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Advances in Applied Mathematics 41 (2008) 307–328

ADVANCES IN Applied Mathematics

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Meinardus' theorem on weighted partitions: Extensions and a probabilistic proof

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Received 26 February 2007; accepted 16 November 2007

Available online 19 March 2008

Abstract

The number c_n of weighted partitions of an integer n, with parameters (weights) b_k, $k \geqslant 1$, is given by the generating function relationship $\sum_{n=0}^{\infty} c_n z^n = \prod_{k=1}^{\infty}(1-z^k)^{-b_k}$. Meinardus (1954) established his famous asymptotic formula for c_n, as $n \to \infty$, under three conditions on power and Dirichlet generating functions for the sequence b_k. We give a probabilistic proof of Meinardus' theorem with weakened third condition and extend the resulting version of the theorem from weighted partitions to other two classic types of decomposable combinatorial structures, which are called assemblies and selections.

MSC: 60J27; 60K35; 82C22; 82C26

Keywords: Combinatorial structures; Local limit theorem

1. Summary

In this paper, we combine Meinardus' approach for deriving the asymptotic formula for the number of weighted partitions with the probabilistic method of Khintchine to develop a unified method of asymptotic enumeration of three basic types of decomposable combinatorial structures: multisets, selections and assemblies. As a byproduct of our approach we weaken one of the three Meinardus conditions. In accordance with these two objectives, the structure of the

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0196-8858/$ – see front matter
doi:10.1016/j.aam.2007.11.001

paper is as follows. Section 2 presents Meinardus' asymptotic formula, the presentation being accompanied by remarks clarifying the context of the three conditions of Meinardus' theorem. In Section 3 we state our main result which consists of asymptotic formulae for numbers of multisets, selections and assemblies. Sections 4, 5 and 6 are devoted to the proof of the main theorem, including the unified representation of basic decomposable random structures, which is the core of the probabilistic method considered. In Section 7 we discuss the striking similarity between the derived asymptotic formulae.

2. Meinardus' theorem

The Euler type generating function $f^{(1)}$ for the numbers $c_n^{(1)}$, $n \geqslant 1$, of weighted partitions of an integer n, with parameters $b_k \geqslant 0$, $k \geqslant 1$, is

$$f^{(1)}(z) := \sum_{n=0}^{\infty} c_n^{(1)} z^n = \prod_{k=1}^{\infty} \left(1 - z^k\right)^{-b_k}, \quad |z| < 1. \tag{1}$$

In this setting, b_k is interpreted as a number of types of summands of size k. (For example, one can imagine that coins of a value k are distinguished by b_k years of their production.) It is also assumed that in a partition, each summand of size k belongs to one of the b_k types. In the case $b_k = 1$ for all $k \geqslant 1$, $c_n^{(1)}$ is the number of standard (non-weighted) partitions of n (with $c_0 = 1$), while the case $b_k = k$, $k \geqslant 1$, conforms to planar partitions, studied by Wright, see [1], and the recent paper [17] by Mutafchiev. The study of the asymptotics of the general generating function (1) was apparently initiated by Brigham who obtained in [6] the asymptotic formula, as $n \to \infty$ for the logarithm of the function, using the Hardy–Ramanujan asymptotic technique. Meinardus' approach [16] to the asymptotics of $c_n^{(1)}$ is based on considering two generating series for the sequence $b_k \geqslant 0$, $k \geqslant 1$: the Dirichlet series $D(s)$ and the power series $G(z)$, defined by

$$D(s) = \sum_{k=1}^{\infty} b_k k^{-s}, \quad s = \sigma + it, \tag{2}$$

$$G(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |z| < 1. \tag{3}$$

We note that the function $f^{(1)}(z)$ converges at the point $|z| < 1$ if and only if the same is true for the function $G(z)$ (see e.g. [7, Lemma 1.15]).

Meinardus [16] established the following seminal asymptotic formula for $c_n^{(1)}$, which is presented in [1]. We denote by $\Re(\bullet)$ and $\Im(\bullet)$ the real and imaginary parts of a number.

Theorem 1 *(Meinardus). Suppose that the parameters $b_k \geqslant 0$, $k \geqslant 1$ of weighted partitions meet the following three conditions*:

(i) *The Dirichlet series* (2) *converges in the half-plane $\sigma > r > 0$ and there is a constant $0 < C_0 \leqslant 1$, such that the function $D(s)$, $s = \sigma + it$, has an analytic continuation to the half-plane*

$$\mathcal{H} = \{s\colon \sigma \geqslant -C_0\} \tag{4}$$

on which it is analytic except for a simple pole at $s = r$ with residue $A > 0$.

(ii) *There is a constant* $C_1 > 0$ *such that*

$$D(s) = O\big(|t|^{C_1}\big), \quad t \to \infty, \tag{5}$$

uniformly in $\sigma \geqslant -C_0$.

(iii) *There are constants* $C_2 > 0$, $\varepsilon > 0$ *such that the function*

$$g(\tau) := G\big(\exp(-\tau)\big), \quad \tau = \delta + 2\pi i\alpha,\ \delta > 0,\ \alpha \in R, \tag{6}$$

satisfies

$$\Re\big(g(\tau)\big) - g(\delta) \leqslant -C_2\delta^{-\varepsilon}, \quad |\arg\tau| > \frac{\pi}{4},\ 0 \neq |\alpha| \leqslant 1/2, \tag{7}$$

for $\delta > 0$ *small enough.*

Then, as $n \to \infty$,

$$c_n^{(1)} \sim C^{(1)} n^{\kappa_1} \exp\left(n^{r/(r+1)}\left(1 + \frac{1}{r}\right)\big(A\Gamma(r+1)\zeta(r+1)\big)^{1/(r+1)}\right), \tag{8}$$

where

$$\kappa_1 = \frac{2D(0) - 2 - r}{2(1+r)}$$

and

$$C^{(1)} = e^{D'(0)}\big(2\pi(1+r)\big)^{-1/2}\big(A\Gamma(r+1)\zeta(r+1)\big)^{\kappa_2},$$

where

$$\kappa_2 = \frac{1 - 2D(0)}{2(1+r)}.$$

Meinardus also gave a bound on the rate of convergence which we have omitted in the statement of Theorem 1.

At this point we wish to make a few clarifying comments on the three Meinardus conditions (i)–(iii).

- The Ikehara–Wiener Tauberian theorem on Dirichlet series cited below tells us that condition (i) implies a bound on the rate of growth, as $k \to \infty$ of the coefficients b_k of the Dirichlet series $D(s)$ in (2).

Theorem 2 *(Wiener–Ikehara). (See [15, Theorem 2.2, p. 122].) Suppose that the Dirichlet series* $D(s) = \sum_{k=1}^{\infty} a_k k^{-s}$ *is such that the function* $D(s) - \frac{A}{s-1}$ *has an analytic continuation to the closed half-plane* $\Re(s) \geqslant 1$. *Then,*

$$\sum_{k=1}^{n} a_k \sim An, \quad n \to \infty. \tag{9}$$

We will use the fact that (9) implies

$$a_k = o(k), \quad k \to \infty. \tag{10}$$

To prove this, we rewrite (9) as

$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}a_n + \frac{1}{n}\sum_{k=1}^{n-1} a_k = A + \varepsilon_n, \quad \varepsilon_n \to 0,\ n \to \infty,$$

which gives

$$\frac{1}{n}a_n + \frac{n-1}{n}(A + \varepsilon_{n-1}) = A + \varepsilon_n.$$

Consequently, $\lim_{n\to\infty} a_n/n = 0$.

Now set $a_k = k^{-r+1} b_k$, $k \geqslant 1$, where b_k, $k \geqslant 1$ satisfy Meinardus' conditions (i) and (ii). Since $C_0, r > 0$, the sequence a_k obeys the conditions of the Wiener–Ikehara theorem, so that we get from (10) the bound:

$$b_k = o\big(k^r\big), \quad k \to \infty. \tag{11}$$

- Functions satisfying Meinardus' condition (ii) are called *of finite order* in the corresponding domain. It is known (see e.g. [20, p. 298]) that the sum D of a Dirichlet series is a function of a finite order in the half-plane of the convergence of the series. Thus, condition (ii) requires that the same holds also for the analytic continuation of D in the domain $\mathcal{H}$.
- We show below that the condition (iii) is associated with bounding the so-called zeta sum known from the theory of the Riemann zeta function. In fact,

$$\Re\big(g(\tau)\big) - g(\delta) = -2\sum_{k=1}^{\infty} b_k e^{-k\delta} \sin^2(\pi k\alpha), \quad \delta > 0,\ \alpha \in \mathbb{R},$$

which allows us to reformulate (7) as

$$2\sum_{k=1}^{\infty} b_k e^{-k\delta} \sin^2(\pi k\alpha) \geqslant C_2\delta^{-\varepsilon}, \quad 0 < \frac{\delta}{2\pi} < |\alpha| \leqslant 1/2, \tag{12}$$

for $\delta > 0$ small enough and some $\varepsilon > 0$.

The verification of condition (iii) in the forthcoming Lemma 1 relies on the lower bound (14) below, for the sum $\sum_{k=1}^{P} \sin^2(\pi k\alpha)$, $\alpha \in \mathbb{R}$. This bound can be derived from the following bound on the zeta sum in the left-hand side of (13) (see [12, Lemma 1, p. 112]):

$$\left|\sum_{k=1}^{P} e^{2\pi i k\alpha}\right| \leqslant \min\left\{P, \frac{1}{2\|\alpha\|}\right\}, \quad P > 1,\ \alpha \in \mathbb{R}, \tag{13}$$

where $\|\alpha\|$ denotes the distance from α to the nearest integer. It follows from (13) that for all $\alpha \in \mathbb{R}$

$$2\sum_{k=1}^{P}\sin^2(\pi k\alpha) \geqslant P - \left|\sum_{k=1}^{P} e^{2\pi ik\alpha}\right| \geqslant P - \min\left\{P, \frac{1}{2\|\alpha\|}\right\},$$

which is convenient to rewrite as

$$2\sum_{k=1}^{P}\sin^2(\pi k\alpha) \geqslant P\left(1 - \min\left\{1, \frac{1}{2P\|\alpha\|}\right\}\right). \tag{14}$$

Under the assumptions in Meinardus' condition (iii),

$$0 \neq \|\alpha\| = |\alpha| \leqslant 1/2, \qquad |\alpha|\delta^{-1} > \frac{1}{2\pi}. \tag{15}$$

Setting in (14)

$$P = P(\alpha, \delta) = \left[\frac{1 + |\alpha|\delta^{-1}}{2|\alpha|}\right] \geqslant 1, \tag{16}$$

where $[x]$ denotes the integer part of x and $\delta > 0$ is small enough, we get the desired bound,

$$2\sum_{k=1}^{P}\sin^2(\pi k\alpha) \geqslant \frac{\delta^{-1}}{2}, \tag{17}$$

provided (15) holds. It follows from the above that for any fixed $k_0 \geqslant 1$, and any $0 < \varepsilon = \varepsilon(\delta; k_0) < 1/2$,

$$2\sum_{k=k_0}^{P}\sin^2(\pi k\alpha) \geqslant \left(\frac{1}{2} - \varepsilon\right)\delta^{-1} := c\delta^{-1}, \tag{18}$$

if $\delta > 0$ is small enough and (15) holds.

In the proof of Lemma 1 below we will also use the fact that under the condition (15), the choice (16) of P provides

$$P\delta \leqslant \frac{1}{2}\left(1 + \frac{1}{|\alpha|\delta^{-1}}\right) < \frac{1}{2}(1 + 2\pi) := d. \tag{19}$$

It seems not to have been noticed that Meinardus' condition (iii) is rather easily satisfied, as is shown in the following lemma.

Lemma 1. *Let the sequence $\{b_k\}$ be such that $b_k \geqslant \rho k^{r-1}$, $k \geqslant k_0$ for some $k_0 \geqslant 1$ and some constants $\rho, r > 0$. Then* (12) *is satisfied.*

Proof. Because of (16), $P \geqslant \frac{1}{2}\delta^{-1}$ and therefore $P > k_0$ for $\delta > 0$ small enough. We have

$$2\sum_{k=1}^{\infty} b_k e^{-k\delta}\sin^2(\pi k\alpha) \geqslant 2\sum_{k=k_0}^{P} \rho k^{r-1} e^{-k\delta}\sin^2(\pi k\alpha)$$

$$\geqslant 2\rho e^{-P\delta}\sum_{k=k_0}^{P} k^{r-1}\sin^2(\pi k\alpha) := Q.$$

In order to get the needed lower bound on Q implied by (12), we need to distinguish between the following two cases: (i) $0 < r < 1$ and (ii) $r \geqslant 1$. Applying (18) and (19) we have in case (i),

$$Q \geqslant \rho e^{-P\delta} P^{r-1} c\delta^{-1} \geqslant \rho e^{-d}(P\delta)^{r-1} c\delta^{-r} \geqslant \rho e^{-d} d^{r-1} c\delta^{-r}$$

and in case (ii),

$$Q \geqslant \rho e^{-d} k_0^{r-1} c\delta^{-1}.$$

Therefore, (12) is satisfied with $\varepsilon = r$ in case (i) and with $\varepsilon = 1$ in case (ii). $\square$

We note that in [17] the validity of condition (iii) was verified in the particular case of planar partitions ($b_k = k$, $k \geqslant 1$), via a complicated analysis of the power series expansion of the function $\Re(g(\tau)) - g(\delta)$.

Example 1. Let $b_k = \rho k^{r-1}$, $\rho, r > 0$, $k \geqslant 1$. Such weighted partitions are associated with the generalized Bose–Einstein model of ideal gas (see [21]). In this case, $D(s) = \rho\zeta(s - r + 1)$, where ζ is the Riemann zeta function. Thus, $D(s)$ has only one simple pole at $s = r > 0$ with the residue $A = \rho$ and it has a meromorphic analytic continuation to the whole complex plane $\mathbb{C}$. These facts together with Lemma 1 show that all three of Meinardus' conditions (i)–(iii) hold. In the case considered the values $D(0) = \rho\zeta(1 - r)$ and $D'(0) = \rho\zeta'(1 - r)$ in the asymptotic formula (8) can be found explicitly from the functional relation for the function ζ, as is explained in [17]. In particular, for standard partitions ($\rho = r = 1$),

$$D(0) = \zeta(0) = -\frac{1}{2}, \qquad D'(0) = \zeta'(0) = -\frac{1}{2}\log 2\pi,$$

while for planar partitions ($\rho = 1, r = 2$),

$$D(0) = \zeta(-1) = -\frac{1}{12}, \qquad D'(0) = \zeta'(-1) = 2\int_0^{\infty} \frac{w\log w}{e^{2\pi w} - 1}\,dw.$$

For an arbitrary $r > 0$, the expressions for $D(0)$, $D'(0)$ include the integral

$$\int_0^{\infty} \frac{w^{r-1}\log w}{e^{2\pi w} - 1}\,dw.$$

Example 2. The purpose of this example is to show that conditions (i) and (ii) of Theorem 1 do not imply condition (iii) in the same theorem. Let

$$b_k = \begin{cases} 1, & \text{if } 4 \mid k, \\ 0, & \text{if } 4 \nmid k. \end{cases}$$

Let $\alpha = 1/4$ in the sum $\sum_{k=1}^{\infty} b_k e^{-\delta k} \sin^2(\pi k \alpha)$. Then, because for all k either $b_k = 0$ or $\sin^2(\pi k/4) = 0$,

$$\sum_{k=1}^{\infty} b_k e^{-\delta k} \sin^2(\pi k/4) = 0$$

and therefore (12) is not satisfied. However,

$$D(s) = \sum_{j=1}^{\infty} (4j)^{-s} = 4^{-s} \zeta(s),$$

which clearly satisfies the first two of Meinardus' conditions because the function 4^{-s} is entire and $|4^{-s}| = 4^{-\sigma} \leqslant 4^{C_0}$ for $s \in \mathcal{H}$, where $\mathcal{H}$ is given by (4).

3. Statement of the main result

Our main result, Theorem 3 below, achieves two objectives: weakening the Meinardus condition (iii) and extending the resulting version of the Meinardus theorem from weighted partitions to other two types of classic decomposable combinatorial structures.

We first recall that a decomposable structure is defined as a union of indecomposable components of various sizes. It is known (see [2,3]) that the three types of decomposable combinatorial structures: multisets, which are also called weighted partitions, selections and assemblies, encompass the variety of classic combinatorial objects. Weighted partitions are defined as in the previous section, selections are defined as weighted partitions in which no component type appears more than once and assemblies are combinatorial objects composed of indecomposable components which are formed from labelled elements. Each decomposable structure is essentially determined by the number of types of its indecomposable components having a given size k. We denote this number by b_k for weighted partitions and selections and by m_k for assemblies. In the case of assemblies we denote $b_k = m_k/k!$, so that in all three cases b_k, $k \geqslant 1$, are parameters defining a structure. In what follows we will use the notation $\bullet^{(i)}$, $i = 1, 2, 3$, for quantities related to weighted partitions, selections and assemblies, respectively. Given a sequence b_k, $k \geqslant 1$, we define $c_n^{(i)} = s_n^{(i)}$ for $i = 1, 2$ and define $c_n^{(3)} = s_n^{(3)}/n!$, where $s_n^{(i)}$ denotes in all three cases the number of combinatorial structures of type i having size n.

Theorem 3. *Suppose that the parameters b_k, $k \geqslant 1$ meet Meinardus' conditions* (i) *and* (ii) *as well as the condition*

(iii′) *For $\delta > 0$ small enough and any $\varepsilon > 0$,*

$$2\sum_{k=1}^{\infty} b_k e^{-k\delta}\sin^2(\pi k\alpha) \geqslant \left(1+\frac{r}{2}+\varepsilon\right)M^{(i)}|\log\delta|,$$

$$\sqrt{\delta} \leqslant |\alpha| \leqslant 1/2, \quad i=1,2,3,$$

where the constants $M^{(i)}$ are defined by

$$M^{(i)} = \begin{cases} \frac{4}{\log 5}, & \text{if } i=1,\\ 4, & \text{if } i=2,\\ 1, & \text{if } i=3. \end{cases}$$

Then the asymptotics for $c_n^{(i)}$, $i=1,2,3$, as $n\to\infty$, are given respectively by Meinardus' formula (8)*, and by the formulae* (20)*,* (21) *below*:

$$c_n^{(2)} \sim C^{(2)} n^{-\frac{r+2}{2r+2}} \exp\left(n^{\frac{r}{r+1}}\left(1+\frac{1}{r}\right)\left(A\left(1-2^{-r}\right)\zeta(r+1)\Gamma(r+1)\right)^{\frac{1}{r+1}}\right), \tag{20}$$

where

$$C^{(2)} = 2^{D(0)}\left(2\pi(1+r)\right)^{-1/2}\left(A\Gamma(r+1)\left(1-2^{-r}\right)\zeta(r+1)\right)^{\frac{1}{2r+2}},$$

and

$$c_n^{(3)} \sim C^{(3)} n^{-\frac{r+2}{2r+2}} \exp\left(n^{\frac{r}{r+1}}\left(1+\frac{1}{r}\right)\left(A\Gamma(r+1)\right)^{\frac{1}{r+1}}\right), \tag{21}$$

where

$$C^{(3)} = e^{D(0)}\left(2\pi(1+r)\right)^{-1/2}\left(A\Gamma(r+1)\right)^{\frac{1}{2r+2}}.$$

Remark. H.-K. Hwang [11, p.109] applied the approach of Meinardus to the study of the asymptotics of the number of summands, say ω_n, in weighted partitions and selections, which he called unrestricted and restricted partitions, respectively. In the first case a local limit theorem for a properly scaled ω_n was obtained in [11] under the three conditions of Meinardus. Regarding restricted partitions, the author claimed the same under Meinardus' conditions (i), (ii) and a condition similar to our (iii′), but the proof contains an error in bounding the function $G_\theta(r)$ on p. 109.

Example 3. This example satisfies all three conditions of Theorem 3, but does not satisfy condition (iii) of Theorem 1. Let b_k, $k\geqslant 1$, be defined by

$$b_k = \begin{cases} 12e^7(\frac{\log k}{k}), & \text{if } 4\nmid k,\\ 12e^7(50+\frac{\log k}{k}-2\frac{\log(k/4)}{k/4}), & \text{if } 4\mid k \text{ and } 16\nmid k,\\ 12e^7(50+\frac{\log k}{k}-2\frac{\log(k/4)}{k/4}+\frac{\log(k/16)}{k/16}), & \text{if } 16\mid k. \end{cases}$$

Note that because $0 \leqslant \frac{\log x}{x} \leqslant e^{-1}$ for $x \geqslant 1$, it follows that $b_k \geqslant 12e^7(\frac{\log k}{k})$, for all $k \geqslant 1$. The Dirichlet series $D(s)$, $s = \sigma + it$, for this choice of b_k converges absolutely for $\sigma > 1$ and in this domain

$$D(s) = 12e^7\bigl(-\bigl(1 - 4^{-s}\bigr)^2\zeta'(s+1) + 50 \cdot 4^{-s}\zeta(s)\bigr), \tag{22}$$

where we have used the fact that $\zeta'(s+1) = -\sum_{k=1}^{\infty} \frac{\log k}{k} k^{-s}$. It is well known that the function $\zeta(s)$ has a simple pole at $s = 1$ and that the Laurent expansion of $\zeta(s+1)$ around $s = 0$ is

$$\zeta(s+1) = \frac{1}{s} + \gamma + \cdots, \tag{23}$$

where γ is Euler's constant. It follows from (23) that the function $\zeta'(s+1)$ has a unique pole at $s = 0$ of order 2. As a result, we derive that in (22) the first term in the parentheses is analytic in the whole complex plane $\mathbb{C}$, while the function D in (22) is analytic in $\mathbb{C}$ except a simple pole at $s = 1$. It is also a known fact that the functions ζ, ζ' satisfy (5) in $\mathbb{C}$, from which we conclude that the same is true for the function D given by (22). To show that condition (iii′) of Theorem 3 is satisfied, we note that, if $\delta > 0$ and $\varepsilon > 0$ are small enough then

$$\begin{aligned}
\sum_{k=1}^{\infty} b_k e^{-k\delta} \sin^2(\pi k\alpha) &\geqslant \sum_{k=1}^{P} b_k e^{-k\delta} \sin^2(\pi k\alpha) \\
&\geqslant 12e^7 e^{-d} \sum_{k=1}^{P} \frac{\log k}{k} \sin^2(\pi k\alpha) \\
&\geqslant 12e^7 e^{-d} \frac{\log P}{P} \sum_{k=3}^{P} \sin^2(\pi k\alpha) \\
&\geqslant 12e^{7-d} \frac{\log(d\delta^{-1})}{d\delta^{-1}} \left(\frac{1}{4} - \varepsilon\right)\delta^{-1} \qquad (24) \\
&> 6\log\bigl(\delta^{-1}\bigr), \quad (2\pi)^{-1}\delta < |\alpha| \leqslant 1/2, \qquad (25)
\end{aligned}$$

where we have used (18) and (19) at (24) and the fact that $3.5 < d < 4$ in the last step. Since in the case considered $r = 1$, the condition (iii′) is indeed satisfied for all three types of random structures. Finally, to show that condition (iii) of Theorem 1 in the form (12) is not satisfied, we set $\alpha = 1/4$ in the left-hand side of (12) to obtain for $\delta \to 0$,

$$\begin{aligned}
\sum_{k=1}^{\infty} b_k e^{-k\delta} \sin^2\left(\pi \frac{k}{4}\right) &\leqslant 12e^7 \sum_{k=1}^{\infty} \frac{\log k}{k} e^{-k\delta} \sim 12e^7 \int_1^{\infty} \frac{\log x}{x} e^{-x\delta}\, dx \\
&= 12e^7 \int_{\delta}^{1} \frac{\log(\delta^{-1}x)}{x} e^{-x}\, dx + 12e^7 \int_1^{\infty} \frac{\log(\delta^{-1}x)}{x} e^{-x}\, dx \\
&= O\bigl(\log^2\bigl(\delta^{-1}\bigr)\bigr).
\end{aligned}$$

Example 4. Consider the assembly of forests, for which components consist of labelled linear trees. The number of such components on k vertices is $m_k = k!$ and so $b_k = 1$, just as for integer partitions. The asymptotic number of labelled linear forests is thereby given by (21) in Theorem 3 with $r = 1$, $A = 1$. We note that the number of labelled linear forests on n vertices equals the number of path coverings of a complete graph on n vertices.

4. A unified probabilistic representation for decomposable combinatorial structures

It has been recently understood (see [8,21]) that the three main types of decomposable random structures: assemblies, multisets and selections, are induced by a class of probability measures on the set of integer partitions, having a multiplicative form. Vershik [21] calls the measures multiplicative, while Pitman [5,18] refers to them as Gibbs partitions. Equivalently, in combinatorics it is common to view the structures above as the ones generated by the conditioning relation (see [2,3]) or by the Kolchin generalized allocation scheme [14]. Our asymptotic analysis is based on the unified Khintchine type probabilistic representation of the number of decomposable structures of size n. Recall that we agree that the number of non-labelled structures (weighted partitions and selections) is denoted by $c_n^{(1)}$ and $c_n^{(2)}$, respectively, and the number of labelled structures (assemblies) by $n!c_n^{(3)}$. In all three cases the probabilistic representation of c_n is constructed as follows. Let f be a generating function of a sequence $\{c_n\}$ associated with some decomposable structure:

$$f(z) = \sum_{n \geqslant 1} c_n z^n.$$

A specific feature of decomposable structures is that the generating function f has the following multiplicative form:

$$f = \prod_{k \geqslant 1} S_k,$$

where S_k is a generating function for some non-negative sequence $\{d_k(j),\ j \geqslant 0,\ k \geqslant 1\}$, i.e.

$$S_k(z) = \sum_{j \geqslant 0} d_k(j) z^{kj}, \quad k \geqslant 1. \tag{26}$$

We now set $z = e^{-\delta + 2\pi i \alpha}$, $\alpha \in [0, 1]$, and use the orthogonality property of the functions $e^{-2\pi i \alpha n}$, $n \geqslant 1$, on the set $\alpha \in [0, 1]$, to get

$$\begin{aligned} c_n &= e^{n\delta} \int_0^1 f\big(e^{-\delta + 2\pi i \alpha}\big) e^{-2\pi i \alpha n}\, d\alpha \\ &= e^{n\delta} \int_0^1 \prod_{k=1}^{n} \big(S_k\big(e^{-\delta + 2\pi i \alpha}\big)\big) e^{-2\pi i \alpha n}\, d\alpha, \quad n \geqslant 1, \end{aligned} \tag{27}$$

where δ is a free parameter. We denote by

$$f_n := \prod_{k=1}^{n} S_k, \quad n \geqslant 1,$$

the truncated generating function. Next, we attribute a probabilistic meaning to the expression in the right-hand side of (27) by defining the independent integer-valued random variables Y_k, $k \geqslant 1$:

$$\mathbb{P}(Y_k = jk) = \frac{d_k(j)e^{-\delta kj}}{S_k(e^{-\delta})}, \quad j \geqslant 0,\ k \geqslant 1, \tag{28}$$

and observing that

$$\phi_n(\alpha) := \prod_{k=1}^{n} \frac{S_k(e^{-\delta+2\pi i\alpha})}{S_k(e^{-\delta})} = \frac{f_n(e^{-\delta+2\pi i\alpha})}{f_n(e^{-\delta})} = \mathbb{E}\big(e^{2\pi i\alpha Z_n}\big), \quad \alpha \in R, \tag{29}$$

is the characteristic function of the random variable

$$Z_n := \sum_{k=1}^{n} Y_k. \tag{30}$$

We have arrived at the desired representation:

$$c_n = e^{n\delta} f_n\big(e^{-\delta}\big)\mathbb{P}(Z_n = n), \quad n \geqslant 1. \tag{31}$$

In accordance with the principle of the probabilistic method considered, we will choose in (31) the free parameter $\delta = \delta_n$ to be the solution of the equation

$$\mathbb{E}Z_n = n, \quad n \geqslant 1, \tag{32}$$

after we show in the next section that for the three classic combinatorial structures the solution to (32) exists and is unique.

It can be easily seen from (26)–(28) that

$$\mathbb{E}Z_n = (\mathbb{E}Z_n)(\delta) = -\big(\log f_n\big(e^{-\delta}\big)\big)', \quad \delta > 0. \tag{33}$$

It is interesting to note that in the context of thermodynamics, the quantity $\log f_n(e^{-\delta})$ has a meaning of the entropy of a system. This important fact that clarifies the choice of the free parameter was observed already by Khintchine [13, Chapter VI], in the course of his study of classic models of thermodynamics.

From this point on, our study will be restricted to the three above mentioned classic combinatorial structures: multisets (weighted partitions), selections and assemblies. Recalling the forms of their generating functions $f^{(i)}$, $i = 1, 2, 3$ (see [2]), and denoting $\mathcal{F}^{(i)}(\delta) = f^{(i)}(e^{-\delta})$, $\delta > 0$, $i = 1, 2, 3$, we obtain

$$\mathcal{F}^{(1)}(\delta) = \prod_{k\geqslant 1} \left(1 - e^{-k\delta}\right)^{-b_k},$$

$$\mathcal{F}^{(2)}(\delta) = \prod_{k\geqslant 1} \left(1 + e^{-k\delta}\right)^{b_k},$$

$$\mathcal{F}^{(3)}(\delta) = \exp\left(\sum_{k\geqslant 1} b_k e^{-k\delta}\right). \tag{34}$$

Now it is easy to derive from (28) and (34) that the following three types of distributions for the random variables $\frac{1}{k}Y_k$ in (28): negative binomial $(b_k; e^{-\delta k})$, binomial $(b_k; \frac{e^{-\delta k}}{1+e^{-\delta k}})$ and Poisson $(b_k e^{-\delta k})$, produce respectively $c_n^{(i)}$, $i = 1, 2, 3$, in the representation (31).

The representation (31) for assemblies was obtained in [9], while the one for general multisets and selections was obtained in [10]. The corresponding truncated generating functions $f_n^{(i)}(z)$ are:

$$f_n^{(1)}(z) = \prod_{k=1}^{n} (1 - z^k)^{-b_k},$$

$$f_n^{(2)}(z) = \prod_{k=1}^{n} (1 + z^k)^{b_k},$$

$$f_n^{(3)}(z) = \exp\left(\sum_{k=1}^{n} b_k z^k\right). \tag{35}$$

Consequently, in the three cases considered Eq. (32) takes the forms (36)–(38) derived from (33):

$$\sum_{k=1}^{n} \frac{k b_k e^{-k\delta_n^{(1)}}}{1 - e^{-k\delta_n^{(1)}}} = n, \tag{36}$$

$$\sum_{k=1}^{n} \frac{k b_k e^{-\delta_n^{(2)} k}}{1 + e^{-\delta_n^{(2)} k}} = n, \tag{37}$$

$$\sum_{k=1}^{n} k b_k e^{-\delta_n^{(3)} k} = n. \tag{38}$$

5. Preliminary asymptotic results

In this section we find asymptotics for solutions to (36)–(38).

Lemma 2. *Suppose that the sequence $b_k \geqslant 0$, $k \geqslant 1$, is such that the associated Dirichlet generating function D satisfies the conditions* (i) *and* (ii) *of Theorem* 1. *Then:*

(i) *As* $\delta \to 0^+$,

$$\mathcal{F}^{(1)}(\delta) = \exp\bigl(A\Gamma(r)\zeta(r+1)\delta^{-r} - D(0)\log\delta + D'(0) + O\bigl(\delta^{C_0}\bigr)\bigr), \tag{39}$$

$$\mathcal{F}^{(2)}(\delta) = \exp\bigl(A\Gamma(r)\bigl(1-2^{-r}\bigr)\zeta(r+1)\delta^{-r} + D(0)\log 2 + O\bigl(\delta^{C_0}\bigr)\bigr), \tag{40}$$

$$\mathcal{F}^{(3)}(\delta) = \exp\bigl(A\Gamma(r)\delta^{-r} + D(0) + O\bigl(\delta^{C_0}\bigr)\bigr), \tag{41}$$

whereas asymptotic expressions for the derivatives

$$\bigl(\log\mathcal{F}^{(i)}(\delta)\bigr)^{(k)}, \quad i=1,2,3,\ k=1,2,3,$$

are given by the formal differentiation of the logarithms of (39)–(41):

$$\begin{aligned}
\bigl(\log\mathcal{F}^{(1)}(\delta)\bigr)^{(k)} &= (-1)^k A\Gamma(r+k)\zeta(r+1)\delta^{-r-k} + (-1)^k(k-1)!D(0)\delta^{-k} + O\bigl(\delta^{C_0-k}\bigr),\\
\bigl(\log\mathcal{F}^{(2)}(\delta)\bigr)^{(k)} &= (-1)^k A\Gamma(r+k)\bigl(1-2^{-r}\bigr)\zeta(r+1)\delta^{-r-k} + O\bigl(\delta^{C_0-k}\bigr),\\
\bigl(\log\mathcal{F}^{(3)}(\delta)\bigr)^{(k)} &= (-1)^k A\Gamma(r+k)\delta^{-r-k} + O\bigl(\delta^{C_0-k}\bigr).
\end{aligned} \tag{42}$$

(ii) *Each of Eqs.* (36)–(38) *has a unique solution* $\delta_n^{(i)}$ *such that* $\delta_n^{(i)} \to 0$, $n\to\infty$, $i=1,2,3$. *Moreover,*

(iii) *As* $n\to\infty$,

$$\begin{aligned}
&\delta_n^{(1)} = \bigl(A\Gamma(r+1)\zeta(r+1)\bigr)^{\frac{1}{r+1}} n^{-\frac{1}{r+1}} + \frac{D(0)}{r+1}n^{-1} + O\bigl(n^{-1-\beta}\bigr),\\
&\quad \textit{where}\quad \beta = \begin{cases} \frac{C_0}{r+1}, & \textit{if } r \geqslant C_0,\\ \frac{r}{r+1}, & \textit{otherwise};\end{cases}
\end{aligned} \tag{43}$$

$$\begin{aligned}
&\delta_n^{(2)} = \bigl(A\Gamma(r+1)\bigl(1-2^{-r}\bigr)\zeta(r+1)\bigr)^{\frac{1}{r+1}} n^{-\frac{1}{r+1}} + O\bigl(n^{-1-\beta}\bigr),\\
&\quad \textit{where } \beta = \frac{C_0}{r+1};
\end{aligned} \tag{44}$$

$$\begin{aligned}
&\delta_n^{(3)} = \bigl(A\Gamma(r+1)\bigr)^{\frac{1}{r+1}} n^{-\frac{1}{r+1}} + O\bigl(n^{-1-\beta}\bigr),\\
&\quad \textit{where } \beta = \frac{C_0}{r+1}.
\end{aligned} \tag{45}$$

(iv) *As* $n\to\infty$, *the derivatives* $f_n^{(i)}(e^{-\delta_n^{(i)}})$, $i=1,2,3$, *have the asymptotic expansions of the right-hand sides of* (39)–(42), *respectively, with* $\delta=\delta_n^{(i)}$.

Proof. (i) First consider the case of weighted partitions. Following the Meinardus approach, we will use the fact that e^{-u}, $u>0$, is the Mellin transform of the Gamma function:

$$e^{-u} = \frac{1}{2\pi i}\int_{v-i\infty}^{v+i\infty} u^{-s}\Gamma(s)\,ds, \quad u>0,\ v>0. \tag{46}$$

Expanding $\log \mathcal{F}^{(1)}(\delta)$ in (34) as

$$\log \mathcal{F}^{(1)}(\delta) = -\sum_{k\geqslant 1} b_k \log\bigl(1 - e^{-\delta k}\bigr) = \sum_{j\geqslant 1} \frac{1}{j} \sum_{k\geqslant 1} b_k e^{-\delta kj}$$

and substituting (46) with $v = 1 + r$ gives

$$\log \mathcal{F}^{(1)}(\delta) = \frac{1}{2\pi i} \int\limits_{1+r-i\infty}^{1+r+i\infty} \delta^{-s}\Gamma(s)\zeta(s+1)D(s)\,ds. \tag{47}$$

By Meinardus' condition (i), the function D has a simple pole at $r > 0$ with residue A, which says that the integrand in (47) has a simple pole at $s = r$ with residue $A\delta^{-r}\Gamma(r)\zeta(r+1)$. Next, from the Laurent expansions at $s = 0$ of the Riemann zeta function $\zeta(s+1) = \frac{1}{s} + \gamma + \cdots$ and the Gamma function $\Gamma(s) = \frac{1}{s} - \gamma + \cdots$, where γ is Euler's constant, and the Taylor series expansions at $s = 0$ of the two remaining factors of the integrand in (47), one concludes that the integrand has also a pole of a second order at $s = 0$ with residue $D'(0) - D(0)\log\delta$. We also recall that the only poles of $\Gamma(s)$ are at $s = -k$, $k = 0, 1, \ldots$. Hence, in the complex domain $-C_0 \leqslant \Re(s) \leqslant 1+r$, with $0 < C_0 < 1$, the integrand has only two poles at 0 and r with the above residuals. We now apply the residue theorem for the integrand in (47), over the above domain. The assumption (5) and the following two properties of zeta and Gamma functions:

$$\zeta(\sigma + 1 + it) = O\bigl(|t|^{C_2}\bigr), \quad t \to \infty,\ C_2 > 0,$$
$$\Gamma(\sigma + it) = O\left(|t|^{C_3} \exp\left(-\frac{\pi}{2}|t|\right)\right), \quad t \to \infty,\ C_3 > 0,$$

uniformly in σ, allow us to conclude that the integral of the integrand considered, over the horizontal contour $-C_0 \leqslant \Re(s) \leqslant 1 + r$, $\Im(s) = t$, tends to zero, as $t \to \infty$, for any fixed δ. Thus, we are able to rewrite (47) as

$$\begin{aligned}\log \mathcal{F}^{(1)}(\delta) &= A\delta^{-r}\Gamma(r)\zeta(r+1) - D(0)\log\delta + D'(0)\\ &\quad + \frac{1}{2\pi i} \int\limits_{-C_0-i\infty}^{-C_0+i\infty} \delta^{-s}\Gamma(s)\zeta(s+1)D(s)\,ds.\end{aligned} \tag{48}$$

Moreover, the previous two bounds and the bound (5) in Meinardus' condition (ii) imply that the integral in (48) is bounded by

$$\begin{aligned}&\left|\frac{1}{2\pi i} \int\limits_{-C_0-i\infty}^{-C_0+i\infty} \delta^{-s}\Gamma(s)\zeta(s+1)D(s)\,ds\right|\\ &\quad = O\left(\delta^{C_0} \int\limits_{-\infty}^{\infty} \exp\left(-\frac{\pi}{2}|t|\right)|t|^{C_1+C_2+C_3}\,dt\right)\\ &\quad = O\bigl(\delta^{C_0}\bigr), \quad \delta \to 0.\end{aligned}$$

This proves (39).

To prove the asymptotic formula for the first derivative $(\log \mathcal{F}^{(1)}(\delta))^{(1)}$, one has to differentiate (48) with respect to δ and then to estimate the resulting integral in the same way as above. Subsequent differentiations produce the asymptotic formulae for $(\log \mathcal{F}^{(1)}(\delta))^{(k)}$, $k = 2, 3$.

The proof of part (i) of the theorem for selections and assemblies is done in a similar way we now briefly describe. Following (34), the representation (47) conforms to

$$\log \mathcal{F}^{(2)}(\delta) = \frac{1}{2\pi i} \int_{1+r-i\infty}^{1+r+i\infty} \delta^{-s}\Gamma(s)\bigl(1-2^{-s}\bigr)\zeta(s+1)D(s)\,ds \tag{49}$$

and

$$\log \mathcal{F}^{(3)}(\delta) = \frac{1}{2\pi i} \int_{1+r-i\infty}^{1+r+i\infty} \delta^{-s}\Gamma(s)D(s)\,ds, \tag{50}$$

for all $\delta > 0$. Accordingly, the integrand in (49) has a simple pole at $s = r > 0$ with residue $A\delta^{-r}\Gamma(r)(1-2^{-r})\zeta(r+1)$, and a simple pole at $s = 0$ with residue $D(0)\log 2$, while the integrand in (50) has two simple poles at $s = r > 0$ and $s = 0$ with residues $A\delta^{-r}\Gamma(r)$ and $D(0)$, respectively. As a result, we obtain (40) and (41).

(ii) We see that the left-hand sides of the Eqs. (36)–(38) are decreasing as $\delta \geqslant 0$ in such a way that for a fixed n, in all the three cases the left-hand sides tend to 0 as $\delta \to +\infty$, while as $\delta \to 0$ the left-hand sides tend to $+\infty$, $\frac{1}{2}\sum_{k=1}^{n} kb_k$ and $\sum_{k=1}^{n} kb_k$, respectively. We now make use of Theorem 2 to get a lower bound (51) below on the sum $\sum_{k=1}^{n} kb_k$ when the sequence $\{b_k\}$ obeys Meinardus' conditions (i) and (ii). We set $a_k := k^{-r+1}b_k$, $k \geqslant 1$, and let $\tilde{D}(s)$ denote the Dirichlet series $\tilde{D}(s) = \sum_{k\geqslant 1} a_k k^{-s}$. Since $C_0, r > 0$, the function $\tilde{D}$ satisfies the conditions of Wiener–Ikehara theorem, with the constant A as in Meinardus' condition (i). Consequently,

$$\sum_{k=1}^{n} k^{-r+1}b_k = \sum_{k=1}^{n} \frac{kb_k}{k^r} \sim An, \quad n \to \infty,$$

from which it follows that for sufficiently large n,

$$\sum_{k=1}^{n} kb_k \geqslant Bn, \tag{51}$$

for some $B > 1$. This can be easily seen from the bound

$$\sum_{k=1}^{n} \frac{kb_k}{k^r} \leqslant \sum_{k=1}^{L-1} \frac{kb_k}{k^r} + \frac{1}{L^r}\sum_{k=L}^{n} kb_k$$

with $L < n$ such that $L^r A > 1$. Moreover, (11) implies that the series $\lim_{n\to\infty} \mathbb{E}Z_n^{(i)}$, $i = 1, 2, 3$, converge for any positive δ.

Combining the above facts, we conclude that each of Eqs. (36)–(38) has a unique solution for sufficiently large n and that the solutions $\delta_n^{(i)} \to 0$, $n \to \infty$, $i = 1, 2, 3$.

(iii) We firstly show that in all three cases,

$$\begin{aligned} &\mathbb{E}Z_n\big(\delta_n^{(i)}\big) = \big(-\log \mathcal{F}^{(i)}\big(e^{-\delta}\big)\big)'\big|_{\delta=\delta_n^{(i)}} + \varepsilon^{(i)}(n), \\ &\varepsilon^{(i)}(n) \to 0,\ n \to \infty,\ i = 1, 2, 3. \end{aligned} \tag{52}$$

In the case of weighted partitions, setting $\hat{\delta}_n = n^{-\frac{r+2}{2(r+1)}}$ gives for sufficiently large n

$$\begin{aligned} \sum_{k=n+1}^{\infty} \frac{k b_k e^{-k\hat{\delta}_n}}{1 - e^{-k\hat{\delta}_n}} &= O\left(\sum_{k=n+1}^{\infty} k b_k e^{-k\hat{\delta}_n}\right) = O\left(\sum_{k=n+1}^{\infty} k^{r+1} e^{-k\hat{\delta}_n}\right) \\ &= O\left(\int_{n+1}^{\infty} x^{r+1} e^{-x\hat{\delta}_n}\,dx\right) \to 0, \quad n \to \infty, \end{aligned} \tag{53}$$

where we have employed (11) and the fact that $n\hat{\delta}_n \to \infty$, $n \to \infty$. From (43) with $k = 1$ we deduce that $\delta_n^{(1)} > \hat{\delta}_n$ for large enough n, which implies that (53) is valid with $\hat{\delta}_n$ replaced by $\delta_n^{(1)}$. This proves (52) for the case considered. Consequently, Eq. (36) can be rewritten as

$$\begin{aligned} &A\Gamma(r+1)\zeta(r+1)\big(\delta_n^{(1)}\big)^{-r-1} + D(0)\big(\delta_n^{(1)}\big)^{-1} + O\big(\big(\delta_n^{(1)}\big)^{C_0-1}\big) + \varepsilon^{(1)}(n) = n, \\ &\varepsilon^{(1)}(n) \to 0,\ n \to \infty. \end{aligned} \tag{54}$$

We outline here the method of solution for asymptotic equations of the form (54) common in applications of Khintchine's method. Denoting the constant coefficient $h := A\Gamma(r+1)\zeta(r+1)$, (54) implies

$$h + D(0)\big(\delta_n^{(1)}\big)^{r} + O\big(\big(\delta_n^{(1)}\big)^{r+C_0}\big) + o\big(\big(\delta_n^{(1)}\big)^{r+1}\big) = n\big(\delta_n^{(1)}\big)^{r+1}. \tag{55}$$

Since $\delta_n^{(1)} \to 0$, $n \to \infty$, we obtain from (55) that $\delta_n^{(1)} \sim h^{\frac{1}{r+1}} n^{-\frac{1}{r+1}}$, $n \to \infty$. Based on this fact and the fact that $0 < C_0 < 1$, we get

$$\begin{aligned} \delta_n^{(1)} &= h^{\frac{1}{r+1}} n^{-\frac{1}{r+1}} + \frac{D(0)}{r+1} n^{-1} + O\big(\big(\delta_n^{(1)}\big)^{r} n^{-1}\big) + O\big(\big(\delta_n^{(1)}\big)^{C_0} n^{-1}\big) \\ &= h^{\frac{1}{r+1}} n^{-\frac{1}{r+1}} + \frac{D(0)}{r+1} n^{-1} + O\big(n^{-1-\beta}\big), \end{aligned}$$

where β is as in (43). For selections and assemblies the analogs of (55) will be, respectively

$$\begin{aligned} &h + O\big(\big(\delta_n^{(2)}\big)^{C_0+r}\big) + o\big(\big(\delta_n^{(2)}\big)^{r+1}\big) = n\big(\delta_n^{(2)}\big)^{r+1}, \\ &h = A\Gamma(r+1)\big(1 - 2^{-r}\big)\zeta(r+1) \end{aligned}$$

and

$$\begin{aligned} &h + O\big(\big(\delta_n^{(3)}\big)^{C_0+r}\big) + o\big(\big(\delta_n^{(3)}\big)^{r+1}\big) = n\big(\delta_n^{(3)}\big)^{r+1}, \\ &h = A\Gamma(r+1). \end{aligned}$$

Now the same reasoning as for weighted partitions leads to the solutions (44), (45).

(iv) In the case of weighted partitions, we have

$$\log f_n^{(1)}\big(e^{-\delta_n^{(1)}}\big) = \log \mathcal{F}_1\big(\delta_n^{(1)}\big) + \sum_{k\geqslant n+1} b_k \log\big(1 - e^{-k\delta_n^{(1)}}\big),$$

where

$$\left|\sum_{k\geqslant n+1} b_k \log\big(1 - e^{-k\delta_n^{(1)}}\big)\right| = O\bigg(\sum_{k\geqslant n+1} b_k e^{-k\delta_n^{(1)}}\bigg) = o(1), \quad n \to \infty,$$

by the argument giving (53). The proof of the remaining parts of the assertion (iv) is similar. □

Remark. As we mentioned before, Meinardus' proof (see [1]) of Theorem 1 relies on application of the saddle point method. In accordance with the principle of the method, the value in question $c_n^{(1)}$ is expressed as

$$c_n^{(1)} = \frac{1}{2\pi i} \int_{-1/2}^{1/2} \mathcal{F}^{(1)}(\tau) e^{n\delta + 2\pi i n\alpha}\, d\alpha, \quad \tau = \delta + 2\pi i\alpha, \tag{56}$$

by virtue of the Cauchy integral theorem. Here the free parameter δ is chosen as the minimal value of the function $\exp(A\Gamma(r)\zeta(r+1)\delta^{-r} + n\delta)$ viewed as an approximation of the absolute value of the integrand in (56) at $\alpha = 0$. This gives $\delta = h^{\frac{1}{r+1}} n^{-\frac{1}{r+1}}$, $h = A\Gamma(r+1)\zeta(r+1)$ which is the principal term of the solution $\delta_n^{(1)}$ of (36). It can be seen that, stemming from this choice of the free parameter, the subsequent steps of Meinardus' proof are considerably more complicated compared with ours. Also note that our choice of the free parameter is in the core of our ability to weaken Meinardus' condition (iii).

Our next assertion reveals that the function $\Re(g(\tau)) - g(\delta)$ in the left-hand side of the Meinardus' condition (iii) is inherent in the employed probabilistic method: the function provides an upper bound for the rate of exponential decay of the absolute value of the characteristic function ϕ_n in (29), as $n \to \infty$, for all three types of random structures considered. The bounds obtained in the forthcoming lemma are used in the proof of our local limit theorem, Theorem 4.

Recall that the function $g(\tau)$ is defined by (3) and (6).

Lemma 3. *Denote*

$$V(\alpha) = V(\alpha;\delta) = \Re\big(g(\tau)\big) - g(\delta) = -2\sum_{k=1}^{\infty} b_k e^{-k\delta} \sin^2(\pi\alpha k),$$
$$\tau = \delta + 2\pi i\alpha,\ \delta > 0,\ \alpha \in \mathbb{R},$$

and let $\delta = \delta_n^{(i)}$, $i = 1, 2, 3$, *be the unique solutions of Eqs.* (36)–(38), *respectively. Then, for all* $\alpha \in \mathbb{R}$,

$$\big|\phi_n^{(i)}(\alpha)\big| \leqslant \big(1 + \varepsilon_n^{(i)}\big) \exp\bigg(\frac{V^{(i)}(\alpha)}{M^{(i)}}\bigg), \quad \varepsilon_n^{(i)} = \varepsilon_n^{(i)}(\alpha) \to 0,\ n \to \infty,\ i = 1, 2, 3,$$

where $V^{(i)}(\alpha)=V(\alpha;\delta_n^{(i)})$ and the constants $M^{(i)}$, $i=1,2,3$, are as in condition (iii′) *of Theorem* 3.

Proof. From (29) we have for $n\geqslant 1$ and $\delta>0$ fixed,

$$\begin{aligned} &\left|\phi_n^{(i)}(\alpha)\right| = \exp\bigl(\Re\bigl(\log f_n^{(i)}\bigl(e^{-\tau}\bigr)\bigr) - \log f_n^{(i)}\bigl(e^{-\delta}\bigr)\bigr) := e^{V_n^{(i)}(\alpha;\delta)},\\ &\tau=\delta+2\pi i\alpha. \end{aligned} \tag{57}$$

Using (35) we now find bounds for $V_n^{(i)}(\alpha;\delta)$ expressed via $V^{(i)}(\alpha;\delta)$ in the three cases considered. By the definition of $V_n^{(i)}(\alpha;\delta)$ as given in (57), we have for weighted partitions,

$$\begin{aligned} V_n^{(1)}(\alpha;\delta) &= \Re\left(-\sum_{k=1}^n b_k\log\left(\frac{1-e^{-\tau k}}{1-e^{-\delta k}}\right)\right)\\ &= -\frac{1}{2}\sum_{k=1}^n b_k\log\left(\frac{1-2e^{-k\delta}\cos(2\pi\alpha k)+e^{-2k\delta}}{(1-e^{-\delta k})^2}\right)\\ &= -\frac{1}{2}\sum_{k=1}^n b_k\log\left(1+\frac{4e^{-\delta k}\sin^2(\pi\alpha k)}{(1-e^{-\delta k})^2}\right)\\ &\leqslant -\frac{1}{2}\sum_{k=1}^n b_k\log\bigl(1+4e^{-\delta k}\sin^2(\pi\alpha k)\bigr)\\ &\leqslant -\frac{\log 5}{2}\sum_{k=1}^n b_k e^{-\delta k}\sin^2(\pi\alpha k),\quad \delta>0,\ \alpha\in\mathbb{R}, \end{aligned}$$

where the last inequality is due to the fact that $\log(1+x)\geqslant(\frac{\log 5}{4})x$, $0\leqslant x\leqslant 4$.

For selections, in a similar manner,

$$\begin{aligned} V_n^{(2)}(\alpha;\delta) &= \Re\left(\sum_{k=1}^n b_k\log\left(\frac{1+e^{-\tau k}}{1+e^{-\delta k}}\right)\right)\\ &= \frac{1}{2}\sum_{k=1}^n b_k\log\left(\frac{1+2e^{-k\delta}\cos(2\pi\alpha k)+e^{-2k\delta}}{(1+e^{-\delta k})^2}\right)\\ &= \frac{1}{2}\sum_{k=1}^n b_k\log\left(1-\frac{4e^{-k\delta}\sin^2(\pi\alpha k)}{(1+e^{-\delta k})^2}\right)\\ &\leqslant -\frac{1}{2}\sum_{k=1}^n b_k\frac{4e^{-k\delta}\sin^2(\pi\alpha k)}{(1+e^{-\delta k})^2}\\ &\leqslant -\frac{1}{2}\sum_{k=1}^n b_k e^{-k\delta}\sin^2(\pi\alpha k),\quad \delta>0,\ \alpha\in\mathbb{R}. \end{aligned}$$

Here the first inequality is due to the fact that $-\log(1-x) \geqslant x$, $0 \leqslant x \leqslant 1$. For assemblies, we get straightforwardly

$$V_n^{(3)}(\alpha;\delta) = -2\sum_{k=1}^{n} b_k e^{-k\delta}\sin^2(\pi\alpha k), \quad \delta > 0,\ \alpha \in R.$$

Finally, setting $\delta = \delta_n^{(i)}$ in the above three expressions, the argument resulting in (53) implies that in all three cases,

$$V_n^{(i)}(\alpha;\delta_n^{(i)}) \leqslant \frac{V^{(i)}(\alpha)}{M^{(i)}} + \varepsilon_n^{(i)}, \quad \varepsilon_n^{(i)} = \varepsilon_n^{(i)}(\alpha) \to 0,\ n \to \infty, \tag{58}$$

uniformly for all $\alpha \in \mathbb{R}$. This completes the proof. □

6. The local limit theorem and completion of the proof

Local limit theorems are viewed as the main ingredient of the Khintchine method. Theorem 4 below says that a local limit theorem holds for all three types of structures obeying the conditions of our Theorem 3.

Theorem 4 *(Local limit theorem). Let $\delta_n^{(i)}$, $i = 1,2,3$, denote the solutions to Eqs.* (36)–(38)*, respectively, and let the random variables $Z_n^{(i)}$, $n \geqslant 1$, be defined as in* (30)*, where the random variables Y_k have distributions given in the paragraph following* (34)*. Assume that condition* (iii′) *of Theorem* 3 *holds for $i = 1,2,3$. Then,*

$$\mathbb{P}\big(Z_n^{(i)} = n\big) \sim \frac{1}{\sqrt{2\pi\,\mathrm{Var}(Z_n^{(i)})}} \sim \frac{1}{\sqrt{2\pi K_2^{(i)}}}\big(\delta_n^{(i)}\big)^{1+r/2}, \quad n \to \infty,\ i = 1,2,3,$$

with constants $K_2^{(i)}$ defined by

$$\begin{aligned} K_2^{(1)} &= A\Gamma(r+2)\zeta(r+1), \\ K_2^{(2)} &= A\big(1-2^{-r}\big)\Gamma(r+2)\zeta(r+1) \quad \textit{and} \\ K_2^{(3)} &= A\Gamma(r+2). \end{aligned}$$

Proof. We will find asymptotics for $\mathbb{P}(Z_n = n)$ as $n \to \infty$ for the three types of random structures. Following the pattern of the Khintchine method (see e.g. [9,10]), we set $\delta = \delta_n$ in (28) and (29) and define $\alpha_0 = \alpha_0(n)$ to be

$$\alpha_0 = \delta_n^{\frac{r+2}{2(r+1)}} \log^2 n. \tag{59}$$

Then we have

$$\mathbb{P}(Z_n = n) = \int_{-1/2}^{1/2} \phi_n(\alpha) e^{-2\pi i n\alpha}\, d\alpha = I_1 + I_2, \tag{60}$$

where $I_1 = I_1(n)$ and $I_2 = I_2(n)$ are defined to be

$$I_1 = \int_{-\alpha_0}^{\alpha_0} \phi_n(\alpha) e^{-2\pi i n\alpha}\, d\alpha \tag{61}$$

and

$$I_2 = \int_{-1/2}^{-\alpha_0} \phi_n(\alpha) e^{-2\pi i n\alpha}\, d\alpha + \int_{\alpha_0}^{1/2} \phi_n(\alpha) e^{-2\pi i n\alpha}\, d\alpha. \tag{62}$$

Defining B_n and T_n by

$$B_n^2 = \frac{d^2}{d\delta^2}\bigl(\log f_n\bigl(e^{-\delta_n}\bigr)\bigr), \qquad T_n = -\frac{d^3}{d\delta^3}\bigl(\log f_n\bigl(e^{-\delta_n}\bigr)\bigr), \tag{63}$$

for n fixed we have the expansion

$$\begin{aligned} \phi_n(\alpha) e^{-2\pi i n\alpha} &= \exp\bigl(2\pi i\alpha(\mathbb{E}Z_n - n) - 2\pi^2\alpha^2 B_n^2 + O\bigl(\alpha^3\bigr)T_n\bigr) \\ &= \exp\bigl(-2\pi^2\alpha^2 B_n^2 + O\bigl(\alpha^3\bigr)T_n\bigr), \quad \alpha \to 0. \end{aligned} \tag{64}$$

It can be checked that $B_n^2 = \mathrm{Var}(Z_n)$ and $T_n = \sum_{j=1}^n \mathbb{E}(Y_j - \mathbb{E}Y_j)^3$, by the argument leading to (33).

Now (42) in Lemma 2 and (63) tell us that for all structures considered

$$\bigl(B_n^2\bigr)^{(i)} \sim K_2^{(i)}\bigl(\delta_n^{(i)}\bigr)^{-r-2}, \tag{65}$$

and

$$T_n^{(i)} \sim K_3^{(i)}\bigl(\delta_n^{(i)}\bigr)^{-r-3},$$

where $K_2^{(i)}, K_3^{(i)} > 0$, $i = 1,2,3$ are constants depending on the type of the structure, while $K_2^{(i)}$, $i = 1,2,3$ are as in the statement of the theorem.

Therefore, considering (59), we find that in all three cases,

$$\lim_{n\to\infty} B_n^2\alpha_0^2 = \infty \quad \text{and} \quad \lim_{n\to\infty} T_n\alpha_0^3 = 0. \tag{66}$$

Combining (64) with (66), we deduce that

$$\phi_n(\alpha) e^{-2\pi i n\alpha} \sim \exp\bigl(-2\pi^2\alpha^2 B_n^2\bigr), \quad n \to \infty,\ |\alpha| \leqslant \alpha_0. \tag{67}$$

Finally, using (61), (66) and (67) gives us

$$I_1 \sim \int_{-\alpha_0}^{\alpha_0} \exp\bigl(-2\pi^2\alpha^2 B_n^2\bigr)\, d\alpha \sim (2\pi B_n)^{-1} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha^2}\, d\alpha = \frac{1}{\sqrt{2\pi B_n^2}}, \quad n \to \infty. \tag{68}$$

The next step of the proof is to show that $I_2 = o(I_1)$, $n \to \infty$. At this step condition (iii′) of Theorem 3 plays a key role. Because of the asymptotic $\sqrt{\delta_n^{(i)}} = o(\alpha_0^{(i)})$, $n \to \infty$, $i = 1, 2, 3$, we can use condition (iii′) to bound the quantity $V^{(i)}(\alpha)$ defined in Lemma 3 by

$$V^{(i)}(\alpha) \leqslant \left(1 + \frac{r}{2} + \varepsilon\right) M^{(i)} \log \delta_n^{(i)}, \quad \alpha_0 \leqslant |\alpha| \leqslant 1/2,\ i = 1, 2, 3.$$

Hence, Lemma 3 and the fact that in condition (iii′) $\varepsilon > 0$, give

$$\left|\phi_n^{(i)}(\alpha)\right| = o\left(\left(\delta_n^{(i)}\right)^{1+\frac{r}{2}}\right)\left(1 + \varepsilon_n^{(i)}\right), \quad \alpha_0 \leqslant |\alpha| \leqslant 1/2,\ n \to \infty,\ i = 1, 2, 3.$$

From the definition (62) and the asymptotic (65) we have

$$I_2 = o\left(\left(\delta_n^{(i)}\right)^{1+r/2}\right) = o(I_1). \tag{69}$$

Lastly from (60), (65), (69) and (68), we derive the following asymptotic expression for $\mathbb{P}(Z_n^{(i)} = n)$, $i = 1, 2, 3$:

$$\mathbb{P}\left(Z_n^{(i)} = n\right) \sim \frac{1}{\sqrt{2\pi (B_n^2)^{(i)}}} \sim \frac{1}{\sqrt{2\pi K_2^{(i)}}}\left(\delta_n^{(i)}\right)^{1+r/2}, \quad n \to \infty. \qquad \square$$

To complete the proof of Theorem 3 it is left to substitute the asymptotic expressions implied by our results for the three factors in the representation (31) when $\delta = \delta_n^{(i)}$.

7. Concluding remarks

(i) Under the stated conditions on parameters b_k, the asymptotic formulae (8), (20), (21), for $c_n^{(i)}$, $i = 1, 2, 3$, have a striking similarity, all of them being of the form

$$c_n \sim \chi_1 n^{\chi_2} \exp\left(\chi_3 n^{\frac{r}{r+1}}\right), \quad n \to \infty,$$

where we have denoted by χ_1, χ_2, χ_3 the constants that depend on the type of a structure and it parameters. A simple analysis that takes into account that $\zeta(r+1) > 1$ reveals that, asymptotically in n, $c_n^{(1)} > c_n^{(2)}$ and $c_n^{(1)} > c_n^{(3)}$, where the first fact follows obviously from the definition of selections.

(ii) The following observation is also in order. It turns out that each one of the three combinatorial structures obeying the above conditions behave very much like the one with parameters $b_k = k^{r-1}$, $r > 0$. According to the classification suggested in [4] the latter structures are called expansive. In this respect, combinatorial structures obeying Meinardus' conditions (as well as their extensions as defined in the present paper) can be viewed as quasi-expansive.

(iii) We hope that the approach of this paper can be applied as well to other enumeration problems, in particular to enumeration of structures with constraints on the number of summands (components) (see e.g. [8,19]).

Acknowledgment

We are grateful to Prof. Harold Shapiro who pointed us to the Wiener–Ikehara theorem during his visit at Technion in 2006.

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