On the Coxeter polynomials of wild stars

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Abstract

The spectral radius of a Coxeter transformation which plays an important role in the representation theory of hereditary algebras [see V. Dlab, C.M. Ringel, Eigenvalues of Coxeter transformations and the Gelfand–Kirillov dimension of the preprojective algebras, Proc. AMS 83 (1990) 228–232] is its important invariant. This paper provides both upper and lower bounds for the spectral radii of the Coxeter transformations of wild stars (i.e. trees that have a single branching point and are neither of Dynkin nor of Euclidean type). In addition, the paper determines limit of the spectral radii of particular infinite sequences of wild stars and shows different classes of graphs with the same limit. The basic idea is to reduce the study of spectral radii of trees to the spectral radii of particular valued graphs with indefinite type of associated generalized Cartan matrix. © 1999 Elsevier Science Inc. All rights reserved.

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1. Definitions and preliminary results

The purpose of the paper is to characterize the spectrum of the trees with one branching point.

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PII: S 0 0 2 4 - 3 7 9 5 ( 9 9 ) 0 0 0 3 3 - 6
Throughout the graph $\Delta$ will be always a finite graph without multiple edges and without loops, i.e. a finite set $I = \{1, 2, \ldots, n\}$ of the vertices, together with a set of (unordered) pairs $(i, j) \in I \times I$, $i \neq j \in I$ (called the edges of $\Delta$).

A valued graph $(\Delta, v)$ is a graph $\Delta$ together with a valuation $v$ defined as follows:

For each edge $i \longrightarrow j$, there are attached two positive integers $v_{i,j}$ and $v_{j,i}$: $i \ (v_{i,j}, v_{j,i}) j$ such that there are positive integers $f_i, i \in I$, satisfying

$$v_{i,j} f_j = v_{j,i} f_i.$$

If $i \ (1, 1) j$, we write simply $i \longrightarrow j$. Furthermore, set $v_{i,j} = v_{j,i} = 0$ if there is no edge between $i$ and $j$. Thus, a graph $\Delta$ is a valued graph $(\Delta, v)$ with a trivial valuation ($v_{i,j} = 0$ or $v_{i,j} = 1$ for all $i, j$).

The matrix $A_{\Delta} = (a_{ij})$, where $a_{ij} = v_{i,j}$ is called the adjacency matrix of the valued graph $(\Delta, v)$. The adjacency matrix $A = A_{\Delta}$ is symmetrizable i.e. $DA$ is a symmetric matrix for a diagonal matrix $D = (d_{ij})$, where $d_{i,i} = f_i$ and $d_{i,j} = 0$ otherwise.

**Definition.** A (valued) star is a valued graph $(\Delta, v)$ such that the underlying graph $\Delta$ is a connected tree (i.e. a graph without cycles) such that there is a vertex $i_0 \in I$ satisfying the following condition: $i \neq i_0$, $v_{i,j} \neq 0$ for at most two $j \in I$.

An orientation $\Omega$ of a valued graph $(\Delta, v)$ given by prescribing, for each edge $(i, j)$ of $(\Delta, v)$, an order (indicated by an arrow: $i \longrightarrow j$ or $j \longrightarrow i$).

Since, for a valued star, our considerations will not depend on a particular orientation, we always choose the orientation such that for all $i \in I$ there is a unique, oriented path from $i_0$. Moreover, we choose an order in such a way that each path $i_0 \longrightarrow i \longrightarrow \cdots \longrightarrow i_t \longrightarrow i_s$ satisfies $i_{s+1} = i_s + 1$ for all $1 \leq s < t$.

Given an orientation $\Omega$ of a star $(\Delta, v)$, we define the Coxeter transformation (see e.g. [3]) $\mathcal{C} : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $\mathcal{C} = \mathcal{C}(\Delta, \Omega)$ as follows:

For all $x \in \mathbb{C}^n$, $\mathcal{C}(x) = \sigma_1 \sigma_2 \cdots \sigma_n(x)$, where $\sigma_i(y) = z$ is given by

$$z_j = y_j \text{ for } j \neq i; \quad z_i = [\sigma_i(y)]_i = \sum_{j \in I} v_{j,i} y_j - y_i.$$

Some valued stars are of particular importance: For example, the Dynkin and some Euclidean graphs are valued stars. All other stars will be called wild stars.

The characteristic polynomial $z_{\Delta}(x)$ of the matrix of a Coxeter transformation $\mathcal{C}$ is called the Coxeter polynomial of $\mathcal{C}$.

The spectrum $\text{Spec}(\Delta, v)$ of the valued graph $(\Delta, v)$ is the set of the eigenvalues of the adjacency matrix $A = A(\Delta) = (a_{ij})$ of $(\Delta, v)$. If $\text{Spec}(\mathcal{C}(\Delta, \Omega))$ is the
spectrum (i.e. the set of all eigenvalues) of the Coxeter transformation $\mathcal{C}$, the spectral radius of $\mathcal{C}$ is given by

$$\rho(\mathcal{C}(A,\Omega)) = \max\{\|\lambda\| : \lambda \in \text{Spec}(\mathcal{C}(A,\Omega))\}.$$  

If the valuation $v$ is trivial, we write simply $\chi_A$ and $\rho(\Delta)$. Throughout our paper, we will use freely the following statements.

**Proposition 1.1** [1]. The Coxeter polynomial of a valued tree does not depend on its orientation.

The following statement is proved for bipartite finite oriented graph without oriented cycle.

**Theorem 1.2** [1]. Let $\Delta = (A, v)$ be a valued tree. Then

(a) Given $0 \neq \lambda$ then $\lambda + \lambda^{-1} \in \text{Spec}(\Delta)$ if and only if $\lambda^2 \in \text{Spec}(\mathcal{C}_d)$. 

(b) $\text{Spec}(\mathcal{C}_d) \subseteq S^1 \cup \mathbb{R}^+$, where $S^1 = \{\lambda : \|\lambda\| = 1\}$.

(c) If $\Delta$ is not Dynkin, then there exists a real number $\lambda \geq 1$ such that $\rho(\Delta) = \lambda + \lambda^{-1}$ and $\rho(\mathcal{C}_d) = \lambda^2$. Moreover, $\Delta$ is Euclidean if and only if $\lambda = 1$.

**Theorem 1.3** [6,7]. Let $m$ be the maximum of the degrees of all vertices of a finite oriented graph $\Delta$ without oriented cycles. Then $\rho(\mathcal{C}_d) \leq m^2 - 2$.

Since the Perron–Frobenius theorem for non-negative matrices yields that $\Delta' \subset \Delta$ implies $\rho(\mathcal{C}_{\Delta'}) < \rho(\mathcal{C}_\Delta)$ (cf. [4]), we get immediately the following corollary.

**Corollary 1.4.** If $\Delta'$ is a subtree of a tree $\Delta$, neither of which is Dynkin, then $\rho(\mathcal{C}_{\Delta'}) < \rho(\mathcal{C}_\Delta)$.

2. Wild stars

Let $p = (p_1, p_2, \ldots, p_s)$, $s \geq 3$, be a sequence of positive integers $p_i$, $1 \leq i \leq s$ and let $n = \sum_{i=1}^s p_i + 1$. Denote by $A_{p_1,p_2,\ldots,p_s}$ the wild star with trivial valuation $v$ consisting of $s$ arms of length $p_1, p_2, \ldots, p_s$, and denote by $\chi_{p_1,p_2,\ldots,p_s}(x)$ and $\rho(\mathcal{C}_{p_1,p_2,\ldots,p_s})$ the characteristic polynomial and the spectral radius of the corresponding Coxeter transformation, respectively.

For a star, we have the following proposition.

**Proposition 2.1** [5]. The Coxeter polynomial of a wild star has exactly two real roots and one irreducible non-cyclotomic factor.
We set \( v_k(x) = x^k - 1/x - 1, k \in \mathbb{Z} \). It is easy to see that the Coxeter transformation of the linear graph \( 1 \xrightarrow{2} \cdots \xrightarrow{n-1} n \) is \( v_{n+1}(x) \).

**Definition.** The valued trees \( \Gamma_1 \) and \( \Gamma_2 \) are called **quasi-cospectral** if they have the same non-cyclotomic irreducible factor of their Coxeter polynomial.

The following lemma allows us to count the Coxeter polynomials of graphs containing only two different types of arms.

**Lemma 2.2.** *The wild stars*

\[
\Gamma_{n,k}^{(r,s)} = A[\underbrace{n,
\ldots,n}_r\underbrace{k,
\ldots,k}_s]^{r\text{-times}}
\]

and the valued lines

\[
\hat{\Gamma}_{n,k}^{(r,s)} = \xrightarrow{n+1}\xrightarrow{n}\xrightarrow{\cdots}\xrightarrow{3}\xrightarrow{2} (1,r) \xrightarrow{1}\xrightarrow{(s,1)} \xrightarrow{n+2}\xrightarrow{n+3}\xrightarrow{n+k}\xrightarrow{n+k+1}
\]

are quasi-cospectral.

**Proof.** Let us denote by \( P_i \) the adjacency matrix of a line of length \( i \) and the vectors \( e_{1,i} = (1,0,\ldots,0)_{i-1}, f_{1,i} = (e_{1,i})^T \).

The adjacency matrix \( A \) of the graph \( \Gamma_{n,k}^{(r,s)} \) is as follows:

\[
A = \begin{bmatrix}
0 & e_{1,n} & e_{1,n} & \ldots & e_{1,n} & e_{1,k} & \ldots & e_{1,k} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
f_{1,n} & P_n & 0 & \ldots & 0 & 0 & \ldots & 0 \\
f_{1,n} & 0 & P_n & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
f_{1,n} & 0 & \ldots & \ldots & P_n & 0 & \ldots & 0 \\
f_{1,k} & 0 & 0 & \ldots & 0 & P_k & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
f_{1,k} & 0 & 0 & \ldots & 0 & 0 & \ldots & P_k \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
f_{1,k} & 0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & \ldots & P_k
\end{bmatrix}
\]

Let

\[
\Gamma_2 = \hat{\Gamma}_{n,k}^{(r,s)} \cup L_n \cup \cdots \cup L_n \cup L_k \cup \cdots \cup L_k,
\]

where \( r \) and \( s \) are the numbers of arms of type 1 and 2, respectively.
where \( L_j \) denotes the linear graph of length \( j \). Clearly, the adjacency matrix \( \hat{A} \) of \( C_2 \) is the matrix

\[
\hat{A} = \begin{pmatrix}
0 & re_{1,n} & 0 & \ldots & 0 & se_{1,k} & 0 & \ldots & 0 \\
fr_{1,n} & P_n & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & P_n & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & 0 & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & P_n & 0 & 0 & \ldots & 0 \\
f_{1,k} & 0 & 0 & \ldots & 0 & P_k & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \ldots & P_k \\
\end{pmatrix},
\]

Denote by \( I_j \) the identity matrix of size \( j \times j \). Clearly, the matrix

\[
T = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_n & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_n & I_n & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & I_n & 0 & \ldots & I_n & 0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & I_n & 0 & \ldots & 0 & I_n & -I_n & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_n & -I_n & \ldots & -I_n & -I_n & I_n & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \ldots & \ldots & 0 & 0 & 0 & I_k & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \ldots & \ldots & 0 & 0 & 0 & I_k & I_k & 0 & \ldots & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 0 & 0 & I_k & 0 & \ldots & I_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & I_k & I_k & -I_k & 0 & \ldots & \ldots & I_k & -I_k & I_k & I_k & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

satisfies \( AT = T\hat{A} \). For example, in the case \( r = 2 \), \( s = 3 \) we have

\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & I_n & -I_n & 0 & 0 & 0 \\
0 & I_n & I_n & 0 & 0 & 0 \\
0 & 0 & 0 & I_k & 0 & 0 \\
0 & 0 & 0 & I_k & I_k & -I_k \\
0 & 0 & 0 & I_k & -I_k & I_k \\
\end{pmatrix}
\]

and
Suppose \( C \) inductively corresponds between the roots of the characteristic polynomial and of the largest positive real root of the polynomial \( q \) corresponding to linear components of \( v \) where \( C \) degree one. Let the interval the adjacency matrix and we get Thus, one may find recursively the polynomial \( \text{Proof.} \) The characteristic polynomial of \( C_2 \) can be obtained by (see \([2]\))

\[
\chi_{C_2}(x) = (x + 1)\chi_{C_1}(x) - x\chi_{C_0}(x).
\]

The Coxeter polynomial \( \chi_{C_i} \) satisfies the recurrence relation

\[
\chi_{C_{i+1}}(x) = (x + 1)\chi_{C_i}(x) - x\chi_{C_{i-1}}(x).
\]

Thus, one may find recursively the polynomial \( \chi_{C_i}(x) \)

\[
\chi_{C_i}(x) = (x + 1)\chi_{C_{i-1}}(x) - x\chi_{C_{i-2}}(x)
\]

and we get

\[
\chi_{C_i}(x) = v_{k-1}(x)(\chi_{C_2}(x) - \chi_{C_1}(x)) + \chi_{C_1}(x).
\]

Denote \( \rho_k \) the spectral radius of Coxeter transformation of \( C_k \) (i.e. \( \rho_k = \rho(\mathcal{C}(C_k)) \)), and write \( \rho = \lim_{k \to \infty} \rho_k \). Taking \( x = \rho_k \), we have \( v_{k-1}(\rho_k)(\chi_{C_2}(\rho_k) - \chi_{C_1}(\rho_k)) = -\chi_{C_1}(\rho_k) \). Since by Theorem 1.3 the polynomial \( \chi_{C_1}(x) \) is bounded on the interval \([\rho_1, m^2 - 1]\), where \( m \) is the maximum of degrees of \( C_1 \), we get

\[
|\chi_{C_2}(\rho_k) - \chi_{C_1}(\rho_k)| = \frac{|\chi_{C_1}(\rho_k)|}{v_{k-1}(\rho_k)} \leq \frac{c}{v_{k-1}(1)} \leq \frac{c}{k},
\]

where \( c > 0 \) is the maximal value of \( |\chi_{C_1}(x)| \) for \( x \in [\rho_1, m^2 - 2] \).

It follows that, for arbitrary \( \epsilon > 0 \), there exists \( k \in \mathbb{N} \), such that \( |\chi_{C_2}(\rho_k) - \chi_{C_1}(\rho_k)| < \epsilon \). Consequently, \( \lim_{k \to \infty} \chi_{C_2}(\rho_k) - \chi_{C_1}(\rho_k) = \chi_{C_2}(\rho) - \chi_{C_1}(\rho) = 0 \). Thus, \( x_0 = \lim_{k \to \infty} \rho_k \) is a positive root of \( \chi_{C_2}(x) - \chi_{C_1}(x) \). \( \blacksquare \)
Lemma 2.4. The Coxeter polynomials of valued trees $\Gamma_1 = \bullet \rightarrow \bullet \rightarrow \frac{1}{2} (v, 1) \rightarrow \frac{3}{2}$ and $\Gamma_2 = \bullet \rightarrow \bullet \rightarrow \frac{1}{2} (v, 1) \rightarrow \frac{3}{2}$ are the same.

Proof. For the adjacency matrix $A_1$ of $\Gamma_1$ is

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & v & 0 \end{pmatrix}$$

and

$$(x_1, x_2, x_3) \xrightarrow{\sigma_1}(x_1, x_2, x_2 - x_3)$$

$$\xrightarrow{\sigma_2}(x_1, x_1 + vx_2 - vx_3 - x_2, x_2 - x_3)$$

$$\xrightarrow{\sigma_3}((v - 1)x_2 - vx_3, x_1 + (v - 1)x_2 - vx_3, x_2 - x_3),$$

i.e. the matrix of the Coxeter transformation of $\Gamma_1$ is

$$\Phi_1 = \begin{pmatrix} 0 & 1 & 0 \\ v - 1 & v - 1 & 1 \\ -v & -v & -1 \end{pmatrix},$$

and $\chi_{\Gamma_1}(x) = x^3 - (v - 2)x^2 - (v - 2)x + 1$. By similar calculation for the graph $\Gamma_2$, we have

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & v \\ 0 & 1 & 0 \end{pmatrix}$$

and the corresponding matrix

$$\Phi_2 = \begin{pmatrix} 0 & 1 & 0 \\ v - 1 & v - 1 & v \\ -1 & -1 & -1 \end{pmatrix},$$

thus $\chi_{\Gamma_2}(x) = \chi_{\Gamma_1}(x) = x^3 - (v - 2)x^2 - (v - 2)x + 1$. □

The following statement shows that Coxeter polynomials of stars with different degree of their branching point may have the same irreducible factors.

Proposition 2.5. The wild stars

(i) $\Gamma_{0,m}^{(0,s)} = A_{\underbrace{[m, m, \ldots, m]}_{s \text{-times}}}$ and $\Gamma_{m-1,1}^{(1,s)} = A_{\underbrace{[m-1, 1, \ldots, 1]}_{s \text{-times}}}$

for $(m > 1)$, as well as
(ii) $\Gamma_{m,k}^{(1,s)} = \Delta_{m,k,k, \ldots, k}$ and $\Gamma_{k-1,m+1}^{(1,s)} = \Delta_{k-1,m+1,m+1, \ldots, m+1}$ for $k > 1$ are quasi-cospectral.

**Proof.** By Lemma 2.2, the non-cyclotomic irreducible factor of the Coxeter polynomials of $\Gamma_{0,m}^{(0,s)}$ and $\tilde{\Gamma}_{0,m}^{(0,s)}$ coincide, where

$$\tilde{\Gamma}_{0,m}^{(0,s)} = \frac{m}{m-1} \cdots \frac{2}{1} \frac{(1,s)}{1} \frac{1}{1}.$$  

Moreover, the irreducible non-cyclotomic factor of the Coxeter polynomial of the graph $\Gamma_{m-1,1}^{(1,s)}$ is the irreducible non-cyclotomic factor of the Coxeter polynomial of the graph

$$\tilde{\Gamma}_{m-1,1}^{(1,s)} = \frac{m}{m-1} \cdots \frac{1}{1} \frac{(s,1)}{m+1} \frac{1}{1}.$$  

The graphs $\tilde{\Gamma}_{0,m}^{(0,s)}$ and $\tilde{\Gamma}_{m-1,1}^{(1,s)}$ can be obtained from the graphs $\frac{3}{2} \frac{2}{1} \frac{(1,s)}{1}$ and $\frac{2}{1} \frac{(s,1)}{3}$ by adding a path. Using formula (1), we have

$$\chi_{\tilde{\Gamma}_{0,m}^{(0,s)}}(x) = \chi_{\tilde{\Gamma}_{m-1,1}^{(1,s)}}(x) = x^{m+1} - (s - 2)x^m - \cdots - (s - 2)x + 1. \quad (2)$$

Similarly, in consequence of Lemma 2.2, we see that the graphs $\tilde{\Gamma}_{m,k}^{(1,s)}$ and $\tilde{\Gamma}_{k-1,m+1}^{(1,s)}$ are both quasi-cospectral with the graph

$$\frac{m+1}{m} \cdots \frac{2}{1} \frac{(s,1)}{m+2} \frac{m+3}{m} \cdots \frac{m+k+1}{m}.$$  

It is known (see [6]) that if $\Delta_{[p_1,p_2, \ldots, p_s]} \neq \Delta_{[p_1,1, \ldots, 1]}$ is neither Dynkin nor Euclidean, then

$$s - 2 < \rho([p_1,p_2, \ldots, p_s]) < s - 1 \quad \text{if} \quad 1 < p_i < \infty \quad \text{for all} \quad 1 \leq i \leq s.$$  

**Remark.** Consider the graph $\Delta_{[2,1, \ldots, 1]}$ with $s > 3$ (which is not Dynkin). According to Lemma 2.3, the irreducible factor of its Coxeter polynomial equals to the irreducible factor of $\chi_{\Gamma}(x)$, where

$$\Gamma = \tilde{\Gamma}_{2,1}^{(1,s-1)} = \frac{3}{2} \frac{2}{1} \frac{(s-1,1)}{4} \frac{1}{1}.$$  

Easy calculation shows that $\chi_{\Gamma}(x) = x^4 - (s - 3)x^3 - (s - 3)x^2 - (s - 3)x + 1$, and $s - 3 < \rho(\Gamma) < s - 2$.

The following statement, determining the limit point of the spectral radii of wild stars, describes the position of the spectral radii between the above bounds.

For brevity, let us denote by $l.p.r.(f(x))$ the largest positive real root of the polynomial $f(x)$. Moreover, $p_i(t) = (p_1(t), p_2(t), \ldots, p_s(t))$ and $m = m_s(t) =$
min \{p_1(t), p_2(t), \ldots, p_s(t)\}, where \(p_i(t)\) for \(1 \leq i \leq s\) is a positive integer valued function of a discrete parameter \(t\).

**Proposition 2.6.** If \(\{A_{i[p_i]}|t \geq 1,\}\) is a sequence of wild stars and if \(\lim_{t \to \infty} m_i(t) = \infty\), then

\[
\lim_{t \to \infty} \rho(\mathcal{C}_{[p_i]}(t)) = s - 1.
\]

and

\[
\lim_{p_i \to \infty} \rho(\mathcal{C}_{[p_i,1,1,\ldots,1]}(s-1 \text{ times})) = s - 2.
\]

**Proof.** In order to prove the statement, we apply Corollary 1.4. Indeed, consider \(I_{0,1}^{0,m} = A_{m, m, \ldots, m}\) as a substar of \(\{A_{i[p_i]}|t \geq 1,\}\) and show that

\[
\lim_{t \to \infty} \rho(\mathcal{C}_{[m,m,\ldots,m]}(s \text{ times})) = \rho(\mathcal{C}_{I_{0,1}^{0,m}}) < s - 1.
\]

Write \(m = m_i(t)\) and by Boldt’s reduction formula [2] we have

\[
\chi_{m,m,\ldots,m}(x) = v_{m+1}^{s-1}(x)(x^{m+1} + 1 - (s - 2)xv_m(x)).
\]

By Lemma 2.3,

\[
\lim_{t \to \infty} \rho(\mathcal{C}_{I_{1,m}}(s \text{ times})) = 1, p.r. \left( \chi_{I_{0,2}^{0,m}}(x) - \chi_{I_{0,1}^{0,m}}(x) \right),
\]

and in view of (2),

\[
\chi_{I_{0,2}^{0,m}}(x) - \chi_{I_{0,1}^{0,m}}(x) = x^3 - (s - 2)x^2 - (s - 2)x + 1 - (x^2 - (s - 2)x + 1)
\]

\[
= x^2(x - (s - 1)),
\]

which establishes the first part of the proposition.

In a similar manner, for \(s' = s - 1\), we get the following relation

\[
\chi_{I_{0,2}^{0,m}}(x) - \chi_{I_{0,1}^{0,m}}(x) = x^3 - (s' - 2)x^2 - (s' - 2)x + 1 - (x^2 - (s' - 2)x + 1)
\]

\[
= x^2(x - (s' - 1)).
\]

Thus,

\[
\lim_{p_i \to \infty} \rho(\mathcal{C}_{[p_i,1,1,\ldots,1]}(s-1 \text{ times})) = s - 2
\]

completes the proof. \(\square\)
If one of the arms of a wild star has bounded length, then the spectral radius is less than \( s - 1 \), where, as before, \( s \) denotes the degree of the branching point of the star. As before, write \( \mathbf{p}_{t-1}(t) = (p_1(t), p_2(t), \ldots, p_{t-1}(t)) \) and \( m = m_{t-1}(t) = \min\{p_1(t), p_2(t), \ldots, p_{t-1}(t)\} \).

**Proposition 2.7.** If \( \{A_{k_{t-1}(t)} \mid t \geq 1,\} \) is a sequence of wild stars and if \( \lim_{t \to \infty} m_{t-1}(t) = \infty \), then

\[
\lim_{t \to \infty} \rho(A_{k_{t-1}(t)}) < s - 1
\]

and

\[
\lim_{t \to \infty} \rho(A_{k_{t-1}(t)}) = l.p.r. \left( x^{k+1} - (s - 2)x^k - \cdots - (s - 2)x - (s - 2) \right).
\]

**Proof.** We proceed similarly as in the proof of Proposition 2.6. Consider \( \overrightarrow{\mathcal{C}_{k, m, \ldots, m}} \) as a substar of \( \overrightarrow{\mathcal{A}_{k, p_{t-1}(t)}} \) and apply Corollary 1.4, Lemmas 2.2 and 2.3. We have, for \( s' = s - 1 \),

\[
\lim_{t \to \infty} \rho(A_{k_{t-1}(t)}) = \lim_{t \to \infty} \rho(A_{k_{t-1}(t)}),
\]

where

\[
\hat{\Gamma}_{k, m} = \begin{array}{cccccccc}
\bullet & \cdots & \bullet & (s, 1) & \bullet & \cdots & \bullet & \bullet \end{array},
\]

Since

\[
\lim_{m \to \infty} \rho(A_{k_{t-1}(t)}) = l.p.r. \left( \chi_{k, m}^{1, s'}(x) - \chi_{k, m}^{1, s'}(x) \right)
\]

\[
= l.p.r. \left( x^{k+3} - (s' - 2)x^{k+2} - (2s' - 3)x^k \right. \\
- \cdots - (2s' - 3)x^2 - (s' - 2)x + 1 \\
- (x^{k+2} - (s' - 2)x^{k+1} - (s' - 2)x^k \\
- \cdots - (s' - 2)x^2 - (s' - 2)x + 1 \\
\left. = l.p.r. \left( x^{k+1} - (s' - 1)x^k - \cdots - (s' - 1)x - (s' - 1) \right) \right)\right.
\]

The Coxeter polynomial of \( \chi_{k, m}^{1, s'}(x) \) can be obtained from the Coxeter polynomial of \( \Gamma_0 = \begin{array}{ccccccc}
\bullet & \cdots & \bullet & (s, 1) & \bullet \end{array}, \) and \( \Gamma_1 = \begin{array}{ccccccc}
\bullet & \cdots & \bullet & (s, 1) & \bullet \end{array}, \) using formula (1). Finally, if

\[
g(x) = x^{k+1} - (s' - 1)x^k - \cdots - (s' - 1)x - (s' - 1),
\]

then \( g(s') = 1 > 0 \) and \( g(s' - 1) < 0 \), and the proof is completed. \( \square \)
The following lemma expresses that the limit point of the spectral radius does not depend in some cases on the number of arms tending to infinity.

**Lemma 2.8.** Let \( \Gamma_1 = \Delta_{(k-1,m,m,...,m)} \) and \( \Gamma_2 = \Delta_{(m,k,m,...,k)} \) be wild stars with the same degree of their branching point. Then, for fixed \( k \in \mathbb{Z} \) and \( m > 1 \),

\[
\lim_{m \to \infty} \rho(\mathcal{G}_{\Gamma_1}) = \lim_{m \to \infty} \rho(\mathcal{G}_{\Gamma_2}).
\]

**Proof.** Denote \( s \) the degree of the branching point of \( \Gamma_1 \) and \( \Gamma_2 \).

Since \( \lim_{m \to \infty} \rho(\mathcal{G}_{\Gamma_1}) = l.p.r. (\chi_{p_{1,s}}^{(k)}(x) - \chi_{p_{1,s}}^{(1)}(x)) \) and \( \lim_{m \to \infty} \rho(\mathcal{G}_{\Gamma_2}) = l.p.r. (\chi_{p_{1,s}}^{(k)}(x) - \chi_{p_{1,s}}^{(0,k)}(x)) \) we can see that, using relation (1) and Lemma 2.8, \( \chi_{p_{1,s}}^{(k)}(x) = \chi_{p_{1,s}}^{(1)}(x) \) and \( \chi_{p_{1,s}}^{(0,k)}(x) = \chi_{p_{1,s}}^{(1)}(x) \). \[\Box\]

Using the above notation and the idea of the proof of Proposition 2.6, one may deduce the following corollary.

**Corollary 2.9.** If \( \lim_{t \to \infty} m_{s-1}(t) = \infty \) then

\[
\lim_{t \to \infty} \rho(\mathcal{G}_{[p_{1,s}\overline{t}(t)]}) = \lim_{t \to \infty} \rho(\mathcal{G}_{[p_{1,s}\overline{t}]}).
\]

We can specialise our results to obtain the following classical results. Denoting by \( p.r.(f(x)) \) the positive root of the polynomial \( f(x) \), then

\[
\lim_{m \to \infty} \rho(\mathcal{G}_{[2,2,m]}) = \lim_{m \to \infty} \rho(\mathcal{G}_{[1,m,m]}) = p.r. (x^2 - x - 1) = (1 + \sqrt{5})/2;
\]

\[
\lim_{m \to \infty} \rho(\mathcal{G}_{[1,3,m]}) = p.r. (x^3 - x^2 - 1) (\sim 1.465);
\]

\[
\lim_{m \to \infty} \rho(\mathcal{G}_{[1,2,m]}) = p.r. (x^3 - x - 1) (\sim 1.3241).
\]

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**References**