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Dynamics of a stochastic density dependent predator–prey system with Beddington–DeAngelis functional response ☆

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ABSTRACT

In this paper, we discuss a stochastic density dependent predator–prey system with Beddington–DeAngelis functional response. First, we show that this system has a unique positive solution as this is essential in any population dynamics model. Then, we investigate the asymptotic behavior of this system. When the white noise is small, the stochastic system imitates the corresponding deterministic system. Either there is a stationary distribution, or the predator population will die out. While if the white noise is large, besides the extinction of the predator population, both species in the system may also die out, which does not happen in the deterministic system. Finally, simulations are carried out to conform to our results.

1. Introduction

The dynamical relationship between prey and their predators has long been and will continue to be one of the dominant themes in ecology due to its universal existence and importance [5,6]. The earliest predator–prey system is the Lotka–Volterra model [19,27], governed by the following differential equations

$$\dot{x}(t) = x(t)\big(a - by(t)\big),$$
$$\dot{y}(t) = y(t)\big(-c + fx(t)\big).$$

Since then some improvements to the original model have been suggested, such as adding a prey self-competition term [21], predator saturation term [22] and predator competition term [3], considering different functional response types: Holling types I–III [13], Hassel–Varley type [11], Beddington–DeAngelis type [4,8] and ratio-dependence type [1], etc.

Recently, Li et al. in [18] studied the dynamics of the density dependent predator–prey system with Beddington–DeAngelis functional response. The model is

$$\begin{cases} \dot{x}(t) = x(t)\left(a_1 - b_1x(t) - \dfrac{c_1y(t)}{m_1 + m_2x(t) + m_3y(t)}\right), \\ \dot{y}(t) = y(t)\left(-a_2 - b_2y(t) + \dfrac{c_2x(t)}{m_1 + m_2x(t) + m_3y(t)}\right), \end{cases} \tag{1.1}$$

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where $x(t)$ and $y(t)$ represent the densities of the prey and the predator, respectively, all the parameters in (1.1) are positive and b_2 is the predator density dependence rate. Assume

$$(\mathrm{H}_0) \quad (c_2 - a_2 m_2)\frac{a_1}{b_1} > a_2 m_1;$$

$$(\mathrm{H}_1) \quad c_2 > a_2 m_2 \quad \text{and} \quad (c_2 - a_2 m_2)\left(\frac{a_1}{b_1} - \frac{c_1}{b_1 m_3} - \frac{a_2 m_3}{b_2 m_2}\right) > a_2 m_1, \quad \text{or}$$

$$a_1 m_3 > c_1 + \frac{b_1 a_2 m_3^2}{b_2 m_2} \quad \text{and} \quad (c_2 - a_2 m_2)\left(\frac{a_1}{b_1} - \frac{c_1}{b_1 m_3} - \frac{a_2 m_3}{b_2 m_2}\right) > a_2 m_1.$$

According to the theory in [18], system (1.1) has a positive equilibrium $E^*(x^*, y^*)$ if (H_0) is satisfied, and it is globally asymptotically stable if (H_1) holds and $b_1 > c_1 m_2 y^*/\Delta(\underline{x}, \underline{y})$, where $\Delta(\underline{x}, \underline{y}) = (m_1 + m_2 x^* + m_3 y^*)(m_1 + m_2 \underline{x} + m_3 \underline{y})$, $\underline{x} = (a_1 - c_1/m_3)/b_1$, $\underline{y} = [c_2 \underline{x}/(m_1 + m_2 \underline{x} + m_3 \overline{y}) - a_2]/b_2$, $\overline{y} = (c_2/m_2 - a_2)/b_2$. In addition, if $c_2 < a_2 m_2$, then $(a_1/b_1, 0)$ is globally asymptotically stable.

However, the model is deterministic, and does not incorporate the effect of environmental noise, which is always present. In reality, parameters involved with the system are not absolute constants, and they always fluctuate around some average values due to continuous fluctuation in the environment. May [21] pointed out that due to continuous fluctuation in the environment, the birth rates, death rates, carrying capacity, competition coefficients and all other parameters involved with the model exhibit random fluctuation to a great lesser extent, and as a result the equilibrium population distribution never attains a steady value, but fluctuates randomly around some average value. There are many authors who have studied the dynamics of predator–prey models with stochastic perturbations [2,7,14–16,25]. Among these, they introduced stochastic perturbations into the birth rate of the prey and the death rate of the predator in different forms of prey–predator systems. For example, Khasminskiĭ and Klebaner gave a precise analysis of Lotka–Volterra system with stochastic perturbations [16]. Cai et al. [7] also investigated the prey–predator system with the perturbation in the Stratonovich sense, and showed the probability distribution of the system state variables. A ratio-dependent prey–predator model with the environmental fluctuations was considered in [25]. They calculated population fluctuation intensity (variance) for the prey and the predator by Laplace transform methods for the stochastic differential equation model.

In this paper, considering the effect of environmental noise, we also introduce stochastic perturbation into the death rate of the prey and the death rate of the predator in system (1.1), and assume that parameters a_1, a_2 are disturbed to $a_1 + \alpha \dot{B}_1(t)$, $a_2 + \beta \dot{B}_2(t)$, respectively. Then we obtain the following stochastic system:

$$\begin{cases} dx(t) = x(t)\left(a_1 - b_1 x(t) - \dfrac{c_1 y(t)}{m_1 + m_2 x(t) + m_3 y(t)}\right)dt + \alpha x(t)\,dB_1(t), \\ dy(t) = y(t)\left(-a_2 - b_2 y(t) + \dfrac{c_2 x(t)}{m_1 + m_2 x(t) + m_3 y(t)}\right)dt - \beta y(t)\,dB_2(t). \end{cases} \tag{1.2}$$

The aim of this paper is to discuss the long-time behavior of system (1.2). We have mentioned that $E^*(x^*, y^*)$ of system (1.1) is globally asymptotically stable under some conditions, which means that the properties of the solution will not be changed under small deterministic perturbation. When it is suffered stochastic perturbations, whether there also exists some structurally stable. But, in this situation, there is no positive equilibrium. Hence, it is impossible that the solution of system (1.2) will tend to a fixed point. In this paper, we show that there is a stationary distribution of system (1.2) mainly according to the theory of Has'minskii [10], if the white noise is small. While if the white noise is large, based on the techniques developed in [23,24], we prove the predator population will die out a.s. and the prey population will either extinct or its distribution converges to a probability measure. It does not happen that both the prey population and the predator population in system (1.1) will die out, which is brought by large white noise, such as weather, epidemic disease. From this point, we say that the stochastic model is more realistic than the deterministic model.

The rest of this paper is organized as follows. In Section 2, we show that there is a unique non-negative solution of system (1.2). In Section 3, we show that there is a stationary distribution under small white noise. While in Section 4, we consider the situation when the white noise is large. We prove that the system is nonpersistent. Finally, we make numerical simulation to conform to our analytical result.

2. Existence of the positive solution

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all P-null sets). Denote $R_+^2 = \{x \in R^2\colon x_1 > 0,\ x_2 > 0\}$.

In any population model, it is essential that the solution of the model is positive. So in this section, we show that there is a unique positive solution of system (1.2).

Lemma 2.1. *For any initial value $(x_0, y_0) \in R_+^2$, there is a positive solution $(x(t), y(t))$, $t \in [0, \tau_e)$ of system (1.2), where τ_e is the explosion time.*

Proof. Consider the following system

$$\begin{cases} du = \left(a_1 - \dfrac{\alpha^2}{2} - b_1 e^u - \dfrac{c_1 e^v}{m_1 + m_2 e^u + m_3 e^v}\right) dt + \alpha\, dB_1(t), \\ dv = \left(-a_2 - \dfrac{\beta^2}{2} - b_2 e^v + \dfrac{c_2 e^u}{m_1 + m_2 e^u + m_3 e^v}\right) dt - \beta\, dB_2(t), \end{cases} \tag{2.1}$$

with initial value $u_0 = \log x_0$, $v_0 = \log y_0$. It is clear that the coefficients of system (2.1) satisfy local Lipschitz condition, then there is a local solution $(u(t), v(t))$, $t \in [0, \tau_e)$ of system (2.1). Therefore, by Itô's formula, it is easy to check that $(x(t) = e^{u(t)}, y(t) = e^{v(t)})$ is the positive solution of system (1.2) with the initial value (x_0, y_0). □

Theorem 2.1. *For any initial value $(x_0, y_0) \in R_+^2$, there is a unique solution $(x(t), y(t))$ of system (1.2) on $t \geqslant 0$, and the solution will remain in R_+^2 with probability 1.*

Proof. Since Lemma 2.1 shows that there is a positive local solution $(x(t), y(t))$, $t \in [0, \tau_e)$ of system (1.2), then to show this solution is global, we only need to show that $\tau_e = \infty$ a.s. Let $m_0 \geqslant 0$ be sufficiently large so that both x_0 and y_0 lie within the interval $[1/m_0, m_0]$. For each integer $m \geqslant m_0$, define the stopping time

$$\tau_m = \inf\bigl\{t \in [0, \tau_e):\ \min\bigl\{x(t), x(t)\bigr\} \leqslant 1/m \text{ or } \max\bigl\{x(t), y(t)\bigr\} \geqslant m\bigr\},$$

where throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m\to\infty} \tau_m$, whence $\tau_\infty \leqslant \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(x(t), y(t)) \in R_+^2$ a.s. for all $t \geqslant 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. For if this statement is false, then there are a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leqslant T\} > \epsilon.$$

Hence there is an integer $m_1 \geqslant m_0$ such that

$$P\{\tau_m \leqslant T\} \geqslant \epsilon \quad \text{for all } m \geqslant m_1. \tag{2.2}$$

Define a C^2-function $V : R_+^2 \to \bar{R}_+$ by

$$V(x, y) = c_2(x - 1 - \log x) + c_1(y - 1 - \log y).$$

The non-negativity of this function can be seen from $u - 1 - \log u \geqslant 0, \forall u > 0$. Using Itô's formula, we get

$$\begin{aligned} dV &= c_2(x-1)\left[\left(a_1 - b_1 x - \frac{c_1 y}{m_1 + m_2 x + m_3 y}\right) dt + \alpha\, dB_1(t)\right] + c_2\alpha^2/2\, dt \\ &\quad + c_1(y-1)\left[\left(-a_2 - b_2 y + \frac{c_2 x}{m_1 + m_2 x + m_3 y}\right) dt - \beta\, dB_2(t)\right] + c_1\beta^2/2\, dt \\ &:= LV\, dt + c_2\alpha(x-1)\, dB_1(t) - c_1\beta(y-1)\, dB_2(t), \end{aligned}$$

where

$$\begin{aligned} LV &= c_2\bigl(-a_1 + \alpha^2/2\bigr) + c_1\bigl(a_2 + \beta^2/2\bigr) + c_2(a_1 + b_1)x + c_1(b_2 - a_2)y \\ &\quad - b_1 c_2 x^2 - b_2 c_1 y^2 + \frac{c_1 c_2 y}{m_1 + m_2 x + m_3 y} - \frac{c_1 c_2 x}{m_1 + m_2 x + m_3 y} \\ &\leqslant c_2\bigl(-a_1 + \alpha^2/2\bigr) + c_1\bigl(a_2 + \beta^2/2 + c_2/m_3\bigr) + c_2(a_1 + b_1)x + c_1(b_2 - a_2)y - b_1 c_2 x^2 - b_2 c_1 y^2 \\ &\leqslant K, \end{aligned}$$

and K is a positive constant. Therefore

$$\int_0^{\tau_m \wedge T} dV\bigl(x(t), y(t)\bigr) \leqslant \int_0^{\tau_m \wedge T} K\, dt + \int_0^{\tau_m \wedge T} c_2\alpha(x-1)\, dB_1(t) - \int_0^{\tau_m \wedge T} c_1\beta(y-1)\, dB_2(t),$$

which implies that,

$$E\bigl[V\bigl(x(\tau_m \wedge T), y(\tau_m \wedge T)\bigr)\bigr] \leqslant V\bigl(x(0), y(0)\bigr) + E\int_0^{\tau_m \wedge T} K\, dt \leqslant V\bigl(x(0), y(0)\bigr) + KT. \tag{2.3}$$

Set $\Omega_m = \{\tau_m \leqslant T\}$ for $m \geqslant m_1$, then by (2.2), we know $P(\Omega_m) \geqslant \epsilon$. Note that for every $\omega \in \Omega_m$, there is at least one of $x(\tau_m, \omega)$, $y(\tau_m, \omega)$ equal either m or $1/m$, then $V(x(\tau_m), y(\tau_m))$ is no less than

$$m - 1 - \log m \quad \text{or} \quad \frac{1}{m} - 1 - \log\frac{1}{m} = \frac{1}{m} - 1 + \log m.$$

Consequently,

$$V\big(x(\tau_m), y(\tau_m)\big) \geqslant (m - 1 - \log m) \wedge \left(\frac{1}{m} - 1 + \log m\right).$$

It then follows from (2.2) and (2.3) that

$$\begin{aligned} V\big(x(0), y(0)\big) + KT &\geqslant E\big[1_{\Omega_m(\omega)} V\big(x(\tau_m), y(\tau_m)\big)\big] \\ &\geqslant \epsilon\left[(m - 1 - \log m) \wedge \left(\frac{1}{m} - 1 + \log m\right)\right], \end{aligned}$$

where $1_{\Omega_m(\omega)}$ is the indicator function of Ω_m. Letting $m \to \infty$ leads to the contradiction that $\infty > V(x(0), y(0)) + KT = \infty$. So we must have $\tau_\infty = \infty$ a.s. □

3. Stationary distribution

In this section, we mainly show that system (1.2) imitates system (1.1), when the stochastic perturbation is small. In accordance with the globally asymptotic stability of $E^*(x^*, y^*)$ of system (1.1), we show that there is a stationary distribution of system (1.2), which can be considered as a stability in the stochastic sense. Before giving the main theorem, we first give a lemma used in the proof of the theorem.

Let $X(t)$ be a homogeneous Markov process in E_l (E_l denotes Euclidean l-space) described by the stochastic equation

$$dX(t) = b(X)\,dt + \sum_{r=1}^{k} g_r(X)\,dB_r(t). \tag{3.1}$$

The diffusion matrix is

$$A(x) = \big(a_{ij}(x)\big), \qquad a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x) g_r^j(x).$$

Assumption B. There exists a bounded domain $U \subset E_l$ with regular boundary Γ, having the following properties:

(B.1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(B.2) If $x \in E_l \setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x \in K} E_x\tau < \infty$ for every compact subset $K \subset E_l$.

Lemma 3.1. *(See [10].) If* (B) *holds, then the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$. Let $f(\cdot)$ be a function integrable with respect to the measure μ. Then*

$$P_x\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f\big(X(t)\big)\,dt = \int_{E_l} f(x)\mu(dx)\right\} = 1$$

for all $x \in E_l$.

Remark 3.1. The proof is given in [10]. Exactly, the existence of a stationary distribution with density is referred to Theorem 4.1, p. 119 and Lemma 9.4, p. 138. The weak convergence and the ergodicity are obtained in Theorem 5.1, p. 121 and Theorem 7.1, p. 130.

To validate (B.1), it suffices to prove that F is uniformly elliptical in U, where $Fu = b(x) \cdot u_x + (tr(A(x)u_{xx}))/2$, that is, there is a positive number M such that

$$\sum_{i,j=1}^{l} a_{ij}(x)\xi_i\xi_j \geqslant M|\xi|^2, \quad x \in U,\ \xi \in R^l$$

(see Chapter 3, p. 103 of [9] and Rayleigh's principle in [26, Chapter 6, p. 349]). To verify (B.2), it is sufficient to show that there exist some neighborhood U and a non-negative C^2-function such that and for any $E_l \setminus U$, LV is negative (for details we refer to [28, p. 1163]).

Remark 3.2. System (1.2) can be written as the form of system (3.1),

$$d\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(t)(a_1 - b_1 x(t) - \frac{c_1 y(t)}{m_1+m_2x(t)+m_3y(t)}) \\ y(t)(-a_2 - b_2 y(t) + \frac{c_2 x(t)}{m_1+m_2x(t)+m_3y(t)}) \end{pmatrix} dt + \begin{pmatrix} \alpha x(t) \\ 0 \end{pmatrix} dB_1(t) + \begin{pmatrix} 0 \\ -\beta y(t) \end{pmatrix} dB_2(t),$$

and the diffusion matrix is

$$A = diag\big(\alpha^2 x^2, \beta^2 y^2\big).$$

Theorem 3.1. *Assume* $(c_2 - a_2 m_2)a_1/b_1 > a_2 m_1$, $b_1 > a_1 m_2/(m_1 + m_2 x^*)$ *and* $\alpha > 0$, $\beta > 0$ *such that* $\delta < \min\{c_2(b_1 - m_2(a_1 - b_1x^*)/m_1)(m_1 + m_3 y^*)(x^*)^2, b_2 c_1(m_1 + m_2 x^*)(y^*)^2\}$, *where* $\delta = c_2 x^* \alpha^2/2 + c_1 y^* \beta^2/2$ *and* (x^*, y^*) *is the equilibrium of system* (1.1)*. Then there is a stationary distribution* $\mu(\cdot)$ *for system* (1.2) *and it has ergodic property.*

Proof. Since $(c_2 - a_2 m_2)a_1/b_1 > a_2 m_1$, then there is a positive equilibrium (x^*, y^*) of system (1.1), and

$$\begin{aligned} a_1 &= b_1 x^* + \frac{c_1 y^*}{m_1 + m_2 x^* + m_3 y^*}, \qquad a_2 = \frac{c_2 x^*}{m_1 + m_2 x^* + m_3 y^*} - b_2 y^*, \\ m_1 + m_2 x^* + m_3 y^* &= \frac{c_1 y^*}{a_1 - b_1 x^*} = \frac{c_2 x^*}{a_2 + b_2 y^*}. \end{aligned} \tag{3.2}$$

Define

$$V(x, y) = c_2\big(m_1 + m_3 y^*\big)\left(x - x^* - x^* \log\frac{x}{x^*}\right) + c_1\big(m_1 + m_2 x^*\big)\left(y - y^* - y^* \log\frac{y}{y^*}\right).$$

By Itô's formula, we get

$$\begin{aligned} LV &= c_2\big(m_1 + m_3 y^*\big)\big(x - x^*\big)\left(a_1 - b_1 x - \frac{c_1 y}{m_1 + m_2 x + m_3 y}\right) + c_2\big(m_1 + m_3 y^*\big)x^*\alpha^2/2 \\ &\quad + c_1\big(m_1 + m_2 x^*\big)\big(y - y^*\big)\left(-a_2 - b_2 y + \frac{c_2 x}{m_1 + m_2 x + m_3 y}\right) + c_1\big(m_1 + m_2 x^*\big)y^*\beta^2/2 \\ &= -b_1 c_2\big(m_1 + m_3 y^*\big)\big(x - x^*\big)^2 - b_2 c_1\big(m_1 + m_2 x^*\big)\big(y - y^*\big)^2 + \delta \\ &\quad + \frac{c_2 m_2 (a_1 - b_1 x^*)(m_1 + m_3 y^*)}{m_1 + m_2 x + m_3 y}\big(x - x^*\big)^2 - \frac{c_1 m_3 (a_2 + b_2 y^*)(m_1 + m_2 x^*)}{m_1 + m_2 x + m_3 y}\big(y - y^*\big)^2 \\ &\leqslant -c_2\left(b_1 - \frac{m_2}{m_1}\big(a_1 - b_1 x^*\big)\right)\big(m_1 + m_3 y^*\big)\big(x - x^*\big)^2 - b_2 c_1\big(m_1 + m_2 x^*\big)\big(y - y^*\big)^2 + \delta, \end{aligned}$$

where (3.2) is used in the second equality and $\delta = c_2 x^* \alpha^2/2 + c_1 y^* \beta^2/2$. Note that $b > a_1 m_2/(m_1 + m_2 x^*)$, then $b_1 - m_2(a_1 - b_1 x^*)/m_1 > 0$. When

$$\delta < \min\left\{c_2\left(b_1 - \frac{m_2}{m_1}\big(a_1 - b_1 x^*\big)\right)\big(m_1 + m_3 y^*\big)\big(x^*\big)^2, b_2 c_1\big(m_1 + m_2 x^*\big)\big(y^*\big)^2\right\},$$

then the ellipsoid

$$-c_2\left(b_1 - \frac{m_2}{m_1}\big(a_1 - b_1 x^*\big)\right)\big(m_1 + m_3 y^*\big)\big(x - x^*\big)^2 - b_2 c_1\big(m_1 + m_2 x^*\big)\big(y - y^*\big)^2 + \delta = 0$$

lies entirely in R^2_+. We can take U to be a neighborhood of the ellipsoid with $\overline{U} \subseteq E_2 = R^2_+$, so for $x \in U \setminus E_2$, $LV \leqslant -K$ (K is a positive constant), which implies condition (B.2) in Lemma 3.1 is satisfied. Besides, there is $M = \min\{\alpha^2 x^2, \beta^2 y^2, (x, y) \in \overline{U}\} > 0$ such that

$$\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j = \alpha^2 x^2 \xi_1^2 + \beta^2 y^2 \xi_2^2 \geqslant M\|\xi^2\|$$

for all $(x, y) \in \overline{U}$, $\xi \in R^2$, which implies condition (B.1) is also satisfied. Therefore, the stochastic system (1.2) has a stationary distribution $\mu(\cdot)$ and it is ergodic. □

Lemma 3.2. *Let $(x(t), y(t))$ be a positive solution of (1.2) with any initial value $(x_0, y_0) \in R^2_+$, then there are $K_1(p)$ and $K_2(p)$ such that*

$$E\big[x^p(t)\big] \leqslant K_1(p), \qquad E\big[y^p(t)\big] \leqslant K_2(p), \quad p > 0.$$

Proof. By Itô's formula, we compute,

$$\begin{aligned} dx^p &= px^p\left(a_1 - b_1x - \frac{c_1y}{m_1 + m_2x + m_3y}\right)dt + p\alpha x^p\,dB_1(t) + \frac{\alpha^2}{2}p(p-1)x^p\,dt \\ &= px^p\left(a_1 + \frac{\alpha^2}{2}(p-1) - b_1x - \frac{c_1y}{m_1 + m_2x + m_3y}\right)dt + p\alpha x^p\,dB_1(t) \\ &\leqslant px^p\big(a_1 + p\alpha^2/2 - b_1x\big)\,dt + p\alpha x^p\,dB_1(t), \end{aligned}$$

and

$$\begin{aligned} dy^p &= py^p\left(-a_2 - b_2y + \frac{c_2x}{m_1 + m_2x + m_3y}\right)dt - p\beta y^p\,dB_2(t) + \frac{\beta^2}{2}p(p-1)y^p\,dt \\ &= py^p\left(-a_2 + \frac{\beta^2}{2}(p-1) - b_2y + \frac{c_2x}{m_1 + m_2x + m_3y}\right)dt - p\beta y^p\,dB_2(t) \\ &\leqslant py^p\big(c_2/m_2 + p\beta^2/2 - b_2y\big)\,dt - p\beta y^p\,dB_1(t). \end{aligned}$$

Taking expectation, we have

$$\begin{aligned} \frac{dE[x^p(t)]}{dt} &\leqslant p\big(a_1 + p\alpha^2/2\big)E\big[x^p(t)\big] - b_1E\big[x^{p+1}(t)\big] \\ &\leqslant p\big(a_1 + p\alpha^2/2\big)E\big[x^p(t)\big] - b_1\big(E\big[x^p(t)\big]\big)^{1+1/p}, \end{aligned}$$

and

$$\begin{aligned} \frac{dE[y^p(t)]}{dt} &\leqslant p\big(c_2/m_2 + p\beta^2/2\big)E\big[y^p(t)\big] - b_2E\big[y^{p+1}(t)\big] \\ &\leqslant p\big(c_2/m_2 + p\beta^2/2\big)E\big[y^p(t)\big] - b_2\big(E\big[y^p(t)\big]\big)^{1+1/p}. \end{aligned}$$

Therefore, by comparison theorem, we get

$$\limsup_{t\to\infty} E\big[x^p(t)\big] \leqslant \left(\frac{a_1 + p\alpha^2/2}{b_1}\right)^p, \qquad \limsup_{t\to\infty} E\big[y^p(t)\big] \leqslant \left(\frac{c_2/m_2 + p\beta^2/2}{b_2}\right)^p,$$

which together with the continuity of $E[x^p(t)]$ implies that there exist $K_1(p) > 0$, $K_2(p) > 0$ such that

$$E\big[x^p(t)\big] \leqslant K_1(p), \qquad E\big[y^p(t)\big] \leqslant K_2(p), \quad t \in [0, \infty). \qquad \square$$

By the ergodic property, for $m > 0$, we have

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t}\int_0^t \big(x^p(s) \wedge m\big)\,ds = \int_{R^2_+} \big(z_1^p \wedge m\big)\mu(dz_1, dz_2) \quad \text{a.s.}, \\ &\lim_{t\to\infty} \frac{1}{t}\int_0^t \big(y^p(s) \wedge m\big)\,ds = \int_{R^2_+} \big(z_2^p \wedge m\big)\mu(dz_1, dz_2) \quad \text{a.s.} \end{aligned} \tag{3.3}$$

Besides, by dominated convergence theorem, we get

$$\begin{aligned} &E\left[\lim_{t\to\infty} \frac{1}{t}\int_0^t \big(x^p(s) \wedge m\big)\,ds\right] = \lim_{t\to\infty} \frac{1}{t}\int_0^t E\big[x^p(s) \wedge m\big]\,ds \leqslant K_1(p), \\ &E\left[\lim_{t\to\infty} \frac{1}{t}\int_0^t \big(y^p(s) \wedge m\big)\,ds\right] = \lim_{t\to\infty} \frac{1}{t}\int_0^t E\big[y^p(s) \wedge m\big]\,ds \leqslant K_2(p), \end{aligned}$$

which together with (3.3) implies

$$\int_{R_+^2} \left(z_i^p \wedge m\right)\mu(dz_1, dz_2) \leqslant K_i(p), \quad i = 1, 2.$$

Letting $m \to \infty$, we get

$$\int_{R_+^2} z_i^p \mu(dz_1, dz_2) \leqslant K_i(p), \quad i = 1, 2.$$

That is to say, functions $f_1(z_1, z_2) = z_1^p$ and $f_2(z_1, z_2) = z_2^p$ are integrable with respect to the measure μ, i.e.

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x^p(s)\,ds = \int_{R_+^2} z_1^p \mu(dz_1, dz_2), \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t y^p(s)\,ds = \int_{R_+^2} z_2^p \mu(dz_1, dz_2) \quad \text{a.s.} \tag{3.4}$$

In addition, it is clear that

$$c_2\,dx + c_1\,dy = \left(a_1c_2x - b_1c_2x^2 - a_2c_1y - b_2c_1y^2\right)dt + c_2\alpha x\,dB_1(t) - c_1\beta y\,dB_2(t),$$

then

$$\begin{aligned} c_2\frac{x(t) - x_0}{t} + c_1\frac{y(t) - y_0}{t} = {} & \frac{a_1c_2}{t}\int_0^t x(s)\,ds - \frac{b_1c_2}{t}\int_0^t x^2(s)\,ds - \frac{a_2c_1}{t}\int_0^t y(s)\,ds - \frac{b_2c_1}{t}\int_0^t y^2(s)\,ds \\ & + \frac{c_2\alpha}{t}\int_0^t x(s)\,dB_1(s) - \frac{c_1\beta}{t}\int_0^t y(s)\,dB_2(s). \end{aligned} \tag{3.5}$$

Let $M_1(t) = \int_0^t x(s)\,dB_1(s)$, $M_2(t) = \int_0^t y(s)\,dB_2(s)$, then $M_1(t)$, $M_2(t)$ are martingales with $M_1(0) = M_2(0) = 0$, and

$$\lim_{t\to\infty} \frac{\langle M_1, M_1\rangle_t}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t x^2(s)\,ds = \int_{R_+^2} z_1^2\mu(dz_1, dz_2) < \infty,$$

$$\lim_{t\to\infty} \frac{\langle M_2, M_2\rangle_t}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t y^2(s)\,ds = \int_{R_+^2} z_2^2\mu(dz_1, dz_2) < \infty,$$

according to (3.4). Hence by strong law of large numbers [20], we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,dB_1(s) = 0, \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t y(s)\,dB_2(s) = 0 \quad \text{a.s.}$$

which together with (3.4) and (3.5), yields

$$\lim_{t\to\infty} \frac{c_2x(t) + c_1y(t)}{t} = \int_{R_+^2} \left(a_1c_2z_1 - b_1c_2z_1^2 - a_2c_1z_2 - b_2c_1z_2^2\right)\mu(dz_1, dz_2) \quad \text{a.s.}$$

Combing upper arguments, we get the following theorem.

Theorem 3.2. *Suppose the conditions in Theorem* 3.1 *hold. Then we have*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x^p(s)\,ds = \int_{R_+^2} z_1^p \mu(dz_1, dz_2), \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t y^p(s)\,ds = \int_{R_+^2} z_2^p \mu(dz_1, dz_2), \quad p > 0,$$

$$\lim_{t\to\infty} \frac{c_2x(t) + c_1y(t)}{t} = \int_{R_+^2} \left(a_1c_2z_1 - b_1c_2z_1^2 - a_2c_1z_2 - b_2c_1z_2^2\right)\mu(dz_1, dz_2) \quad \text{a.s.}$$

Note that

$$dx \leqslant x(a_1 - bx_1)\,dt + \alpha x\,dB_1(t),$$

and

$$dx \geqslant x\left(a_1 - \frac{c_1}{m_3} - b_1 x\right)dt + \alpha x\,dB_1(t),$$

then by the theory in [14], we get

$$\limsup_{t\to\infty}\frac{\log x(t)}{t} \leqslant 0, \qquad \limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,ds \leqslant \frac{a_1 - \alpha^2/2}{b_1} \quad \text{a.s.,}$$

if $a_1 > \alpha^2/2$;

$$\liminf_{t\to\infty}\frac{\log x(t)}{t} \geqslant 0, \qquad \liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,ds \geqslant \frac{a_1 - c_1/m_3 - \alpha^2/2}{b_1} \quad \text{a.s.,}$$

if $a_1 > c_1/m_3 + \alpha^2/2$. Therefore, if $a_1 > c_1/m_3 + \alpha^2/2$, we have

$$\lim_{t\to\infty}\frac{\log x(t)}{t} = 0 \quad \text{a.s.} \tag{3.6}$$

Moreover,

$$\frac{\log x(t) - \log x_0}{t} = a_1 - \frac{\alpha^2}{2} - b_1\frac{1}{t}\int_0^t x(s)\,ds - c_1\frac{1}{t}\int_0^t \frac{y(s)}{m_1 + m_2 x(s) + m_3 y(s)}\,ds + \alpha\frac{B_1(t)}{t},$$

which together with (3.4), (3.6) and $\lim_{t\to\infty} B_1(t)/t = 0$ a.s. implies

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{y(s)}{m_1 + m_2 x(s) + m_3 y(s)}\,ds = \frac{a_1 - \alpha^2/2}{c_1} - \frac{b_1}{c_1}\int_{R_+^2} z_1\mu(dz_1, dz_2) \quad \text{a.s.}$$

Now, we consider $y(t)$. It is clear that

$$dy \leqslant y\left(-a_2 + \frac{c_2}{m_2} - b_2 y\right)dt - \beta y\,dB_2(t),$$

then

$$\limsup_{t\to\infty}\frac{\log y(t)}{t} \leqslant 0, \qquad \limsup_{t\to\infty}\frac{1}{t}\int_0^t y(s)\,ds \leqslant \frac{c_2/m_2 - a_2 - \beta^2/2}{b_2} \quad \text{a.s.}$$

if $c_2/m_2 > a_2 + \beta^2/2$. Note that

$$\frac{\log y(t)}{t} = \frac{\log y_0}{t} - \left(a_2 + \frac{\beta^2}{2}\right) - \frac{b_2}{t}\int_0^t y(s)\,ds + \frac{c_2}{t}\int_0^t \frac{x(s)}{m_1 + m_2 x(s) + m_3 y(s)}\,ds - \beta\frac{B_2(t)}{t},$$

then

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \frac{x(s)}{m_1 + m_2 x(s) + m_3 y(s)}\,ds \geqslant \frac{a_2 + \beta^2/2}{c_2} + \frac{b_2}{c_2}\int_{R_+^2} z_2\mu(dz_1, dz_2) \quad \text{a.s.}$$

Therefore, we have

Theorem 3.3. *Assume the conditions in Theorem* 3.1 *hold and* $a_1 > c_1/m_3 + \alpha^2/2$, $c_2/m_2 > a_2 + \beta^2/2$. *Then we have*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{y(s)}{m_1 + m_2 x(s) + m_3 y(s)}\,ds = \frac{a_1 - \alpha^2/2}{c_1} - \frac{b_1}{c_1}\int_{R_+^2} z_1 \mu(dz_1, dz_2),$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \frac{x(s)}{m_1 + m_2 x(s) + m_3 y(s)}\,ds \geqslant \frac{a_2 + \beta^2/2}{c_2} + \frac{b_2}{c_2}\int_{R_+^2} z_2 \mu(dz_1, dz_2) \quad a.s.$$

4. Non-permanence

In this section, we show the dynamics of system (1.2) with large white noise. Large white noise may lead to the extinction of the two species, which does not happen in the deterministic system.

First, we give the property of the solutions of a one-dimensional stochastic equation. Consider the following stochastic equation [17]:

$$dX_t = a(X_t)\,dW_t + b(X_t)\,dt. \tag{4.1}$$

Lemma 4.1. *Let* X_t *be a solution of Eq.* (4.1), *and*

$$s(\xi) = \int_0^\xi \exp\left\{-\int_0^z \frac{2b(r)}{a^2(r)}\,dr\right\} dz.$$

If $s(-\infty) > -\infty$ *and* $s(\infty) = \infty$, *then* $\lim_{t\to\infty} X_t = -\infty$ *a.s.*

Theorem 4.1. *Let* $(x(t), y(t))$ *be a solution of system* (1.2). *Then*:

(1) *If* $c_2/m_2 < a_2 + \beta^2/2$ *and* $a_1 > \alpha^2/2$, *then* $\lim_{t\to\infty} y(t) = 0$ *a.s. and the distribution of* $x(t)$ *converges weakly to the probability measure with density* $f^*(\zeta) = C\zeta^{2(a_1-\alpha^2/2)/\alpha^2-1}e^{-2b_1\zeta/\alpha^2}$, *where* $C = (2b_1/\alpha^2)^{2(a_1-\alpha^2/2)/\alpha^2}/\Gamma(2(a_1-\alpha^2/2)/\alpha^2)$, *and*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds = \frac{a_1 - \alpha^2/2}{b_1} \quad a.s.$$

(2) *If* $a_1 < \alpha^2/2$, *then* $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} y(t) = 0$ *a.s.*

Proof. It is clear that

$$\begin{aligned} d\log y &= \left(-a_2 - \beta^2/2 + c_2/m_2 - b_2 y\right)dt - \beta\,dB_2(t) \\ &\leqslant (-a_2 + c_2/m_2 - b_2 y)\,dt - \beta\,dB_2(t). \end{aligned}$$

Let

$$d\Psi(t) = \left(-a_2 + c_2/m_2 - b_2 e^{\Psi(t)}\right)dt - \beta\,dB_2(t),$$

then $\log y(t) \leqslant \Psi(t)$ a.s. and it is easy to compute $s(-\infty) > -\infty$ and $s(\infty) = \infty$ when $c_2/m_2 < a_2 + \beta^2/2$. Then $\lim_{t\to\infty} \Psi(t) = -\infty$ a.s. according to Lemma 4.1. Thus by the comparison theorems for stochastic equations and the positivity of the solution, we get

$$\lim_{t\to\infty} y(t) = 0 \quad \text{a.s.}$$

That is to say, for $\forall 0 < \epsilon < a_1 - \alpha^2/2$, there exist a constant $T_1 = T_1(\omega)$ and a set Ω_ϵ such that $P(\Omega_\epsilon) > 1 - \epsilon$ and $y(t) \leqslant m_1\epsilon/c_1$ for $t \geqslant T_1$ and $\omega \in \Omega_\epsilon$. Then

$$x(t)\big(a_1 - \epsilon - b_1 x(t)\big)\,dt + \alpha x(t)\,dB_1(t) \leqslant dx(t) \leqslant x\big(a_1 - b_1 x(t)\big)\,dt + \alpha x(t)\,dB_1(t),$$

and

$$\big(a_1-\alpha^2/2-\epsilon-b_1x(t)\big)\,dt+\alpha\,dB_1(t)\leqslant d\log x(t)\leqslant\big(a_1-\alpha^2/2-b_1x(t)\big)\,dt+\alpha\,dB_1(t). \tag{4.2}$$

Consider the following equation

$$d\Phi(t)=\big(a_1-\alpha^2/2-b_1e^{\Phi(t)}\big)\,dt+\alpha\,dB_1(t). \tag{4.3}$$

If $a_1>\alpha^2/2$, Eq. (4.3) has the density $g_*(\zeta)$ such that

$$\frac{1}{2}\alpha^2g_*'(\zeta)=\big(a_1-\alpha^2/2-b_1e^{\zeta}\big)g_*(\zeta). \tag{4.4}$$

Therefore from (4.2) and the arbitrary of ϵ, we get the distribution of $\log x(t)$ converges weakly to the probability measure with density g_*. Thus, from (4.4), we obtain the distribution of $x(t)$ converges weakly to the probability measure with density $f^*(\zeta)=C_2\zeta^{2(a_1-\alpha^2/2)/\alpha^2-1}e^{-2b_1\zeta/\alpha^2}$, where $C_2=(2b_1/\alpha^2)^{2(a_1-\alpha^2/2)/\alpha^2}/\Gamma(2(a_1-\alpha^2/2)/\alpha^2)$. Besides, from the ergodic theorem and (4.4) it follows that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,ds=\int_{-\infty}^{\infty}e^{\zeta}g_*(\zeta)\,d\zeta=\int_{-\infty}^{\infty}\frac{a_1-\alpha^2/2}{b_1}g_*(\zeta)\,d\zeta=\frac{a_1-\alpha^2/2}{b_1}\quad\text{a.s.}$$

Thus, the proof of case (1) is completed. Now, we prove case (2). Obviously,

$$\begin{aligned}d\log x(t)&=\left(a_1-\frac{\alpha^2}{2}-b_1x(t)-\frac{c_1y(t)}{m_1+m_2x(t)+m_3y(t)}\right)dt+\alpha\,dB_1(t)\\&\leqslant\big(a_1-\alpha^2/2\big)\,dt+\alpha\,dB_1(t).\end{aligned}$$

Since $a_1-\alpha^2/2<0$, we have $\lim_{t\to\infty}\log x(t)=-\infty$, and so $\lim_{t\to\infty}x(t)=0$ a.s. which implies

$$\begin{aligned}d\log y(t)&=\left(-a_2-\frac{\beta^2}{2}-b_2y(t)-\frac{c_2x(t)}{m_1+m_2x(t)+m_3y(t)}\right)dt+\beta\,dB_2(t)\\&\leqslant -a_2/2\,dt+\beta\,dB_2(t).\end{aligned}$$

By the same arguments as above, we have $\lim_{t\to\infty}y(t)=0$ a.s. according to $a_2>0$. □

Remark 4.1. The case (1) of Theorem 4.1 shows the situation when the predator population will die out, and the prey population is persistent for system (1.2), which also happens in system (1.1) if $c_2/m_2<a_2$. But, from the condition of case (1) of Theorem 4.1, we can see this phenomena can also happen in system (1.2), even if $c_2/m_2>a_2$. In addition, we show that large white noise can make the extinction of the two species in system (1.2) (see the case (2) of Theorem 4.1), which does not happen forever in system (1.1). Therefore, the white noise can bring more asymptotic behavior, and the large white noise may be bad weather, serious epidemic, etc. in a real world, which are responsible for the extinction of the species.

5. Simulations

In order to conform to the results above, we numerically simulate the solution of system (1.2). By the method mentioned in [12], we consider the discretized equation:

$$\begin{cases}x_{k+1}=x_k+x_k\left[\left(a_1-b_1x_k-\dfrac{c_1y_k}{m_1+m_2x_k+m_3y_k}\right)\Delta t+\alpha\epsilon_{1,k}\sqrt{\Delta t}+\dfrac{1}{2}\alpha^2\big(\epsilon_{1,k}^2\Delta t-\Delta t\big)\right],\\ y_{k+1}=y_k+y_k\left[\left(-a_2-b_2y_k+\dfrac{c_2x_k}{m_1+m_2x_k+m_3y_k}\right)\Delta t+\sigma_2\epsilon_{2,k}\sqrt{\Delta t}+\dfrac{1}{2}\sigma_2^2\big(\epsilon_{2,k}^2\Delta t-\Delta t\big)\right].\end{cases}$$

Choosing suitable parameters in the system, by Matlab we get the simulation figures with initial value $(x(0),y(0))=(0.6,0.4)$ and time step $\Delta t=0.002$.

In Figs. 1–4, we choose $a_1=0.4$, $a_2=0.2$, $b_1=0.7$, $b_2=0.2$, $c_1=0.1$, $c_2=0.2$, $m_1=0.1$, $m_2=0.5$, $m_3=0.3$ and change the values of α and β, then $x^*\doteq 0.49876$, $y^*\doteq 0.20972$, $(c_2-a_2m_2)a_1/b_1=2/35$, $a_2m_1=1/50$, $a_1m_2/(m_1+m_2x^*)\doteq 0.57244$, $c_2[b_1-m_2(a_1-b_1x^*)/m_1](m_1+m_3y^*)(x^*)^2\doteq 0.00361$, $b_2c_1(m_1+m_2x^*)(y^*)^2\doteq 0.00031$. Hence $(c_2-a_2m_2)a_1/b_1>a_2m_1$, $b_1>a_1m_2/(m_1+m_2x^*)$. In this situation, the equilibrium $E^*(x^*,y^*)$ of system (1.1) is globally asymptotically stable. But, the white noise may make system (1.2) appearing different phenomena. In detail, $\alpha=0.05$, $\beta=0.07$ in Figs. 1 and 2,

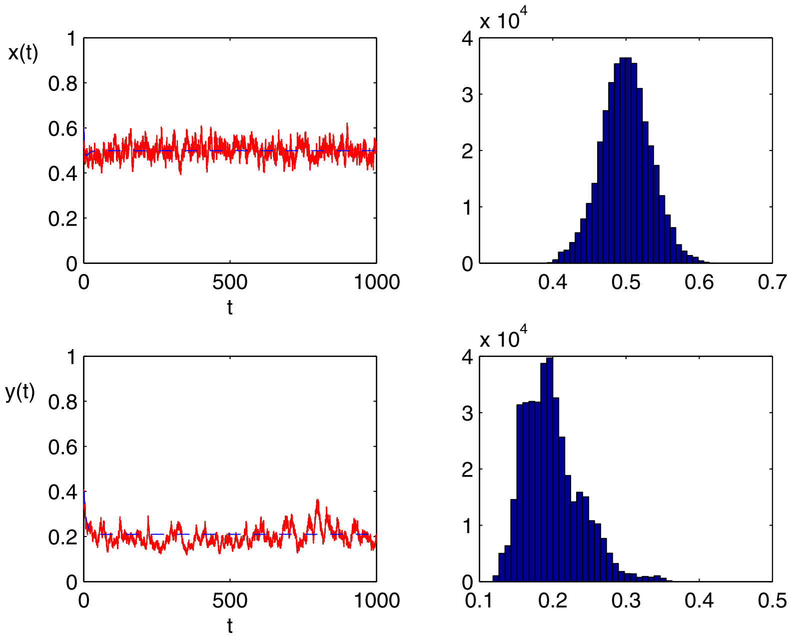

Fig. 1. The solution of the stochastic system and its histogram. The red lines represent the solution of system (1.2), and the blue lines represent the solution of corresponding undisturbed system (1.1). The pictures on the right are the histogram of system (1.2). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

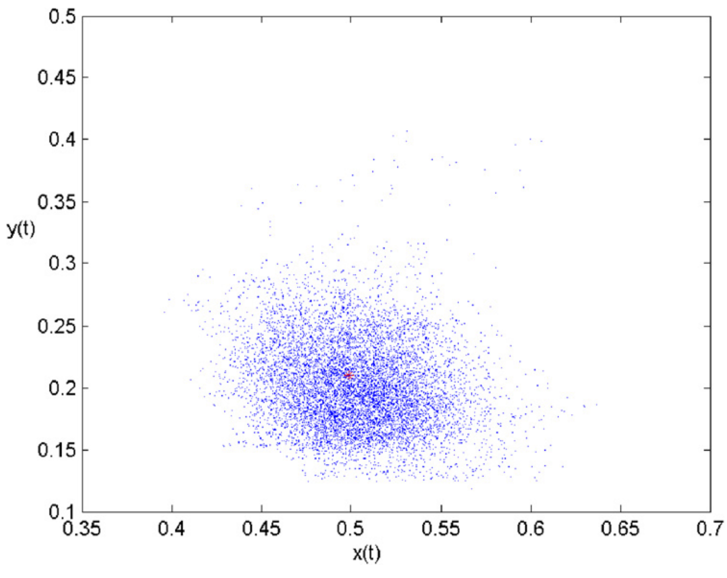

Fig. 2. Population distribution around point $E^*(x^*, y^*) \doteq (0.49876, 0.20972)$ corresponding to Fig. 1. (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

then $c_2 x^* \alpha^2/2 + c_1 y^* \beta^2/2 \doteq 0.00018$, and so the condition $c_2 x^* \alpha^2/2 + c_1 y^* \beta^2/2 < \min\{c_2[b_1 - m_2(a_1 - b_1 x^*)/m_1](m_1 + m_3 y^*)(x^*)^2, b_2 c_1(m_1 + m_2 x^*)(y^*)^2\}$ is also satisfied. Therefore, as Theorem 3.1 said, there is a stationary distribution (see the histogram on the right in Fig. 1). In addition, the left pictures in Fig. 1 show that the solution of system (1.2) is fluctuating in a small neighborhood. Moreover, from Fig. 2, we find that 95% or more of the population distribution lie within a neighborhood, which can be imagined a circular or elliptic region centered at $E^*(x^*, y^*)$ (see the red point in Fig. 2). All of these imply system (1.2) is stochastic stability. In Fig. 3, we assume the predator population is suffered large white noise. We choose $\alpha = 0.05$, $\beta = 0.7$, then $0.4 = c_2/m_2 < a_2 + \beta^2/2 = 0.445$. As the first case in Theorem 4.1 expected, the predator population will die out a.s., and the prey population tends to the value $(a_1 - \alpha^2/2)/b_1 \doteq 0.56964$ in time average. In Fig. 4, we choose $\alpha = 0.9$, $\beta = 0.07$ such that $a_1 < \alpha^2/2$, which means the prey population is suffered large white noise. In this situation, both the prey population and the predator population are going to die out a.s.

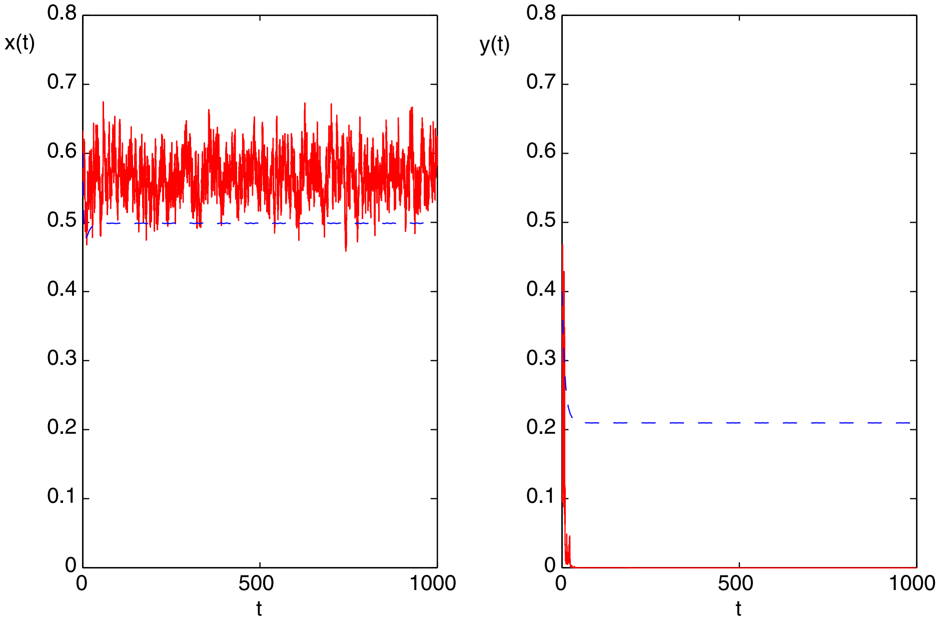

Fig. 3. The solution of system (1.2) with the predator population suffered the large white noise, in which case the equilibrium $E^*(x^*, y^*)$ system (1.1) is globally asymptotically stable.

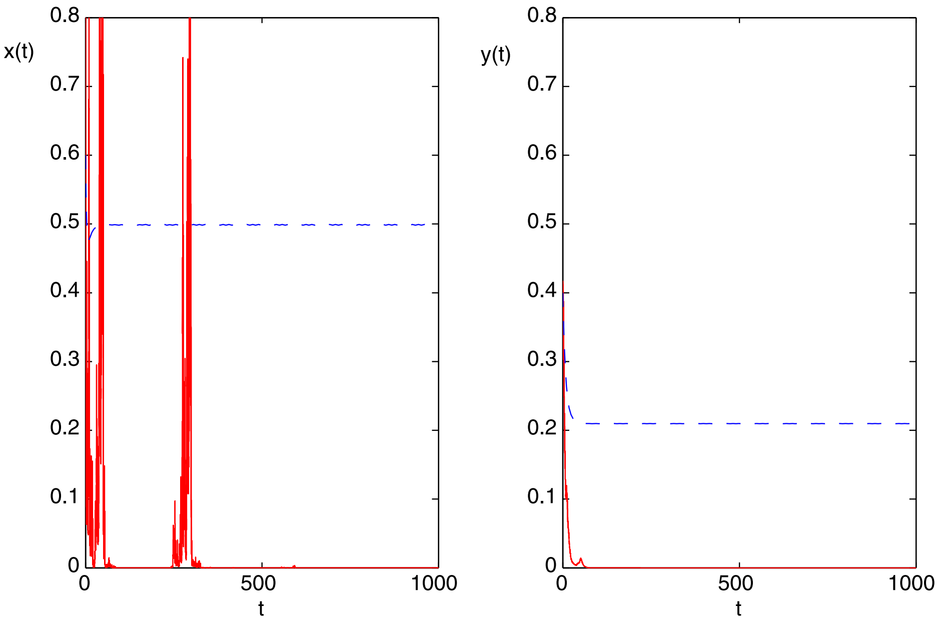

Fig. 4. The solution of system (1.2) with the prey population suffered the large white noise, in which case the equilibrium $E^*(x^*, y^*)$ of system (1.1) is globally asymptotically stable.

In Fig. 5, we choose the same parameters as in Fig. 1, but the value of a_2, in which $a_2 = 5$ such that $a_2 m_2 = 0.25 > c_2$. In this situation, $(a_1/b_1, 0)$ of system (1.1) is globally asymptotically stable, and the solution of system (1.2) has the similar phenomena as Fig. 2.

From these figures, we can see when the white noise is small, system (1.2) imitates system (1.1) (see Figs. 1, 2 and 5). But when the white noise is large, it will bring the extinction of the species, which does not happen in the deterministic system (see Figs. 3 and 4). Consequently, the stochastic system incorporates more asymptotic behavior. In real world, the large white noise may be bad weather, serious epidemic, etc., which can be considered as the decisive factors responsible for the extinction of populations.

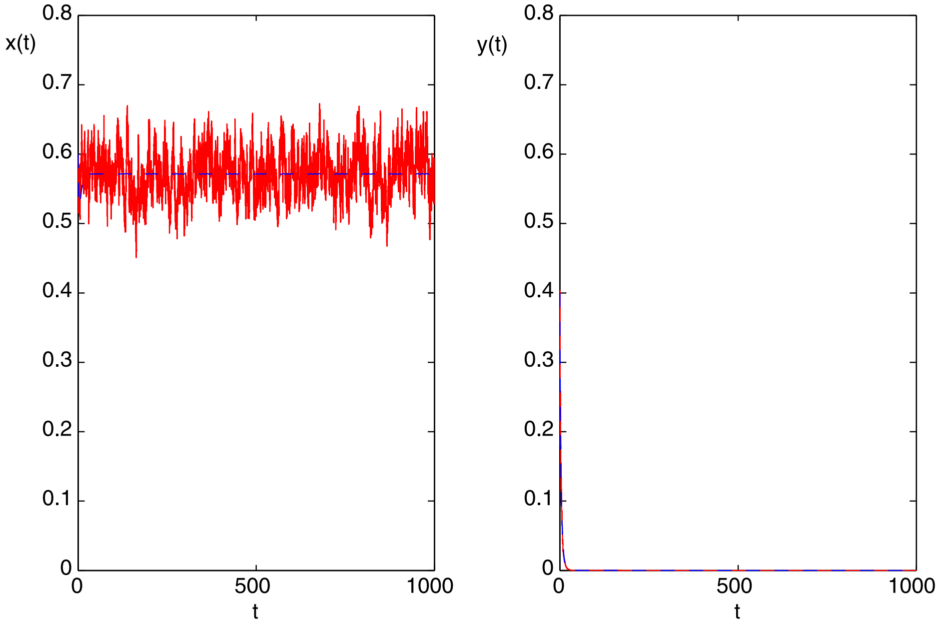

Fig. 5. The solution of system (1.2) with the small white noise, in which case $(a_1/b_1, 0)$ of system (1.1) is globally asymptotically stable.

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