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CHARACTERISTIC NUMBERS OF G-MANIFOLDS II

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1. Introduction and results

In this note we investigate to which extent K-theory characteristic numbers determine bordism classes of unitary G-manifolds. We show that, roughly, the part with cyclic isotropy groups is determined by characteristic numbers. This result is best possible.

Let G be a compact Lie group. In Section 2 we review the characteristic numbers we need. We construct a natural transformation

$$B : U_G^*(X) \to K_G^*(X)[[a_1, a_2, ...]]$$

of multiplicative equivariant cohomology theories (the Boardman map). Here $U_G^*(X)$ is the cobordism theory defined by an equivariant Thom spectrum [4], and the range of B is the ring of formal power series in $a_1, a_2, ...$ over equivariant K-theory [12]. If X is a point, we use the notation

$$U_G^k(\text{Point}) = U_G^k = U_{-k}^G; \qquad K_G(\text{Point}) = R(G).$$

We call G *cyclic* if powers of a suitable element are dense in G, i.e., G is a product of a torus and a finite cyclic group.

Let $S \subset U_G^*$ be the set of Euler classes of complex G-modules without trivial direct summand (compare [4, Theorem 3.1]). Let S denote also the image of S under B. The basic algebraic result of this note is:

Theorem 1. *Let G be cyclic. Then the localised Boardman map*

$$S^{-1}B : S^{-1}U_G^* \to S^{-1}R(G)[[a_1, a_2, ...]]$$

is injective.

We list the consequences of this fact. Let

$$r_H : U_G^* \to U_H^* \to S_H^{-1}U_H^*$$

be the composition of the restriction to the subgroup H and the localisation (this time corresponding to Euler classes of H-modules). Let

$$r : U_G^* \to \oplus S_H^{-1} U_H^*$$

be the sum of the r_H, where H runs through a complete set of non-conjugate closed subgroups of G. We have an analogous map for K-theory and a commutative diagram

$$\begin{tikzcd} U_G^* \arrow[r, "B"] \arrow[d, "r"'] & R(G)[[a]] \arrow[d, "r'"] \\ \oplus S_H^{-1} U_H^* \arrow[r, "S_H^{-1}B"] & \oplus S_H^{-1} R(H)[[a]] \end{tikzcd}$$

(we abbreviate $[[a_1, a_2, \ldots]] = [[a]]$).

Lemma 1. *$S_H^{-1}R(H)[[a]]$ is non-zero if and only if H is cyclic. It is an integral domain.*

This is in contrast to the cobordism theory, where $S_H^{-1}U_H^*$ is always non-zero [4, Theorem 3.1]. The map r' is injective, this is essentially the fact that a representation is determined by its character. The map r probably is also injective; I have checked this for many groups G and will write about it elsewhere. From [9, Theorem 1] it follows that $r \otimes Q$ is injective if G is finite. From Theorem 1 and Lemma 1 we get the result that elements x and y in U_G^* are distinguished by B, i.e. by K-theory characteristic numbers, if and only if they are distinguished by the maps r_H for cyclic H in G. In particular (using injectivity of r) the map $B : U_G^* \to R(G)[[a]]$ is injective for cyclic G.

There are corresponding statements for geometric bordism. Let $\mathfrak{U}_k^G(X)$ be the bordism group of k-dimensional singular unitary G-manifolds in X and let $i : \mathfrak{U}_k^G(X) \to U_k^G(X)$ be the Pontrjagin–Thom construction. Call $\chi = Bi : \mathfrak{U}_*^G \to U_*^G \to R(G)[[a]]$ the characteristic number map. The localisation $U_*^G \to S^{-1}U_*^G$ can also be written $U_*^G(C) \to U_*^G(C,SF)$, where this is part of the exact homology sequence of the pair (C,SF); here C is the cone on SF and SF is the classifying space for numerable G-spaces without stationary points [7, Satz 5]. This interpretation is also meaningful for $\mathfrak{U}_*^G$. The map $i : \mathfrak{U}_*^G(C,SF) \to U_*^G(C,SF)$ is always injective [4, Theorem 3.1, Proposition 4.1]. Hence from the work of Hamrick and Ossa [10] and Theorem 1 we conclude:

Theorem 2. *Let G be cyclic. Then the characteristic number map is injective. The map $B : U_G^* \to R(G)[[a_1, a_2, \ldots]]$ is injective.*

The injectivity of B in Theorem 2 implies, as in the proof of [6, Satz 8], the following theorem.

Theorem 3. *Let V be a complex G-module and SV its unit sphere. Let G be cyclic. Then*

$$B : U_G^*(SV) \to K_G^*(SV)[[a_1, a_2, ...]]$$

is injective.

Theorem 3 in turn implies that for cyclic G, K-theory characteristic numbers determine bordism classes of unitary manifolds without fixed points, and as a special case we see that the integrality theorem holds (compare the last sentence of [4, §4]). The proof is completely analogous to the proof of [6, Satz 6, Satz 7] and need not be written once more.

2. Characteristic numbers. The Boardman map

Let G be a compact Lie group. For a G-space X we denote by $K_G(X)$ the Grothendieck ring of numerable complex G-vector bundles (even if X is not compact). The i^{th} exterior power of vector bundles extends to a map $\lambda^i : K_G(X) \to K_G(X)$. If one puts, with an indeterminate t,

$$\lambda_t(x) = 1 + \lambda^1(x)t + \lambda^2(x)t^2 + \dots ,$$

then

$$\lambda_t(x+y) = \lambda_t(x)\lambda_t(y).$$

One defines natural transformations $\gamma^i : K_G(X) \to K_G(X)$ by putting

$$\gamma_t(x) = \lambda_{t/(1-t)}(x),$$

$$\gamma_t(x) = 1 + \gamma^1(x)t + \gamma^2(x)t^2 + \dots .$$

One still has

$$(1) \qquad \gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$$

(cf. [1, III §1]).

Let $\xi_m : E(m) \to B(m)$ be the universal m-dimensional complex G vector bundle. Let $R = \mathbf{Z}[c_1, c_2, ...]$ be the polynomial ring over the integers in a countable number of variables c_i. If V is a complex G-module, we define a ring homomorphism $i_{m,V} : R \to K_G(B_m)$ by putting

$$i_{m,V}(c_k) = \gamma^k(\xi_m - V - m + |V|),$$

where $|V|$ is the complex dimension of V, and V, m and $|V|$ are trivial bundles with fibre V, m and $|V|$ and trivial G-action on m and $|V|$.

Let $k_{m,n} : B(m) \times B(n) \to B(m+n)$ be the classifying map of $\xi_m \times \xi_n$. Let $d : R \to R \otimes R$ be the diagonal

$$(2) \qquad d(c_k) = \sum_{i=0}^{k} c_i \otimes c_{k-i}, \qquad c_0 = 1.$$

Then the maps $i_{m,V}$ are multiplicative in the sense that

$$(3) \qquad \mu(i_{m,V} \otimes i_{n,W})d = k^*_{m,n}\, i_{m+n,V \oplus W},$$

where μ denotes (exterior) multiplication in K_G-theory. (For the proof of (3) one uses (1) and (2).)

We now come to the construction of the Boardman map, which will give us the K_G-theory characteristic numbers for unitary G-manifolds. Let $U^*_G(X)$ be the cobordism ring for the pointed G-space X defined by the equivariant Thom spectrum as in [4]. Let X be compact, and let $x \in U^{2n}_G(X)$ be represented by

$$f : V^c X \to M(n + |V|).$$

Here $V^c = V \cup \{\infty\}$ is the one-point compactification of V with base point ∞,

$$V^c X = V^c \times X/(V^c \times \{0\} \cup \{\infty\} \times X)$$

and $M(k)$ is the Thom space of ξ_k.) We consider the following composition of homomorphisms

$$R \xrightarrow{(a)} K_G(B(n+|V|)) \xrightarrow{(b)} \widetilde{K}_G(M(n+|V|)) \xrightarrow{(c)} \widetilde{K}_G(V^c X) \xrightarrow{(d)} \widetilde{K}_G(X),$$

where (a) is $i_{|V|+n,V}$, (b) is the Thom homomorphism for $\xi_{n+|V|}$, (c) is induced by f, and (d) is the suspension isomorphism. We remark that the Thom homomorphisms and suspension homomorphisms are defined such that the corresponding Euler classes (= restriction of the Thom class to the zero section) are given by the alternating sum over the exterior powers of the conjugate bundle. One verifies that the homomorphism above is independent of the representative f of x. So we get a homomorphism

$$b = b_{2n} : U^{2n}_G(X) \to \mathrm{Hom}(R, K_G(X)).$$

In view of formula (3) the b_{2n} are multiplicative in the following sense: Give $\mathrm{Hom}(R, K_G(X))$ the ring structure induced by the diagonal d of R and the multiplication μ of $K_G(X)$. Then the b_{2n} are compatible with the multiplication in $U^*_G(X)$. For computational purposes it is better to use a dualized version of the groups $\mathrm{Hom}(R, K_G(X))$. Develop the product

$$U = \prod_{i=1}^{\infty} (1 + c_1 t_i + c_2 t_i^2 + \dots),$$

formally $\Sigma_r c^r b^r$ according to the monomials $c^r = c_1^{r_1} = c_1^{r_2} \dots$. Then b^r is symmetric in the $t_1, t_2, \dots$ and can be expressed as a polynomial with integer coefficients in the elementary symmetric functions $a_1, a_2, \dots$ of the $t_1, t_2, \dots$. So we consider $b^r \in \mathbf{Z}[a_1, a_2, \dots]$. If K is any $\mathbf{Z}$-algebra, define a map

$$\alpha : \mathrm{Hom}(R, K) \to K[[a_1, a_2, \dots]]$$

(where $K[[a_1, a_2, ...]]$ is the algebra of formal power series over K in $a_1, a_2, ...$) by

$$\alpha(f) = \sum_r f(c^r) b^r .$$

Then α is a ring homomorphism (for details see [3, X. 1.22]). Note that for $r = (k, 0, 0, ...)$ we get $b^r = a_k$. The composition of α and b defines the Boardman map

$$B : U_G^{2n}(X) \to K_G(X)\,[[a_1, a_2, ...]].$$

If $e(\eta)$ is the cobordism Euler class of a line bundle η over X, then

$$(4) \qquad Be(\eta) = (1-\bar{\eta}) + (1-\bar{\eta})^2 a_1 + (1-\bar{\eta})^3 a_2 + \dots .$$

Here $\bar{\eta}$ is the complex conjugate of η. (We have not distinguished a bundle and the corresponding element in $K_G(X)$.) For odd degrees we define B through the suspension isomorphism. To sum up: The Boardman map

$$(5) \qquad B : U_G^*(X) \to K_G^*(X)[[a_1, a_2, ...]]$$

is a natural transformation of multiplicative cohomology theories such that (4) holds.

Remark. In our presentation of B everything is $\mathbf{Z}_2$-graded (U^* through even and odd degrees). It is also possible to use $\mathbf{Z}$-graded K_G-theory and homogeneous power series (degree $a_i = -2i$) such that B has degree zero by giving c_i degree $2i$, $i_{m,V}$ degree zero, and Thom homomorphisms the degree "dimension of the bundle", and using a graded $\mathrm{Hom}(R, K)$.

If M is a compact unitary G-manifold, then one has the index homomorphism $\mathrm{Ind} : K_G(M) \to R(G)$ (cf. [2, p. 498]). If $x \in U_G$ is the element represented by M via the Pontrjagin–Thom construction, then $b(x)$ is the homomorphism which maps a monomial in the c_i onto Ind of the corresponding monomial in the $\gamma^i(|\tau| - \tau)$, where τ is the tangent bundle of M and $|\tau|$ its complex dimension. (In fact, by the definition of unitary manifold one has to add some trivial bundles ϵ to τ, and then one considers complex structures on $\tau \oplus k\epsilon$ which are G-invariant.) If $G = \{1\}$ is the trivial group, then $B : U^* \to \mathbf{Z}[[a_1, a_2, ...]]$ is injective by the theorem of Stong [13] and Hattori [11]. Note further that B by construction is compatible with the restriction of group actions to subgroups.

Remark. The construction of χ in [6, p. 292] is wrong. Normal and tangent bundle have to be interchanged.

3. Proof of Theorem 1

If G is abelian, the irreducible representations are one-dimensional. In order to get $S^{-1}K_G^*(X)\,[[a]]$ we therefore have to invert power series of the form (4). But then the localisation is equivalent to inverting elements $(1-V) \in R(G)$ for non-trivial irreducible G-modules V. Let T be multiplicatively generated by such $1-V$. We have to show that

$$S^{-1}B : S^{-1}U_G^* \to T^{-1}R(G)[[a]]$$

is injective.

We recall the canonical elements of $S^{-1}U_G^*$ that we found in [8]. Let η be the canonical line bundle over infinite complex projective space CP^∞. Let V be a non-trivial irreducible G-module. There is an isomorphism $U_G^*(CP^\infty) = U_G^*[[C]]$ of U_G^*-algebras; the element C corresponds to the Euler class $e(\eta)$. Hence we can write

$$(6) \qquad e(V \otimes \eta) = a_0(V) + a_1(V)C + a_2(V)C^2 + \ldots .$$

Let $\lambda : U_G^* \to S^{-1}U_G^*$ be the localisation and put $h_i(V) = e(V)^{-1}\lambda a_i(V)$. Let $\{V_j;\ j \in J\}$ be a complete set of non-trivial non-isomorphic irreducible G-modules. Then we have by [8, Satz 2]:

Lemma 2. *The canonical map of*

$$U_*[h_i(V_j): i \geq 1, j \in J] \otimes \mathbf{Z}[e(V_j), e(V_j)^{-1} : j \in J]$$

into $S^{-1}U_G^*$ is an isomorphism of U_*-algebras. One has $\lambda a_0(V_j) = e(V_j)$.

The proof of Theorem 1 now runs along the following lines: We show that the elements $S^{-1}B(\lambda a_i(V_j)), j \in J, i \geq 0$, are algebraically independent over $S^{-1}B(U_* \cdot 1)$. Since $S^{-1}B|U_* \cdot 1$ is injective by the theorem of Hattori and Stong, we conclude that $S^{-1}B$ is injective on the U_*-subalgebra A generated by the elements $\lambda a_i(V_j)$. But every element of $S^{-1}U_G^*$ is algebraic over A. Hence $S^{-1}B$ is injective.

To proceed, we have to compute $S^{-1}B(\lambda a_i(V_j))$. We apply $S^{-1}B$ to the defining equation (6) for the $a_i(V)$ and use (4). We get the equality

$$(7) \qquad (1-\bar{V}\otimes\bar{\eta})+(1-\bar{V}\otimes\bar{\eta})^2 a_1 + \ldots = Be(V\otimes\eta) = B(a_0(V)) + B(a_1(V))BC + \ldots$$

with

$$(8) \qquad BC = (1-\bar{\eta}) + (1-\bar{\eta})^2 a_1 + \ldots$$

Hence we have to expand the first series in (7) as a power series in (8) and the resulting coefficients are the $B(a_i(V))$. The required expansion is a difficult task; we avoid it by introducing new coordinates. The Boardman map B maps the coefficients U_* injectively into the polynomial ring

$$P := Z[a_1, a_2, \ldots] \subset R(G)[[a_1, a_2, \ldots]].$$

We identify U_* with this subring of P and extend $S^{-1}B$ to

$$S^{-1}U_G^* \otimes_{U_*} P \to S^{-1}R(G)[[a_1, a_2, \ldots]]$$

and similarly from $S^{-1}U_G^*(CP^\infty) = S^{-1}(U_G^*[[C]])$ to $(S^{-1}U_G^* \otimes_{U_*} P)[[C]]$.

Let

$$g(x) = x + a_1 x^2 + \dots$$

and $h(x)$ its inverse in the sense that $g(h(x)) = x = h(g(x))$. We expand $e(V \otimes \eta)$ in terms of $h(e(\eta))$, and get

$$e(V \otimes \eta) = \sum a_i(V) e(\eta)^i = \sum a_i(V)(gh(e(\eta)))^i = \sum b_i(V) h(e(\eta))^i \tag{9}$$

with suitable $b_i(V)$. From (9) we conclude that $b_i(V)$ has the form

$$b_i(V) = a_i(V) + r_i(V), \tag{10}$$

where $r_i(V)$ is a polynomial in $a_0(V), \dots, a_{i-1}(V)$ with coefficients in P. Therefore the $S^{-1}B(a_i(V))$ are algebraically independent over P if and only if the $S^{-1}B(b_j(V))$ are independent over P. To make this statement meaningful we use Lemma 1 together with the fact that

$$P \to R(G)[[a]] \to T^{-1}R(G)[[a]]$$

is injective.

Proof of Lemma 1. Write $G = \mathbf{Z}_m \times T^n$, where T^n is an n-dimensional torus ($n \geqslant 0$) and $\mathbf{Z}_m$ the cyclic group of order m. Then

$$R(G) \cong R(\mathbf{Z}_m) \otimes R(T), \qquad R(\mathbf{Z}_m) \cong \mathbf{Z}[x]/(x^m - 1),$$

$$R(T^n) \cong \mathbf{Z}[x_1, \dots, x_n, 1/x_1, \dots, 1/x_n],$$

where $x, x_1, \dots, x_n$ are suitable irreducible modules. If we invert the $(1-x^j)$, $1 \leqslant j < m$, we get a ring without zero divisor, isomorphic to a subring of $Q[\omega]$, $\omega = \exp(2\pi i/m)$, by evaluating at a generator of $\mathbf{Z}_m$. The tensor product with $R(T)$ then has no zero divisor and further localisation cannot change this fact. So $T^{-1}R(G)$ is as claimed for cyclic G. For non-abelian G one shows that $T^{-1}R(G) = 0$ as in [5, p.36], using equivariant K-theory instead of cobordism theory. If G is abelian, but not cyclic, then a direct calculation shows that $T^{-1}R(G) = 0$; for instance, for $G = \mathbf{Z}_p \times \mathbf{Z}_p$ the product of the elements $(1 - V_j)$, $j \in J$, is zero in $R(G)$.

We return to the elements $b_i(V)$. From the definition we get

$$Be(V \otimes \eta) = \sum B(b_i(V))\,(Bhe(\eta))^i, \tag{11}$$

$$Bhe(\eta) = hBe(\eta) = hg(1-\bar{\eta}) = 1-\bar{\eta}. \tag{12}$$

That means, in order to compute $Bb_i(V)$ we have to expand

$$(1-\overline{V\eta}) + (1-\overline{V\eta})^2 a_1 + \dots$$

in terms of $(1-\bar{\eta})$, and this can be achieved using the formal Taylor expansion. If we put $f(x) = g(1-\bar{V}x)$, then

$$Bb_i(V) = (i!)^{-1} \bar{V}^k g^{(k)} (1-\bar{V}), \tag{13}$$

where $g^{(k)}$ is the k^{th} formal derivative. So we have reduced our problem to showing that the power series $g^{(k)}(1-V_j)$, $j \in J$, $k \geqslant 0$, are algebraically independent over P. We use the following generalisation of Vandermonde's determinant.

Lemma 3. *Let $y_1, \ldots, y_k$ be non-zero pairwise different elements of an integral domain. Consider the determinant $D(y_1, \ldots, y_k, r, n)$ with the following rk rows: The j^{th} row is*

$$(n-s+t,\, s)\, y_i^{n-s+t}, \qquad 0 \leqslant t < rk,$$

with $(a, b) = (a+1) \cdot \ldots \cdot (a+b)$ for $b \geqslant 1$ and $(a, b) = 1$ for $b = 0$, if $j = ks + i$. Then $D(y_1, \ldots, y_k, r, n)$ is non-zero.

Proof. The usual evaluation of the Vandermonde determinant essentially works also in this case. We sketch the procedure. Multiply the j^{th} column by y_1 and subtract from the $(1+j)^{\text{th}}$. Expand with respect to the first row. Use suitable row operations and extract common row factors $(x_j - x_1)$ and $(j-1)x_1$ for $j = 2, \ldots, r$ to get a determinant of the same form as before with $rk-1$ rows, except that $y_1, \ldots, y_k$ are replaced by $y_2, \ldots, y_k, y_1$, and n is replaced by $n-1$ and the last row is missing. Finally this procedure leads to the Vandermonde determinant.

The proof of Theorem 1 will now be finished with the following algebraic result.

Lemma 4. *Let K be an integral domain. Let $V_1, \ldots, V_k \in K$ be pairwise distinct and different from 0 and 1. Then the power series $b_j(V_t)$, $1 \leqslant t \leqslant k$, $j \geqslant 0$ in $K[[a_1, a_2, \ldots]]$ defined by the identity*

$$\sum_{i \geqslant 0} (1-V_j x)^{i+1} a_i = \sum_{i \geqslant 0} b_i(V_j)(1-x)^i, \qquad a_0 = 1,$$

are algebraically independent over the polynomial ring $K[a_1, a_2, \ldots]$.

Proof. Ley us work over the quotient field Q of K. Assume we have an algebraic relation $R(b(1), \ldots, b(r_1)) = 0$, where $b(1), \ldots, b(r_1)$ are without loss of generality the $b_j(V_t)$ with $1 \leqslant t \leqslant k$ and $0 \leqslant j < r$ and R is a polynomial with coefficients in $Q[a_1, a_2, \ldots]$. These coefficients involve only a finite number of indeterminates a_s, say only $a_1, \ldots, a_m$. Choose $n > m$. The power series $b_j(V_t)$ have the form

$$u_0 + u_1 a_1 + u_2 a_2 + \ldots, \qquad u_i \in Q.$$

We look at the coefficients of the $b(j)$ of $a_n, a_{n+1}, \ldots, a_{n+kr-1}$. They form a determinant which up to a non-zero factor is just the one considered in Lemma 3 and hence different from zero. We can therefore express the $b(j)$ as linear combination of series $b'(j)$, where $b'(j)$ involves a_{n+j} with a non-zero coefficient and does not involve a_{n+i}, $i \neq j$, $a \leqslant i < kr$. If we rewrite the relation $R(b(1), \ldots, b(r_1)) = 0$ in terms of the $b'(j)$ and put $a_k = 0$ for $k \geqslant n+kr$, we would get an algebraic relation of polynomials $b'(j) = \lambda_j a_j + r_j$, $r_j \in Q[a_1, \ldots, a_m]$, $j > m$, over $Q[a_1, \ldots, a_m]$, which is impossible.

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