek a new trajectory $\tilde{q}(t)$ such that

$$
\tilde{q}(0) = 0, \quad \frac{d\tilde{q}}{dt}(0) = 0 \quad (8.4) \\
\tilde{q}(t_1) = q(t_1), \quad \frac{d\tilde{q}}{dt}(t_1) = \frac{dq}{dt}(t_1).
$$

To make the differential equation for which $\tilde{q}(t)$ is a solution an autonomous one, we reparametrize the new trajectory in terms of the arc
length along \( p(t) \). Let \( l(p(\tau)) \) denote the arc length of \( p(t) \) up to time \( \tau \), that is,

\[
l(p(\tau)) = \int_0^\tau \left| \frac{dp}{dt}(s) \right| ds.
\]  

(8.5)

We define a real variable \( \rho = l(p(t))\left[ l(\rho(t_1))\right]^{-1} \), so that \( 0 \leq \rho \leq 1 \) for \( 0 \leq t \leq t_1 \). Employing the method of cubic splines we determine \( a \) and \( b \) in \( QH \) such that

\[
h(\rho) = a\rho^3 + b\rho^2
\]  

(8.6)

satisfies

\[
\begin{align*}
  h(0) &= 0, & \frac{dh}{d\rho}(0) &= 0 \\
  h(1) &= q(t_1), & \frac{dh}{d\rho}(1) &= \frac{dq}{dt}(t_1)\left(\frac{dp}{dt}(t_1)\right)^{-1}.
\end{align*}
\]  

(8.7)

The boundary value problem consisting of (8.6) with conditions (8.7) leads to a system of linear algebraic equations in \( QH \) with solutions

\[
\begin{align*}
  a &= -2h(1) + \frac{dh}{d\rho}(1) = -2q(t_1) + \frac{dq}{dt}(t_1)\left(\frac{dp}{dt}(t_1)\right)^{-1} \\
  b &= 3h(1) - \frac{dh}{d\rho}(1) = 3q(t_1) - \frac{dq}{dt}(t_1)\left(\frac{dp}{dt}(t_1)\right)^{-1}.
\end{align*}
\]  

(8.8)

The mapping \( h \) defines a wing of trajectories over \( \mathcal{O} \) as the graph of the function

\[
\hat{\phi} : \mathcal{O} \to QH
\]

\[
p_* \mapsto h\left(\frac{\ell(p_*)}{\ell(p(t_1))}\right).
\]

We define the \( Q \)-component of the modified nonlinearity on this wing by

\[
Q\hat{F}_\delta(p_*, \hat{\phi}(p_*)) = \frac{dh}{d\rho} \frac{dp}{dt} + A\hat{\phi}(p_*)
\]  

(8.9)

and proceed to estimate its Lipschitz constant in terms of the parameter \( \delta \).

To carry out Step (5) the following elementary result will be used repeatedly.
Lemma 8.2. Let \( g: \mathcal{D} \to \mathbb{R} \) and \( G: \mathcal{D} \to Y \) be continuous functions on a compact set \( \mathcal{D} \subset X \), where \( X \) and \( Y \) are normed spaces. If \( g \) and \( G \) are both Lipschitz continuous then the function \( gG \) is also Lipschitz continuous with

\[
\text{Lip}(gG) \leq \sup_{x \in \mathcal{D}} (|g| \text{Lip}(G) + \text{Lip}(g) \sup_{x \in \mathcal{D}} \|G\|). \tag{8.10}
\]

If \( g(u) \neq 0 \) for all \( u \in \mathcal{D} \) then \( g^{-1} \) is Lipschitz continuous with

\[
\text{Lip}(g^{-1}) \leq \left( \inf_{x \in \mathcal{D}} |g(x)| \right)^{-2} \text{Lip}(g). \tag{8.11}
\]

Proof. In the case of the product we write

\[
\| g(u) G(u) - g(v) G(v) \|_Y \\
= \| g(u) G(u) - g(v) G(u) + g(v) G(u) - g(v) G(v) \|_Y \\
\leq \| G(u) \|_Y | g(u) - g(v) | + | g(v) | \| G(u) - G(v) \|_Y \\
\leq \left( \text{Lip}(g) \sup_{u \in \mathcal{D}} \|G(u)\|_Y + \text{Lip}(G) \sup_{u \in \mathcal{D}} |g(u)| \right) \|u - v\|_X.
\]

So \( gG \) is Lipschitz and (8.10) holds. For \( g^{-1} \) we write

\[
\| g^{-1}(u) - g^{-1}(v) \| = \| g(u) g(v) \|^{-1} | g(v) - g(u) | \\
\leq \left( \inf_{u \in \mathcal{D}} |g(u)| \right)^{-2} \text{Lip}(g) \|u - v\|_X.
\]

By the compactness of \( \mathcal{D} \) the infimum is positive, so \( g^{-1} \) is Lipschitz, and (8.11) holds.

Lemma 8.3. The function \( Q\tilde{F}_\delta: \text{wing} \to H \) defined by Eq. (8.9) is Lipschitz continuous in the \( L^2 \)-norm with \( \text{Lip}(Q\tilde{F}_\delta) \to 0 \) as \( \delta \to 0 \).

Proof. It follows from (8.7), (8.8), and (8.9) that

\[
Q\tilde{F}_\delta = (3a\rho^2 + 2b\rho) \frac{d\rho}{dt} + A(a\rho^3 + b\rho^2) \\
= (6\rho - 6\rho^2) \frac{d\rho}{dt} q(t_1) + (3\rho^2 - \rho) \frac{d\rho}{dt} \left[ \frac{d\rho}{dt} (t_1) \right]^{-1} \frac{dq}{dt} (t_1) \\
+ (-2\rho^2 + 3\rho^2) Aq(t_1) + (\rho^3 - \rho^2) \left[ \frac{d\rho}{dt} (t_1) \right]^{-1} A \frac{dq}{dt} (t_1).
\]
Recall that through \( t_1 = t_1(p_\star) \), each of the terms

\[
p(t_1), \rho = l(p_\star)[l(p(t_1))]^{-1}, \quad \frac{dp}{dt}(t_1), \quad \frac{dq}{dt}(t_1)
\]

depends implicitly on \( p_\star \). The proof proceeds by systematically applying Lemma 8.2 to pairs of terms in \( Q \tilde{F}_\delta \). Table 8.1 summarizes the asymptotic behavior as \( \delta \to 0 \) of the key functions. The first two columns identify the pair of functions whose scalar product requires a Lipschitz estimate. The next four columns contain the estimates which along with (8.10) give the result in the last column. The final result of the first row is used in the second. Those of the second and third are combined in the fourth to yield

\[
\text{Lip} \left( \rho \frac{dp}{dt} q(t_1) \right) \leq O(\text{Lip}(\Phi)),
\]

\[
\text{Lip} \left( \rho \frac{dp}{dt} Aq(t_1) \right) \leq O(\text{Lip}(A\Phi)).
\]

The procedure in the fourth row can be repeated with \( g = \rho \) and \( G = \rho (dp/dt) q(t_1) \) (and also \( \rho Aq(t_1) \)). Combining these results with Lemma 6.2, one has that

\[
\text{Lip} \left( \rho^j \frac{dp}{dt} q(t_1) \right) \to 0, \quad j = 1, 2,
\]

\[
\text{Lip} \left( \rho^j Aq(t_1) \right) \to 0, \quad j = 2, 3,
\]
as \( \delta \to 0 \). This same procedure can be applied to the results in rows six and eight to obtain

\[
\text{Lip} \left( \rho^j \frac{dp}{dt} \left[ \frac{dp}{dt}(t_1) \right]^{-1} \frac{dq}{dt}(t_1) \right) \leq O(\text{Lip}(A\Phi) + \text{Lip}(F)), \quad j = 1, 2,
\]

\[
\text{Lip} \left( \rho^j \left[ \frac{dp}{dt}(t_1) \right]^{-1} A \frac{dq}{dt}(t_1) \right) \leq O(\text{Lip}(A^2\Phi) + \text{Lip}(AF)), \quad j = 2, 3,
\]
as \( \delta \to 0 \). By Lemmas 6.2 and 8.1 these Lipschitz constants all tend to zero as \( \delta \to 0 \).

All that remains then is the verification of the preliminary estimates in columns three through six. This is provide in Table 8.2 utilizing in some instances the following notation. By Lemma 8.1 one may take \( \delta_\star \) so small that

\[
\text{Lip}(F|_{\Phi_{\delta_\star}}) \leq \frac{1}{2} \max_{1 < i < m} |\lambda_i| = L_1.
\]
**TABLE 8.1**

<table>
<thead>
<tr>
<th>The factors</th>
<th>$O(\sup G)$</th>
<th>$O(\text{Lip}(G))$</th>
<th>$O(\text{Lip}(g))$</th>
<th>$O(\text{Lip}(\Phi))$</th>
<th>$O(\text{Lip}(\Phi)^\delta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>$\delta^{-1}$</td>
<td>$\delta^{-2}$</td>
<td>$1$</td>
<td>$1$</td>
<td>$\delta^{-1}$</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>$\delta^{-1}$</td>
<td>$\delta^{-2}$</td>
<td>$1$</td>
<td>$1$</td>
<td>$\delta^{-1}$</td>
</tr>
<tr>
<td>$\rho = \frac{d\rho}{dt}$</td>
<td>$\delta^{-1}$</td>
<td>$\delta^{-2}$</td>
<td>$1$</td>
<td>$1$</td>
<td>$\delta^{-1}$</td>
</tr>
<tr>
<td>$A\frac{d\alpha}{dt}$</td>
<td>$\delta^{-1}$</td>
<td>$\delta^{-2}$</td>
<td>$1$</td>
<td>$1$</td>
<td>$\delta^{-1}$</td>
</tr>
</tbody>
</table>

**CONSTRUCTION OF INERTIAL MANIFOLD**
### TABLE 8.2
Verification of Preliminary Results

| Row | $O(\sup |g|)$ | $O(\text{Lip}(G))$ | $O(\text{Lip}(g))$ | $O(\sup \| G \|)$ |
|-----|----------------|---------------------|----------------------|-----------------
<p>| 1   | $l(p(t_1)) \geq \text{dist}(R_\delta, \mathcal{P}(A^2) \setminus R_{2\delta})$ | $= \delta$ | Lip\left(\frac{dp}{dt}\right)$ | By Lemma 8.2 |
|     | $\leq \max_{1 \leq i \leq n} |\lambda_i| + \text{Lip}(F|<em>{\text{R}</em>{2\delta}})$ | $\leq 2L_1$ for $L_1$ as in (8.12) | Lip\left(\frac{dp}{dt}\right)(t_1) \leq L_1 \delta^2$ | $| \frac{dp}{dt} |<em>{p=0} = 0$, $\sup | \frac{dp}{dt} | \leq 4L_1 \delta$ |
| 2   | $\sup \left| \frac{dp}{dt} \right| \leq \sup | \frac{dp}{dt} |$ | Since $q(t_1) = \Phi(p_1(p</em>\delta))$: | By the result of the first row: | Since $\Phi(0) = 0$, $\sup | \Phi | \leq 2 \text{Lip}(\Phi) \delta$, $\sup | A\Phi | \leq 2 \text{Lip}(A\Phi) \delta$. |
|     | $\leq \frac{4L_1 \delta}{\delta} = 4L_1$ | Lip$(\Lambda(q(t_1))) \leq L_1 \text{Lip}(\Lambda)$ | Lip\left(\frac{dp}{dt}\right) \leq (2L_1 + 4L_1 L_1) \delta^{-1}$ | $| p |<em>{s,t}$, $| P</em>{w} S u |$ are monotonic. Thus $l(p(t_1)) \leq m \delta$. |
| 3   | As in row 1 | $l(p(t_1)) \leq L_3$ | As in row 1 | From row 2 |
|     | for $L_3$ as in (8.13) | $\leq 2L_1$ for $L_1$ as in (8.12) | | $\sup | \frac{dp}{dt} q(t_1) | \leq \sup | \frac{dp}{dt} | \sup | q(t_1) | \leq 8L_1 \text{Lip}(\Phi) \delta$ |
| 4   | Obvious | This is the result of row 2 in Table 8.1 | This is the result of row 3 in Table 8.1 | |</p>
<table>
<thead>
<tr>
<th>Row</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>[\text{Lip}(|\frac{dp}{dt}|) \geq \max_{1 \leq i \leq n}</td>
</tr>
<tr>
<td>6</td>
<td>[\sup \left( \left| \frac{dp}{dt} \right| \left| \frac{dp}{dt}(t_1) \right|^{-1} \right) \leq \sup \left{ \frac{dp}{dt} \right} \cdot \sup \left{ \frac{dp}{dt}(t_1) \right}^{-1} = A\Phi(p_1) + Q^k(p_1 + \Phi(p_1)), \text{ This is the result of row 5 in Table 8.1.}]</td>
</tr>
<tr>
<td>7</td>
<td>[\text{As in row 5 of Table 8.2.}]</td>
</tr>
<tr>
<td>8</td>
<td>[\sup \left[ \frac{l(p(t_1))}{|dp/dt(t_1)|} \right] \leq \sup \frac{l(p(t_1))}{\inf |dp/dt|} \leq \frac{m\delta}{L_1 \delta} \leq 2 \text{ Lip} \left( \frac{dq}{dt}(t_1) \right) \delta ]</td>
</tr>
</tbody>
</table>

By Lemma 8.2, and row 1, column 2 of Table 8.2: \[\text{Lip} \left( \left\| \frac{dp}{dt}(t_1) \right\| \right) \leq \frac{2L_1}{L_1^2 \delta} \]
$L_2, L_3$ denote positive real numbers such that the Lipschitz constants for the functions

$$p_1 : \mathcal{D} \to \mathcal{PH}, \quad l_1 : \mathcal{D} \to \mathbf{R}$$

$$p_\alpha \mapsto p(t_1), \quad p \mapsto l(p(t_1)).$$

satisfy

$$\text{Lip}(p_1) \leq L_2, \quad \text{Lip}(l_1) \leq L_3 \quad (8.13)$$

for all $\delta < \delta_*$.  

Step (6) requires an examination of the geometry of the extended domain.

**Lemma 8.4.** If $QF_\delta$ is extended so that

$$QF_\delta = \begin{cases} QF & \text{in } \mathcal{R}_\delta \\ QF_\delta & \text{on the wing} \\ 0 & \text{in } \mathcal{D}(A) \setminus \mathcal{R}_{2\delta} \end{cases}$$

then $QF_\delta$ is Lipschitz continuous over the extended domain and $\text{Lip}(QF_\delta) \to 0$ as $\delta \to 0$.

**Proof.** There are three combinations to consider.

**Case 1.** $u_1 \in \mathcal{R}_\delta, u_2 \in \mathcal{D}(A) \setminus \mathcal{R}_{2\delta}$. Since $F(0) = 0$ we have

$$\|QF_\delta(u_1) - QF_\delta(u_2)\| = \|QF(u_1)\| \leq \text{Lip}(F|_{\mathcal{R}_\delta}) \delta$$

$$\leq \text{Lip}(F|_{\mathcal{R}_\delta}) \|u_1 - u_2\|.$$

**Case 2.** $u_1 \in \mathcal{R}_\delta, u_2$ on the wing. We first obtain a geometric result using the Lipschitz constant of $\Phi$. In the line segment connecting $Pu_1$ and $Pu_2$ lies a point $p_3$ with $\|Pwp_3\| = \delta$ (see Fig. 8.1). Let $u_3 = p_3 + \Phi(p_3), a = u_2 - u_3, b = u_1 - u_3, c = u_1 - u_2$, and $L = \text{Lip}(\Phi)$. Since $\Phi(p_3) = \Phi(p_3)$ we have

$$\|a\|^2 = \|Pa\|^2 + \|Qa\|^2 \leq (1 + L^2)\|Pa\|^2 \leq (1 + L^2)\|Pc\|^2 \leq (1 + L^2)\|c\|^2,$$

so that if $K_2 = \sqrt{1 + L^2}$,

$$\|u_2 - u_3\| \leq K_2 \|u_1 - u_2\|. \quad (8.14)$$
Let $\beta$ be the angle determined by the vectors $a$, $c$. By the law of cosines, and (8.14)

$$
\| b \|^2 = \| a \|^2 + \| c \|^2 - 2 \cos \beta \| a \| \| c \|
$$

$$
\leq \begin{cases} 
(2 + L^2) \| c \|^2 & \text{if } 0 \leq \beta \leq \pi/2 \\
(2 + L^2 - 2 \cos \beta \sqrt{1 + L^2}) \| c \|^2 & \text{if } \pi/2 < \beta \leq \pi
\end{cases}
$$

$$
\leq (2 + L^2 + 2 \sqrt{1 + L^2}) \| c \|^2
$$

for all $\beta$. Setting $K_1 = (2 + L^2 + 2 \sqrt{1 + L^2})^{1/2}$, we have

$$
\| u_1 - u_3 \| \leq K_1 \| u_1 - u_2 \|. \quad (8.15)
$$

Now the estimates in (8.14) and (8.15) are independent of $u_1$, $u_2$, so it follows that

$$
\| Q\hat{F}_\delta(u_1) - Q\hat{F}_\delta(u_2) \| = \| QF(u_1) - Q\hat{F}_\delta(u_2) \|
$$

$$
\leq \| F(u_1) - QF(u_3) \| + \| Q\hat{F}_\delta(u_3) - Q\hat{F}_\delta(u_2) \|
$$

$$
\leq \text{Lip}(F|_{\mathcal{S}_2}) \| u_1 - u_3 \| + \text{Lip}(\hat{F}_\delta) \| u_3 - u_2 \|
$$

$$
\leq (K_1 \text{Lip}(F|_{\mathcal{S}_2}) + K_2 \text{Lip}(\hat{F}_\delta)) \| u_1 - u_2 \|
$$

for all $u_1$, $u_2$ in this case.

Case 3. $u_1$ on the wing, $u_2 \in \mathcal{D}(A) \setminus \mathcal{S}_2$. If $\| P_w u_2 \| \leq 2\delta$, we may argue as in Case 1. If $\| P_w u_2 \| > 2\delta$, we may argue as in Case 2.

The following Lipschitz extension result due to Kirzbraun and Valentine [13], completes the construction of the $Q$-component of $\hat{F}$, and hence Step (6).
THEOREM 8.5. Let $H$ be a Hilbert space. If $\Omega \subset H$ and $F: \Omega \to H$ is Lipschitz continuous, then $F$ has a Lipschitz extension $\hat{F}: \to H$ with $\text{Lip}(F) = \text{Lip}(\hat{F})$.

To construct the $P$-component of $\hat{F}$ we need only perform the truncation in Step (7). This involves fixing a smooth function $\eta: \mathbb{R}^+ \to [0, 1]$ such that $\eta(s) = 1$ for $0 \leq s \leq 1$, $\eta(s) = 0$ for $s \geq 2$, $|\eta'(s)| \leq 2$ for $s \geq 0$. Define $\eta_\delta(s) = \eta(s/(3\delta))$ and the $P$-component of the modified nonlinearity by

$$P\hat{F}_\delta(u) = \eta_\delta(\| A^2 u \|) PF(u).$$

Since $F(0) = 0$ we have

$$\sup_{u \in \mathfrak{A}_{\delta}^2} \{ \| F(u) \| \} \leq 6\delta \text{Lip}(F | \mathfrak{A}_{\delta}^2).$$

An application of Lemma 8.2 yields

$$\text{Lip}(P\hat{F}_\delta | \mathfrak{A}_{\delta}^2) \leq \text{Lip}(\eta_\delta) \sup_{\mathfrak{A}_{\delta}^2} (\| F \|) + \sup_{\mathfrak{A}_{\delta}^2} (\| \eta \|) \text{Lip}(F | \mathfrak{A}_{\delta}^2)$$

$$\leq 2(3\delta)^{-1} \cdot 6\delta \text{Lip}(F | \mathfrak{A}_{\delta}^2) + 1 \cdot \text{Lip}(F | \mathfrak{A}_{\delta}^2) = 5 \text{Lip}(F | \mathfrak{A}_{\delta}^2).$$

Thus $\text{Lip}(P\hat{F}_\delta) \to 0$ as $\delta \to 0$, and Step (7) is complete.

The combination of Steps (1) through (7) comprises a proof of Theorem 7.1.

APPENDIX

We will need several lemmas for the proof of Lemma 6.2. The first is a uniform estimate on the $P$-component of $S_i u_1 - S_i u_2$.

**Lemma A.1.** Suppose $Z(u_1 - u_2) < m$ for all $u_1, u_2 \in \mathcal{N}$. There exist positive constants $\gamma_\ast$, $c_1$, $c_2$ such that for each $\tilde{p}(t) = P(S_i u_1 - S_i u_2)$ with $u_1, u_2 \in \mathcal{N} \cap \mathcal{W}^3(\varphi) \cap \mathcal{R}_\gamma$, $\gamma < \gamma_\ast$, the estimate

$$c_1 \exp \left( -\int_0^t \lambda_k(\tau) \, d\tau \right) \leq \| A^2 \tilde{p}(t) \| \leq c_2 \exp \left( -\int_0^t \lambda_k(\tau) \, d\tau \right)$$

holds for some $k$ with $1 \leq k \leq m$.

**Proof:** It is a direct consequence of Lemma 5.3 that

$$S_i u_1 - S_i u_2 = \exp \left( -\int_0^t \lambda_k(\tau) \, d\tau \right) \{ c\psi_k + o(1) \}$$
for some \( k \) with \( 1 \leq k \leq m \). Let \( \tilde{c}_1 = c \min\{\tilde{\lambda}_i^* : 1 \leq i \leq m\} \). Then since \( \psi_k \in PH \), one has

\[
[\tilde{c}_1 - o(1)] \exp \left( - \int_0^t \lambda_k(\tau) \, d\tau \right) \leq \|A^2\tilde{p}(t)\|.
\]

Now the initial conditions \( u_1, u_2 \) act as parameters for the family of variational equations they satisfy. By continuity and the compactness of the manifolds with boundary the decaying term can be chosen uniformly in the initial condition. By taking a small enough \( \gamma_* \) we may assume a sufficiently large time period has passed to reduce the \( o(1) \) term to half of \( \tilde{c}_1 \). Similarly we obtain the uniform upper bound.

**Lemma A.2.** For \( \lambda_k(s), \tilde{\lambda}_k^* \) defined as in Section 3.2 there exist \( \gamma_{**} \) such that \( |\lambda_k(s) - \tilde{\lambda}_k^*| < \alpha \) for all \( u_1(0), u_2(0) \in \mathcal{R}_{\gamma_*} \) and all \( s \geq 0 \).

**Proof.** This follows from the fact that the potential determining the eigenvalues \( \lambda_k(s) \) is given by

\[
a(x, t) = \int_0^t f'\left( u_2 - \alpha(u_2 - u_1) \right) \, dx.
\]

**Proof of Lemma 6.2.** By the continuity of solutions with respect to initial conditions we have that \( \mathcal{W}^u(\varphi) \subset \mathcal{N} \). Since \( \mathcal{W}^u(\varphi) \) is tangent to \( PH \) at \( \varphi \) it suffices to show that \( \mathcal{N} \cap \mathcal{W}^s(\varphi) \) is also tangent to \( PH \) at \( \varphi \).

Let \( \delta_* \) be given by Lemma A.1, and let \( L : \mathbb{R}^+ \to \mathbb{R}^+ \) be given by Lemma 8.1. Consider initial conditions \( u_1 \) and \( u_2 \) in \( \mathcal{N} \cap \mathcal{W}^s(\varphi) \cap \mathcal{R}_\delta \) where \( \delta < \delta_* \). Let

\[
\tilde{q}(t) = Q(S, u_1 - S, u_2).
\]

By the variation of constants formula one has

\[
\tilde{q}(t) = e^{-A^t}\tilde{q}(0) + \int_0^t e^{-A(t-s)}Q[F(S,u_1) - F(S,u_2)] \, ds.
\]

Applying the operator \( A^2 \) to both sides one obtains

\[
\|A^2\tilde{q}(t)\| \leq \|e^{-A^t}Q\| \|A^2\tilde{q}(0)\|
\]

\[
+ \int_0^t \|e^{-A(t-s)}Q\| \|A^2[F(S,u_1) - F(S,u_2)]\| \, ds \quad (A.1)
\]

where

\[
\|e^{A^t}Q\| = \sup \{\|e^{-A^t}Qu\| : \|u\| = 1\} \leq e^{-\lambda_{m-1}^* t}.
\]
By the Lipschitz condition for $F$ provided by Theorem 8.1 one has

$$\int_0^t \| e^{-A(t-s)}Q \| \| A^2 [F(S_t u_1) - F(S_t u_2)] \| \, ds$$

$$\leq L \int_0^t e^{-\lambda_{m+1}(t-s)} \| A^2 (S_t u_1 - S_t u_2) \| \, ds$$

$$\leq L \int_0^t e^{-\lambda_{m+1}(t-s)} \| A^2 \hat{q}(s) \| \, ds + L \int_0^t e^{-\lambda_{m+1}(t-s)} \| A^2 \tilde{q}(s) \| \, ds. \quad (A.2)$$

It follows from Lemmas A.1 and A.2 that

$$\int_0^t e^{-\lambda_{m+1} s} \| A^2 \tilde{p}(s) \| \, ds$$

$$\leq c_2 \int_0^t \exp \left( \lambda_{m+1} s - \int_0^s \lambda_k(\tau) \, d\tau \right) \, ds$$

$$\leq \sup_{0 < c < \infty} \left\{ \frac{c_2}{\lambda_{m+1} - \lambda_k(s)} \right\}$$

$$\times \int_0^t \exp \left( \lambda_{m+1} s - \int_0^s \lambda_k(\tau) \, d\tau \right) [\lambda_{m+1} - \lambda_k(s)] \, ds$$

$$\leq c_3 \left[ \exp \left( \lambda_{m+1} t - \int_0^t \lambda_k(\tau) \, d\tau \right) - 1 \right], \quad (A.3)$$

where $c_3 = c_2 [\lambda_{m+1} - \lambda_k - \alpha]^{-1} > 0$. Combining the estimates (A.1), (A.2), and (A.3) one has

$$e^{\lambda_{m+1} t} \| A^2 \tilde{q}(t) \| \leq \| A\tilde{q}(0) \| - c_3 L + c_3 L \exp \left( \lambda_{m+1} t - \int_0^t \lambda_k(\tau) \, d\tau \right)$$

$$+ L \int_0^t e^{\lambda_{m+1} s} \| A^2 \tilde{q}(s) \| \, ds. \quad (A.4)$$

Let $v(t) = e^{\lambda_{m+1} t} \| A^2 \tilde{q}(t) \|$ and

$$K(t) = K_0 + c_3 L \exp \left( \lambda_{m+1} t - \int_0^t \lambda_k(\tau) \, d\tau \right),$$

where $K_0 = \| A^2 \tilde{q}(0) \| - c_3 L$. One may then rewrite (A.4) as

$$v(t) \leq K(t) + L \int_0^t v(s) \, ds.$$
By a standard Gronwall inequality in [4] it follows that

$$v(t) \leq K(t) + L^{Lt} \int_0^t K(s) e^{-Lt} ds.$$  \hspace{1cm} (A.5)

Now consider $\delta < \delta_1$ where $2L(\delta_1) < \alpha$ so that

$$0 < \lambda_{m+1} - \lambda_k^+ - \alpha/2 < \lambda_{m+1} - \lambda_k(s) - L.$$  

Estimating the integral in (A.5) as in (A.3), one has

$$L^{Lt} \int_0^t K(s) e^{-Lt} ds \leq L^{Lt} \left\{ K_0 L^{-1} \left[ 1 - e^{-Lt} \right] 
+ c_3 c_4^{-1} L^2 \left[ \exp \left( \lambda_{m+1} t - \int_0^t \lambda_k(\tau) d\tau - Lt \right) - 1 \right] \right\}$$

$$\leq K_0 e^{Lt} - K_0 - c_3 c_4^{-1} L^2 e^{Lt}$$

$$+ c_3 c_4^{-1} L^2 \exp \left( \lambda_{m+1} t - \int_0^t \lambda_k(\tau) d\tau \right),$$

where $c_4 = \lambda_{m+1} - \lambda_k^+ - \alpha/2$. Applying the lower estimate provided by Lemma A.1 one obtains

$$\| A^2 \tilde{q}(t) \| \| A^2 \tilde{p}(t) \|^{-1}$$

$$\leq e^{-\lambda_{m+1} t} \left\{ K_0 + c_3 L \exp \left( \lambda_{m+1} t - \int_0^t \lambda_k(\tau) d\tau \right)
+ K_0 e^{Lt} - K_0 + c_3 c_4^{-1} L^2 \exp \left( \lambda_{m+1} t - \int_0^t \lambda_k(\tau) d\tau \right)
- c_3 c_4^{-1} L^2 e^{Lt} \right\} c_1 \exp \left( \int_0^t \lambda_k(\tau) d\tau \right)^{-1}$$

$$\leq (1 + c_4^{-1} L) L c_3 c_1^{-1}$$

$$+(K_0 - c_3 c_4^{-1} L^2) c_1^{-1} \exp \left( -\lambda_{m+1} t + \int_0^t \lambda_k(\tau) d\tau + Lt \right)$$

$$\leq (1 + c_4^{-1} L) L c_3 c_1^{-1} + \| A^2 \tilde{q}(0) \| c_1^{-1} e^{-ct}.$$  

By the compactness of the manifold $\mathcal{N}$ in $H^4$ and the continuity of the function

$$H^4 \to \mathbb{R} : u \mapsto \| A^2 u \|,$$
there exists $B \in \mathbb{R}^+$ with $\|A^2(\bar{q}(0))\| < B$. Take $\delta_2 < \delta_1$ so small and $t_1$ so large that

$$(1 + c_4^{-1} L) Lc_3 c_1^{-1} < \varepsilon/2, \quad Bc_1^{-1} e^{-c_4 t} < \varepsilon/2$$

for all $\delta < \delta_2$. Consider the compact set

$$\mathcal{C} = \{ \tilde{u} = u_1 - u_2 : u_i \in \mathcal{N} \cap \mathcal{W}^s(\varphi) \cap (\mathcal{R}_\delta - \mathcal{R}_{\delta/2}) \}.$$

The result now holds for all differences of solutions starting in $\mathcal{R}_\delta$, after time $t_1$. It suffices then to take

$$\delta = \min_{\tilde{u} \in \mathcal{C}} \{ \| A^2(S_{t_1} u_1 - S_{t_1} u_2) \| \}.$$

ACKNOWLEDGMENTS

The results presented here originally appeared as part of the author's Ph.D. thesis. He would like to take this opportunity to thank once again his advisor, George Sell for not only suggesting the problem, but also for a generous measure of guidance and support. Gratitude is due many others as well, but of particular help were discussions with John Baxter, Pavol Brunovsky, Ciprian Foias, Norman Meyers, and Mario Taboada.

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