Global and singular solutions to the generalized Proudman–Johnson equation

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A R T I C L E   I N F O

Article history:
Received 31 July 2009
Revised 5 March 2010
Available online 21 March 2010

Dedicated to Professor Adrian Constantin on the occasion of his 40th birthday

M S C:
35D05
74H35
35Q35

K e y w o r d s:
The generalized Proudman–Johnson equation
Global weak solutions
Blow-up rate

A B S T R A C T

We show that there is a class of solutions to the generalized Proudman–Johnson equation which exist globally for all parameters \(a\) having the form \(-\frac{n+3}{n+1}\) for \(n \in \mathbb{N}\), thereby extending a result of Bressan and Constantin (2005) [2]. Furthermore, we present new proofs of existence of solutions developing spontaneous singularities and compute the corresponding blow-up rates.

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1. Introduction

In this paper, we shall be concerned with the Cauchy problem

\[
\begin{align*}
    u_t(t,x) + \left( \frac{u^2}{2} \right)_x &= \frac{a + 3}{4} \left( \frac{1}{x} \int_x^\infty u_x(t,\zeta)^2 \, d\zeta \right), & \quad t > 0, \\
    au(0,x) &= \tilde{u}(x), & \quad x \in \mathbb{R},
\end{align*}
\]  

(1)
where the parameter \( a \in [-2, -1) \) is of the form

\[
a = \frac{-n + 3}{n + 1} \quad \text{for} \ n \in \mathbb{N},
\]

(2)
as well as with the periodic inviscid generalized Proudman–Johnson equation [27,30]

\[
\begin{cases}
  u_{txx} + uu_{xxx} = au_xu_{xx}, & x \in S = \mathbb{R}/\mathbb{Z}, \ t > 0, \\
  u(0, x) = \bar{u}(x), & x \in S
\end{cases}
\]

(3)
for any \( a < -1 \).

Formal differentiation of (1) with respect to the spatial variable \( x \in \mathbb{R} \) yields

\[
(u_t + uu_x)_x = \frac{a + 3}{2} u_x^2,
\]

(4)
whereupon yet another differentiation gives

\[
\begin{cases}
  u_{txx} + uu_{xxx} = au_xu_{xx}, & t > 0, \ x \in \mathbb{R}, \\
  u(0, x) = \bar{u}(x)
\end{cases}
\]

(5)
which is, by comparison with (3), the inviscid generalized Proudman–Johnson equation on the real line.

1.1. Significance of the generalized Proudman–Johnson equation

The generalized Proudman–Johnson equation (3), (5) can be motivated in several ways. The first motivation comes from the study of the two-dimensional incompressible Euler equations. For parameters \( a \in \mathbb{R} \cup \{+\infty\} \), Eq. (3) was first derived by Okamoto and Zhu [30] and further investigated in [27,6,14,35,36]. Remarkably, the parameter \( a \) interconnects several well-studied equations within the framework of the generalized Proudman–Johnson equation (3), (5).

For \( a = 1 \), it reduces to the original Proudman–Johnson equation [32,7], which is obtained via the separation of spatial variables for the stream function \( \psi \) of the two-dimensional incompressible Euler equations:

\[
\psi(t, x, y) = yu(t, x),
\]

where \( x \) is contained in a finite interval, and \( y \in \mathbb{R} \). (More generally, in dimensions \( d \geq 2 \), this ansatz was used by [33] to obtain a stagnation-point class of solutions which are periodic in one coordinate direction and which are related to (3) via \( a = \frac{3-d}{d-1} \), cf. Remark 1 below.)

Moreover, the generalized Proudman–Johnson equation encompasses the well-known Burgers equation [3] for \( a = -3 \), which is a successful model in gas dynamics; if \( a = -2 \), the generalized Proudman–Johnson equation becomes the Hunter–Saxton equation [18,23,24,37] arising in the study of nematic liquid crystals. In this connection, it is of interest to point out that both the periodic Burgers equation and the periodic Hunter–Saxton equation, aside from their significance as physical models, have a deep geometric meaning as well: The former being the geodesic equation for the right-invariant \( L^2 \)-metric on the group of orientation-preserving diffeomorphisms \( D \) on the unit circle \( S = \mathbb{R}/\mathbb{Z} \) [12,21], and the second one, the geodesic equation on \( D \) modulo the subgroup of rigid rotations, equipped with the right-invariant \( H^1(S) \)-metric [25,20,23,24].

The equation \( u_{xx} + uu_{xxx} = 0 \), arrived at if \( a = 0 \), emerges at the intersection of projective geometry and gravity [31].
An important heuristic motivation for inserting the parameter $a$ lies in the close analogy of the Proudman–Johnson equation $u_{xxx} + uu_{xx} = u_xu_{xx}$ with the vorticity equation in three space dimensions (obtained by taking the curl of the velocity in the three-dimensional incompressible Euler equations). By introducing the parameter $a \in \mathbb{R} \cup \{+\infty\}$, one can carefully study the interplay between the “convection term” $(uu_{xxx})$ and the “stretching term” $(u_xu_{xx})$ leading to creation or depletion of finite-time singularities [28] (cf. also [29] for a similar vorticity model equation).

Rather interestingly, the link of the inviscid generalized Proudman–Johnson equation with the Calogero equation [4,31],

$$\partial_t u_x = uu_{xx} + \Phi(u_x),$$  \tag{6}

where $\Phi(z)$ is an arbitrary scalar function, does not seem to have been realized before. Eq. (6) is obviously related to (5) by time reversal ($t \mapsto -t$) and by setting $\Phi(z) = -\frac{a+1}{2}z^2$.

Several authors [26,31] have analyzed a special class of Calogero equations referred to as the generalized Hunter–Saxton equation,

$$u_{tx} = uu_{xx} + \varepsilon u_x^2,$$

with $\varepsilon \in \mathbb{R}$. (The Hunter–Saxton equation is then given as the special case $\varepsilon = \frac{1}{2}$.) In [31], the generalized Hunter–Saxton equation was transformed into a Liouville equation in order to construct general solutions.

It is also worth mentioning that the generalized Proudman–Johnson equation arises as the short-wave limit for the so-called $b$-equation [16,17] given by

$$u_t - \alpha^2 u_{txx} + (b + 1)uu_x = \alpha^2 (bu_xu_{xx} + uu_{xxx}), \quad (\alpha, b) \in \mathbb{R}^2,$$  \tag{7}

which is a nonlinear dispersive wave equation comprising the famous Korteweg–de Vries (KdV) equation ($\alpha = 0, b = 2$) [22], the Camassa–Holm equation ($\alpha = 1, b = 2$) [5], and the Degasperis–Procesi equation ($\alpha = 1, b = 3$) [15]. All three equations are bi-Hamiltonian and arise in the modeling of shallow water waves, with the KdV equation being appropriate for waves of small amplitude, while the Camassa–Holm and Degasperis–Procesi equations pertain to waves of medium amplitude (cf. the discussions in [19,13]) and thus also accommodate wave-breaking phenomena (which is not the case for the KdV equation). Moreover, the Camassa–Holm and the Degasperis–Procesi equations admit peaked (periodic as well as solitary) traveling waves capturing the main feature of the exact traveling wave solutions of greatest height of the governing equations for water waves (cf. [8,11]).

The short-wave (or high-frequency) limit equation is obtained via the change of variables $(t, x) \mapsto (\epsilon t, \epsilon x)$ and by letting $\epsilon$ tend to infinity in the resulting equation. The generalized Proudman–Johnson equation with parameter $a$ is thus the short-wave limit of the $b$-equation (7) for any $\alpha \neq 0$ and $b = -a$.

Let us finally mention that [34] obtained a class of Einstein–Weyl spaces from the generalized Proudman–Johnson equation for $a = \frac{1}{3}$.

1.2. Known results concerning global existence and blow-up

Okamoto demonstrated that solutions to the periodic Proudman–Johnson equation on the unit interval exist locally in time and are unique in the Sobolev space $H^s([0,1]), s \geq 1$. Solutions exhibit distinct long-time features according to the parameter values $a \in (-\infty, 1) \cup \{+\infty\}$.

**Theorem 1.** (See [27].)

- Suppose that $a \in (-\infty, -2)$ and that $\int_0^1 \bar{u}_x(x)^3 \, dx < 0$. Then $\|u_x(t, \cdot)\|_{L^2([0,1])}$ blows up in finite time.
• Suppose that \( a \in [-2, -1) \). Then \( u_x(t, \cdot) \) stays bounded in the \( L^2([0, 1]) \)-norm, yet \( \| u_x(t, \cdot) \|_{L^\infty([0, 1])} \) becomes unbounded in finite time.

• Suppose that \( a \in [-1, 0) \) and that \( \tilde{u}_{xx} \in L^{-\frac{d}{2}}([0, 1]) \). Then the solution exists globally in time.

• Suppose that \( a \in [0, 1) \) and that \( \tilde{u}_{xxx} \in L^{\frac{d}{d+2}}([0, 1]) \). Then the corresponding solution exists globally in time.

• Suppose that \( a = \infty \). Then there is blow-up in finite time if and only if the Lebesgue measure

\[
\left| \left\{ x \in [0, 1]: \tilde{u}(x) = \max_{y \in [0, 1]} \tilde{u}(y) \right\} \right| \leq 1/2.
\]

**Remark 1.** Independently of Okamoto [27], Saxton and Tüğlay [33] established global existence for a stagnation-point class of unidirectionally periodic solutions to the Euler equations in \( \mathbb{R}^d \), \( d \geq 3 \),

\[
\partial_t[u_t + uu_x](t, x) = \frac{d}{d-1} \left\{ u_x(t, x)^2 - \int_{\mathbb{R}} u_x(t, y)^2 dy \right\},
\]

which are related to the periodic generalized Proudman–Johnson equation (3) by \( a = \frac{3-d}{d-1} \), as well as to the Calogero equation (6) if time is reversed and \( \Phi(z) = \frac{z^2}{2} \). Due to the restriction to (integer) dimensions \( d \geq 3 \), their result on global existence is strictly contained in the proof of global existence for parameters \( a \in (-1, 0) \) of [27].

**Remark 2.** Solutions to the periodic generalized Proudman–Johnson equation are also interesting in that they preserve certain geometric properties of their initial data. Let us define \( \mathcal{F}^* \) consisting of all odd and periodic functions \( f \in H^2(\mathbb{R}^d \setminus (-\frac{3}{2}, \frac{3}{2})) \) with \( \sup_{x \in [-\frac{3}{2}, \frac{3}{2}]} |f_x(x)| = f_x(0) \) such that \( f \) is convex on \((-1/2, 0)\) and concave on \((0, 1/2)\).

**Proposition 2.** (See [14, 36].) If \( \tilde{u} \in \mathcal{F}^* \), then the corresponding solution to the periodic generalized Proudman–Johnson equation \( u(t, \cdot) \) remains in \( \mathcal{F}^* \) as long as the solution exists.

### 1.3. Contents

The remainder of article is divided into two main sections: Section 2 presents a class of global solutions to the generalized Proudman–Johnson equation on the real line if \( a = -\frac{n+3}{n+1}, n \in \mathbb{N} \), and Section 3 offers proofs of blow-up in the periodic case together with exact blow-up rates if \(-\infty < a < -1\).

### 2. Global existence of solutions

#### 2.1. The solution concept on the real line

We will treat a special class of solutions to (1) which is given below. For notational convenience, let us set

\[
\mathbb{N} \ni n = n(a) := \frac{-2}{a + 1} \geq 2 \quad (\text{cf. (2)}).
\]

**Definition 3.** A function \( u^{(a)}(t, x) \) defined on \([0, T] \times \mathbb{R}\) is a solution of the generalized Proudman–Johnson equation (1) with parameter \( a \) if the subsequent conditions are fulfilled.

(i) \( u^{(a)} \in C([0, T] \times \mathbb{R}; \mathbb{R}) \) and \( u^{(a)}(0, x) = \tilde{u}^{(a)}(x) \) point-wise on \( \mathbb{R} \).
(ii) For each $t \in [0, T]$, the map $x \mapsto u^{(a)}(t, x)$ is absolutely continuous with $u_x^{(a)}(t, .) \in \Omega^n(\mathbb{R}) := L^2(\mathbb{R}) \cap L^n(\mathbb{R})$. Moreover, the map $t \mapsto u_x^{(a)}(t, .)$ belongs to the space $L^\infty([0, T]; \Omega^n(\mathbb{R}))$.

(iii) The map $t \mapsto u^{(a)}(t, .) \in L^2_{loc}(\mathbb{R})$ is absolutely continuous and satisfies (1) for almost every $t \in [0, T]$.

**Remark 3.** As a consequence of the Riesz–Thorin interpolation theorem [1], condition (ii) implies that $u_x^{(a)}(t, .)$ has to be in $L^r(\mathbb{R})$ for all $2 \leq r \leq n$.

### 2.2. The method of characteristics

Observe that, by (4) and the definition of $n$,

$$u_t + uu_x = -\frac{1}{n} u_x^2,$$

and define the characteristic $t \mapsto \xi(t, y)$ as the solution to the ODE

$$\frac{\partial}{\partial t} \xi(t, y) = u(t, \xi(t, y)), \quad \xi(0, y) = y. \quad (8)$$

By (4), the evolution of the gradient $u_x$ along each characteristic is described by

$$\begin{cases}
\frac{\partial}{\partial t} u_x(t, \xi(t, y)) = -\frac{1}{n} u_x(t, \xi(t, y))^2, \\
u_x(0, y) = \bar{u}_x(y).
\end{cases} \quad (9)$$

Notice that the solution to the ODE

$$z'(t) = -\frac{1}{n} z(t)^2, \quad z(0) = \bar{z},$$

is given by

$$z(t) = \frac{n\bar{z}}{n + 2t}. \quad (10)$$

If $\bar{z} \geq 0$, then solutions to the ODE will exist for all times, whereas if $\bar{z} < 0$, this solution approaches $-\infty$ as

$$t \uparrow T_n(\bar{z}) = \frac{n}{-\bar{z}}. \quad (11)$$

By (10), the solution of (9) can thus be represented by

$$u_x(t, \xi(t, y)) = \frac{\bar{u}_x(y)}{n + \bar{u}_x(y)t}. \quad (12)$$

This tells us that a monotonously increasing $\bar{u}$ yields a global solution, while there will be a finite-time singularity if the set $\{x \in \mathbb{R}: \bar{u}(x) < 0\}$ is not empty. Accordingly, we will turn our attention to this case. Now define

$$T_n := \inf_{\{x \in \mathbb{R}: \bar{u}_x(x) < 0\}} \left\{ \frac{n}{-\bar{u}_x(x)} \right\} \geq 0. \quad (13)$$
If \( \liminf_{x \to \infty} \bar{u}_x(x) > -\infty \), then the method of characteristics can be employed to construct the unique solution to (1) for \( 0 \leq t < T_n \), as will be explained below.

From (8) and (12) we have

\[
\partial_t \xi_y(t, y) = u_x(t, \xi) \xi_y = \frac{n\bar{u}_x(y)}{t} \cdot \xi_y(t, y), \quad t \in (0, T_n),
\]

(14)

\[
\xi_y(0, y) = 1, \quad y \in \mathbb{R}.
\]

(15)

The unique solution to this linear ODE is given by

\[
\xi_y(t, y) = \left[ 1 + \frac{t}{n} \bar{u}_x(y) \right]^n > 0 \quad \text{as long as } 0 \leq t < T_n.
\]

(16)

This shows that for each \( t \in [0, T_n) \), the map \( y \mapsto \xi(t, y) \) is an absolutely continuous strictly increasing diffeomorphism of the real line, since \( 1 + \frac{t}{n} \bar{u}_x(y) > 0 \) in this time span. Differentiating (16) with respect to time, we obtain

\[
\xi_{ty}(t, y) = \bar{u}_x(y) \left[ 1 + \frac{t}{n} \bar{u}_x(y) \right]^{n-1}
\]

\[
= \bar{u}_x(y) + \sum_{k=1}^{n-1} \binom{n-1}{k} \left( \frac{t}{n} \right)^k \bar{u}_x(y)^{k+1}.
\]

(17)

It is here that we utilize the assumption of \( \frac{2}{n+1} = n \) being a natural number, which facilitates the representation of \( \xi_{ty} \) as a finite binomial series (so that we may dispense with considerations on the convergence radius of the Newton series, see Remark 5). Integration of (17) in the spatial variable \( y \) yields

\[
\xi_t(t, y) = \bar{u}(y) + \varphi(t, y),
\]

(18)

where

\[
\varphi(t, y) := \frac{1}{2} \sum_{k=1}^{n-1} \left( \binom{n-1}{k} \left( \frac{t}{n} \right)^k \int_{\mathbb{R}} \text{sign}(y-x)\bar{u}_x(x)^{k+1} \, dx \right).
\]

Thus, by virtue of (8), we infer that along the characteristic curve

\[
\xi(t, y) = y + \int_0^t \xi_t(s, y) \, ds = y + t\bar{u}(y) + \int_0^t \varphi(s, x) \, ds,
\]

(19)

the value of the solution \( u \) is given by

\[
uu
\]

\[
uu(\bar{u}(y) + \varphi(t, y) \quad \text{for all } 0 \leq t < T_n.
\]

(20)
Remark 4. The approach of this subsection works as long as \( n + t \bar{u}_x(x) > 0 \), but fails as soon as \( t = T_n \) (13), which is a consequence of the spontaneous singularity appearing as time advances toward the maximal existence time \( T_n \) of the strong solution (20):

\[
\liminf_{t \uparrow T_n} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty.
\]

Equivalently, at this instance the characteristic curve \( \xi(t, \cdot) \) might not be an increasing diffeomorphism of the line any more (meaning that \( \xi_y(t, y) = 0 \) in some nonempty interval \( y \in (a, b) \) for \( t = T_n \)), rendering it impossible that (20) gives rise to a well-defined solution \( u(t, x) \) for \( t \geq T_n \).

2.3. Modified characteristics

Since the method of characteristics breaks down as time approaches \( T_n \), we shall adopt a different approach allowing for global solutions (cf. \[2\]). Let \( \bar{u} \in C(\mathbb{R}) \) be such that its distributional derivative \( \bar{u}_x \in L^n(\mathbb{R}) \), and define \( \psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) by

\[
\psi_y(t, y) = \left( \sum_{k=1}^{n-1} \left( \frac{n-1}{k} \right) \frac{k t^{k-1}}{n^k} \bar{u}_x(y)^{k+1} \cdot \chi_{\{\bar{u}_x(y) > -\frac{n}{t}\}} \right),
\]

or equivalently,

\[
\psi(t, y) = \frac{1}{2} \sum_{k=1}^{n-1} \left( \frac{n-1}{k} \right) \frac{k t^{k-1}}{n^k} \int_{\{\bar{u}_x(x) > -\frac{n}{t}\}} \text{sign}(y - x) \bar{u}_x(x)^{k+1} \, dx.
\]

Expressed in a different way: if \( x_0 \in \mathbb{R} \) satisfies \( \bar{u}_x(x_0) < 0 \) such that \( u(t, \xi(t, x_0)) \) blows up as \( t \to T_n(\bar{u}_x(x_0)) \), then this point \( x_0 \) is deleted from the domain of integration defining \( \psi(t, \cdot) \) for all \( t \geq T_n(\bar{u}_x(x_0)) \), because

\[
\bar{u}_x(x_0) > -\frac{n}{t} \quad \text{if and only if} \quad T_n(\bar{u}_x(x_0)) > t.
\]

Let us also comment that \( \psi(t, y) \) is well-defined by (22) by the hypothesis that \( \bar{u}_x \in L^n(\mathbb{R}) \) and the Riesz–Thorin interpolation theorem (cf. Remark 3).

We now define the modified characteristic

\[
\xi(t, y) = y + t \bar{u}(y) + \int_0^t (t - s) \psi(s, y) \, ds,
\]

so that the value of the solution \( u \) along this curve should be

\[
u(t, \xi(t, y)) = \bar{u}(y) + \int_0^t \psi(s, y) \, ds,
\]

in order that the characteristic equation \( \xi_t(t, y) = u(t, \xi(t, y)) \) hold true.
2.4. Global solutions

After these preparations, we can state and prove

**Theorem 4.** Given any absolutely continuous function $\bar{u} : \mathbb{R} \to \mathbb{R}$ such that its distributional derivative $\bar{u}_x \in L^n(\mathbb{R}) = L^2(\mathbb{R}) \cap L^n(\mathbb{R})$, formulae (22), (23), and (24) provide a global solution to (1).

**Proof.** To structure the proof, we proceed in four steps.

**First step.** Because $\psi_y(t,.) \in L^n$, the map $y \mapsto \xi(t, y)$ is absolutely continuous for any fixed time $t \geq 0$. We claim that, moreover, this map is nondecreasing on $\mathbb{R}$ with $\lim_{y \to \pm \infty} \xi(t, y) = \pm \infty$. Indeed, if $\bar{u}_x(y) > -\frac{n}{t}$, then $\bar{u}_x(y) > -\frac{n}{2}$ for all $s \in [0, t]$, so that in this case,

$$
\psi_y(s, y) = \sum_{k=1}^{n-1} \left( \begin{array}{c} n-1 \\ k \end{array} \right) \frac{k^{k-1}}{n^k} \bar{u}_x(y)^{k+1} \quad \forall s \in [0, t].
$$

Since

$$
\xi_y(t, y) = 1 + t \bar{u}_x(y) + \int_0^t (t-s) \psi_y(s, y) \, ds,
$$

we conclude

$$
\xi_y(t, y) = 1 + t \bar{u}_x(y) + \sum_{k=1}^{n-1} \left( \begin{array}{c} n-1 \\ k \end{array} \right) \frac{t^{k+1}}{(k+1)n^k} \bar{u}_x(y)^{k+1} = n^{-n} \left[ n + t \bar{u}_x(y) \right]^n. \quad (25)
$$

In the other case, that is, if $\bar{u}_x(y) \leq -\frac{n}{t}$, there exists a point in time $t_0$ such that

$$
\bar{u}_x = -\frac{n}{t_0} \quad \text{for some } t_0 \in (0, t]. \quad (26)
$$

Hence by (21),

$$
\psi_y(s, y) = \sum_{k=1}^{n-1} \left( \begin{array}{c} n-1 \\ k \end{array} \right) \frac{k^{k-1}}{n^k} \bar{u}_x(y)^{k+1}
$$

for all times $s \in [0, t_0)$, while $\psi_y(s, y)$ vanishes if $s \in [t_0, t]$. Then

$$
\xi_y(t, y) = 0, \quad (27)
$$

because

$$
\xi_y(t, y) = 1 + t \bar{u}_x(y) + \int_0^{t_0} (t-s) \psi_y(s, y) \, ds
$$

$$
= (t-t_0) \bar{u}_x(y) + \int_0^{t_0} (t-t_0) \psi_y(s, y) \, ds
$$
\[
+ 1 + t_0 \ddot{u}_x(y) + \int_{0}^{t_0} (t_0 - s) \psi_y(s, y) \, ds
\]

\[
\overset{(25)}{=} n^{-n} \left[ n + t_0 \ddot{u}_x(y) \right] + (t - t_0) \left[ \ddot{u}_x(y) + \int_{0}^{t_0} \psi_y(s, y) \, ds \right]
\]

\[
\overset{(26)}{=} \left( t - t_0 \right) \ddot{u}_x(y) \left[ 1 + \frac{t_0}{n} \ddot{u}_x(y) \right]^{n-1}
\]

\[
\overset{(26)}{=} 0.
\]

These relations confirm the monotonicity of \( \xi(t, \cdot) \) for all times. Next we proceed to the description of the limiting behavior of \( \xi(t, \cdot) \) at spatial infinity. Fix a time \( t > 0 \). The Lebesgue measure \( |l(t)| \) of the set \( l(t) := \{ y \in \mathbb{R}: \bar{u}_x(y) \leq \frac{1-n}{t} \} \) is bounded since \( \bar{u}_x \in L^n(\mathbb{R}) \). For any element \( y \) of the complement \( K(t) \setminus l(t) \), obviously \( \bar{u}_x(y) > \frac{1-n}{t} \), and thus \( \xi_y(t, y) \geq n^{-n} \) by (25). This implies that for two given points \( y_2 > y_1 \),

\[
\xi(t, y_2) - \xi(t, y_1) = \int_{y_1}^{y_2} \xi_y(t, y) \, dy \geq \int_{[y_1, y_2] \cap K(t)} \xi_y(t, y) \, dy \geq n^{-n} \int_{[y_1, y_2] \cap K(t)} dy \geq \frac{y_2 - y_1 - l(t)}{n^n},
\]

which demonstrates the desired limiting behavior of \( \xi(t, \cdot) \).

**Second step.** \( u(t, x) \) is well-defined by formula (24): As the map \( \xi(t, \cdot) \) is surjective and monotone, it is sufficient to demonstrate that if \( \xi(t, y_1) = \xi(t, y_2) \) for some \( y_2 > y_1 \), then the values of \( u \) defined by (24) must also be equal. In this case, as a matter of fact, \( \xi_y(t, y) = 0 \) for all \( y \in [y_1, y_2] \) by the monotonicity of \( \xi(t, \cdot) \), and this implies that \( \xi(t, y) = \xi(t, y_1) \) in the same space interval. Thereby the previous deliberations (cf. (25), (27)) assert that for every fixed \( y \in [y_1, y_2] \), there exists a time \( t(y) \in [0, t] \) with the property \( \bar{u}_x(y) = -\frac{n}{t(y)} \). This leads to

\[
\psi_y(s, y) = \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{k s^{k-1}}{n^k} \bar{u}_x(y)^{k+1} \cdot \chi(0, t(y))(s)
\]

for \( s \in [0, t] \), so that, by differentiation of the right-hand side of (24), we have

\[
\frac{\partial}{\partial x} \left[ \ddot{u}(y) + \int_{0}^{t} \psi(s, y) \, ds \right] = \ddot{u}_x(y) + \int_{0}^{t} \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{k s^{k-1}}{n^k} \bar{u}_x(y)^{k+1} \, ds
\]

\[
= -\frac{n}{t(y)} + \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{t(y)^k (-n)^{k+1}}{n^k (t(y))^{k+1}}
\]

\[
= -\frac{n}{t(y)} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} = 0
\]
for \( y \in (y_1, y_2) \), whence the right-hand side of (24) takes equal values when evaluated at \((t, y_1)\) and \((t, y_2)\). This proves that \( u \) is well-defined.

As for the continuity of the map \( y \mapsto \xi(t, y) \) for every fixed positive time, the arguments of [2] carry over to our case without alterations.

**Third step.** In this step, we establish that \( u_x(t, \cdot) \in L^\infty(\mathbb{R}) \). We first show that at every point \( z = \xi(t, y) \) where \( \xi_y(t, y) > 0 \), \( u_x(t, z) \in \mathbb{R} \). As a matter of fact, at such a point \( z \) the right-hand side of (24) is differentiable:

\[
\bar{u}_x(y) + \int_0^t \psi_y(s, y) \, ds = \bar{u}_x(y) \left[ 1 + \frac{t}{n} \bar{u}_x(y) \right]^{n-1}.
\]

This entails

\[
u_x(t, z) = \frac{\bar{u}_x(y)[1 + \frac{t}{n} \bar{u}_x(y)]^{n-1}}{\xi_y(t, y)} = \frac{\bar{u}_x(y)}{[1 + \frac{t}{n} \bar{u}_x(y)]}.
\]

Choosing points \( z_1 = \xi(t, y_1), z_2 = \xi(t, y_2) \) such that \( \xi_y(t, y) > 0 \) in \([y_1, y_2]\), we have

\[
\int_{z_1}^{z_2} u_x(t, z)^n \, dz = \int_{y_1}^{y_2} u_x(t, \xi(t, y))^n \xi_y(t, y) \, dy = \int_{y_1}^{y_2} \bar{u}_x(y)^n \, dy.
\]

Consequently, if we join all of these intervals \([y_i, y_{i+1}]\), we obtain

\[
\int_{\mathbb{R}} u_x(t, z)^n \, dz = \int_{\{\bar{u}_x(z) > -\frac{n}{t}\}} \bar{u}_x(y)^n \, dy.
\]

which is finite due to our assumption that \( \bar{u}_x \in L^\infty(\mathbb{R}) \). Note also that for intervals \((\eta_1, \eta_2) \subset \mathbb{R} \) with \( \eta_1 = \xi(t, y_1), \eta_2 = \xi(t, y_2) \) such that \( \xi(t, y) = \xi(t, y_1) \) for all \( x \in (x_1, x_2) \),

\[
\int_{\eta_1}^{\eta_2} u_x(t, \eta)^n \, d\eta = 0.
\]

If \( n \) is an even number, (29) reveals that \( u_x(t, \cdot) \in L^n(\mathbb{R}) \). On the other hand, if \( n \) is an odd number, then

\[
\int_{z_1}^{z_2} \left| u_x(t, z) \right|^n \, dz = \int_{[z_1, z_2] \cap \{u_x(t, z) \geq 0\}} u_x(t, z)^n \, dz - \int_{[z_1, z_2] \cap \{u_x(t, z) < 0\}} u_x(t, z)^n \, dz \\
\equiv (29) \int_{[y_1, y_2] \cap \{u_x(t, \xi(t, y)) \geq 0\}} \bar{u}_x(y)^n \, dy - \int_{[y_1, y_2] \cap \{u_x(t, \xi(t, y)) < 0\}} \bar{u}_x(y)^n \, dy
\]
$$
 \int_{y_1}^{y_2} |u_x(y)|^n dy + \int_{y_1}^{y_2} |u_x(y)|^n dy
 = \int_{y_1}^{y_2} |u_x(y)|^n dy.
$$

Hence if we unite these intervals \([y_i, y_{i+1}]\), we see that also for odd \(n\), \(u_x(t, .) \in L^n(\mathbb{R})\).

In addition, we have

$$
\int_{\mathbb{R}} u_x(t, z)^2 dz = \int_{\{u_x(y) < -\frac{n}{2}\}} \tilde{u}_x(y)^2 \left[ 1 + \frac{r}{n} \tilde{u}_x(y) \right]^{n-2} dy,
$$

and so, summing again,

$$
\int_{\mathbb{R}} u_x(t, z)^2 dz = \int_{\{u_x(y) > -\frac{n}{2}\}} \tilde{u}_x(y)^2 \left[ 1 + \frac{r}{n} \tilde{u}_x(y) \right]^{n-2} dy
 = \frac{n}{n-1} \int_{\{\tilde{u}_x(y) > -\frac{n}{2}\}} \psi_y(t, y) dy.
$$

As \(\tilde{u}_x \in L^n(\mathbb{R})\), we have, by Remark 3 (Riesz–Thorin interpolation theorem), that \(u_x(t, .) \in L^2(\mathbb{R})\), and so \(u_x(t, .) \in L^n(\mathbb{R})\).

**Fourth step.** Finally, we shall prove that (24) indeed defines a solution to (1). Note that

$$
\frac{n-1}{n} \int_{-\infty}^{\xi(t, y)} u_x(t, \zeta)^2 d\zeta = \int_{\{w \in (-\infty, y]: \tilde{u}_x(w) > -\frac{n}{2}\}} \psi_y(t, w) dw,
$$

and

$$
\frac{n-1}{n} \int_{\xi(t, y)}^{\infty} u_x(t, \zeta)^2 d\zeta = \int_{\{w \in [y, \infty): \tilde{u}_x(w) > -\frac{n}{2}\}} \psi_y(t, w) dw.
$$

By comparing the coefficients of (30) with those of (22), one sees that

$$
\psi(t, y) = \frac{n-1}{2n} \int_{\mathbb{R}} \text{sign}(\xi(t, y) - \zeta) u_x(t, \zeta)^2 d\zeta.
$$

Note that, due to \(n = \frac{-2}{a+1}\), \(n-1 = \frac{a+3}{4}\). If \(x = \xi(t, y)\) is such that \(\xi_y(t, y) > 0\), we get \((u_t + uu_x)(t, x) = \psi(t, y)\) by differentiating (24) with respect to the time variable \(t\) (using the chain rule and the characteristic equation). In combination with (31), this implies

$$
(u_t + uu_x)(t, \xi(t, y)) = \frac{a+3}{4} \int_{\mathbb{R}} \text{sign}(\xi(t, y) - \zeta) u_x(t, \zeta)^2 d\zeta.
$$
which after differentiating twice with respect to the space variable \( x \) gives exactly the generalized Proudman–Johnson equation (5) with parameter \( a = \frac{n+3}{n+1}, \ n \in \mathbb{N} \). This completes the proof of the theorem. \( \square \)

**Remark 5.** It is an interesting problem to prove global existence of solutions to the generalized Proudman–Johnson equation for arbitrary real-valued numbers \( \varrho = \frac{-2}{a+1} \). The main obstruction to our approach in this case is the fact that the Newton series \( (1+z)^\varrho = \sum_{k=0}^{\infty} \left( \frac{\varrho}{k} \right) z^k, \ \varrho \in \mathbb{R} \), converges only if \( |z| \leq 1 \). In view of the equation for the derivative of the characteristic (16), this restriction on the radius of convergence would amount to requiring

\[-\frac{n}{t} \leq \tilde{u}_x(y) \leq \frac{n}{t}\]

for all times \( t \in [0, T_n] \).

### 3. Blow-up proofs and rates

Now, consider the generalized Proudman–Johnson equation with periodic boundary conditions (3). Writing \( v = u_x \) (with the understanding that for the initial data, \( \tilde{u}_x = \tilde{v} \)), we have, because of periodicity,

\[v_t + u v_x - \frac{a+1}{2} v^2 = -\frac{a+3}{2} \int_0^1 v^2 \, dx. \tag{32}\]

The following lemma was proved in [30].

**Lemma 1.** Let \( v(t, x) \) be the solution of (32) with periodic boundary conditions. Then

\[
\frac{d}{dt} \int_0^1 v^2 \, dx = (a+2) \int_0^1 v^3 \, dx, \tag{33}
\]

and

\[
\frac{d}{dt} \int_0^1 v^3 \, dx = \frac{3a+5}{2} \int_0^1 v^4 \, dx - \frac{3(a+3)}{2} \left( \int_0^1 v^2 \, dx \right)^2. \tag{34}
\]

In a similar fashion, one can in fact show the more general

**Lemma 2.** It holds that

\[
\frac{d}{dt} \int_0^1 v^n \, dx = A_1 \int_0^1 v^{n+1} \, dx - A_2 \left( \int_0^1 v^2 \, dx \right) \left( \int_0^1 v^{n-1} \, dx \right), \tag{35}
\]

with \( A_1 = \frac{na + (n+2)}{2} \) and \( A_2 = \frac{n(a+3)}{2} \).
Now, we are in a position to prove the following blow-up result.

**Theorem 5.** Assume that \( a < -\frac{5}{2} \) and \( \int_0^1 \bar{v}^3(x) \, dx < 0 \), then the solution of (32) blows up in finite time in the following sense that there exists a finite \( T \) such that

\[
\int_0^1 v^3(t, x) \, dx \to -\infty \quad \text{as } t \to T.
\]

**Proof.** First, we consider the case that \( a < -3 \). As

\[
\left( \int_0^1 v^2 \, dx \right)^2 = \| v \|_{L^4}^4 \leq \| v \|_{L^4}^4 = \int_0^1 v^4 \, dx,
\]

one has

\[
\frac{d}{dt} \int_0^1 v^3 \, dx \leq \frac{3a + 5}{2} \int_0^1 v^4 \, dx - \frac{3a + 9}{2} \int_0^1 v^4 \, dx = -2 \int_0^1 v^4 \, dx.
\]

Setting \( V(t) = \int_0^1 v^3(t, x) \, dx \), then (36) reads as

\[
\frac{d}{dt} V(t) \leq -2(V(t))^{\frac{4}{3}}.
\]

Thus, it follows that

\[
\frac{1}{V(t)} \geq \left( \frac{2t}{3} + \frac{1}{(V(0))^{1/3}} \right)^3, \quad t \geq 0.
\]

By our assumption that \( \int_0^1 \bar{v}^3(x) \, dx < 0 \), it is easy to show that

\[
V(t) \to -\infty \quad \text{as } t \to \frac{-3}{2(V(0))^{1/3}}.
\]

Next, we consider the case that \(-3 \leq a < -\frac{5}{2}\). By (34), we have that

\[
\frac{d}{dt} \int_0^1 v^3 \, dx = \frac{3a + 5}{2} \int_0^1 v^4 \, dx - \frac{3(a + 3)}{2} \left( \int_0^1 v^2 \, dx \right)^2
\]

\[
\leq \frac{3a + 5}{2} \int_0^1 v^4 \, dx
\]

\[
\leq \frac{3a + 5}{2} \left( \int_0^1 v^3 \, dx \right)^{\frac{4}{3}}.
\]

Then, along the exact proof given above, we have \( V(t) \to -\infty \) in finite time. \( \square \)
In fact, we have the following result.

**Theorem 6.** Let \( n \geq 3 \) be a positive odd integer. Assume that \( a < -\frac{n+2}{n} \) and \( \int_0^1 \tilde{v}^n(x) \, dx < 0 \), then the solution to (3) blows up in finite time.

**Proof.** Since

\[
\|v\|_{L_{n+1}^1}^{n+1} = \|v\|_{L_{n+1}^1}^{n-1} \|v\|_{L_{n+1}^2}^{2} \geq \|v\|_{L_{n-1}^1}^{n-1} \|v\|_{L_{2}^2}^{2},
\]

we have, for \( a < -3 \),

\[
\frac{d}{dt} \int_0^1 v^n \, dx = \frac{na + (n+2)}{2} \int_0^1 v^{n+1} \, dx - \frac{n(a+3)}{2} \left( \int_0^1 v^2 \, dx \right) \left( \int_0^1 v^{n-1} \, dx \right)
\]

\[
\leq \frac{na + (n+2)}{2} \int_0^1 v^{n+1} \, dx - \frac{n(a+3)}{2} \left( \int_0^1 v^{n+1} \, dx \right)
\]

\[
= (1-n) \int_0^1 v^{n+1} \, dx \leq (1-n) \left( \int_0^1 v^n \, dx \right)^{\frac{n+1}{n}}.
\]

Next, we consider the case that \(-3 \leq a < -\frac{n+2}{n}\). By (32), we have

\[
\frac{d}{dt} \int_0^1 v^n \, dx = \frac{na + (n+2)}{2} \int_0^1 v^{n+1} \, dx - \frac{n(a+3)}{2} \left( \int_0^1 v^2 \, dx \right) \left( \int_0^1 v^{n-1} \, dx \right)
\]

\[
\leq \frac{na + (n+2)}{2} \left( \int_0^1 |v|^n \, dx \right)^{\frac{n+1}{n}}
\]

\[
\leq \frac{na + (n+2)}{2} \left( \int_0^1 v^n \, dx \right)^{\frac{n+1}{n}}.
\]

In either case, the solution blows up in the following sense:

\[
\int_0^1 v^n(t, x) \, dx \rightarrow -\infty \quad \text{as} \quad t \rightarrow T < \infty. \quad \Box
\]

**Corollary 7.** Let \( a \leq -3 \). Assume that there exists a positive integer \( q \) such that \( \int_0^1 \tilde{v}^{2q+1}(x) \, dx < 0 \), then the solution to (32) blows up in finite time.
3.1. An abstract lemma

An important tool for the following proofs of finite-time singularities and the computation of blow-up rates is the subsequent abstract lemma [9,10].

**Lemma 3.** Let $T > 0$ and $f_x \in C^1([0, T); H^1([0, 1]))$. Then for every $t \in [0, T)$, there exists at least one point $\xi(t) \in [0, 1]$ with

$$m(t) := \inf_{x \in [0,1]} \{ f_x(t, x) \},$$

and the function $m$ is almost everywhere differentiable on $(0, T)$ with

$$m'(t) = f_x(t, \xi(t)) \quad \text{a.e. on } (0, T).$$

Thus, by setting $m(t) := \inf_{x \in [0,1]} \{ \tilde{v}(t, x) \}$, (32) translates into

$$\frac{dm(t)}{dt} = a + \frac{1}{2} m^2(t) - a + \frac{1}{2} \int_0^1 v^2(t, x) \, dx.$$  \hfill (38)

Hence we can give another blow-up condition.

**Theorem 8.** (See [35,36].) Let $-3 \leq a < -1$. Assume that $\tilde{u} \in H^3([0, 1])$ and $m(0) = \min_{x \in [0,1]} \tilde{v}(x) < 0$. Then there exists a finite time $T$ such that $m(t) = \min_{x \in [0,1]} u_x(t, x) \to -\infty$ as $t \to T$.

3.2. Computing the blow-up rates

In order that the abstract lemma can be made use of, we shall henceforth assume that

- $\tilde{u} \in H^3([0, 1])$ and
- $\tilde{u}_{xx}$ is not a constant, which implies that $m(0) = \inf_{x \in [0,1]} \tilde{u}_x(x) < 0$.

**Theorem 9.** Assume that $a < -3$ and $k > -(a + 3)$ and that the initial datum satisfies

$$m^2(0) - k \int_0^1 \tilde{v}(x)^2 \, dx > 0.$$  \hfill (39)

Then $m(t)$ blows up in finite time $T$.

Before giving the proof of Theorem 9, we verify

**Lemma 4.** Under the assumptions of Theorem 9, we have

$$m^2(t) - k \int_0^1 v^2(t, x) \, dx \geq 0$$

for all $t$ as long as the solution exists.
Proof. If \( m^2(t) - k \int_0^1 v^2(t, x) \, dx > 0 \) for all \( t \leq +\infty \), then we are done. Alternatively, there exists a finite point in time \( t_0 \) such that

\[
m^2(t) - k \int_0^1 v^2(t, x) \, dx > 0 \quad \forall t < t_0 \quad \text{but} \quad m^2(t_0) - k \int_0^1 v^2(t_0, x) \, dx = 0.
\]

Since \( a < -3 \) and

\[
m(t) \int_0^1 v^2(t) \, dx \leq \int_0^1 v^3(t) \, dx,
\]

we have

\[
\frac{d}{dt} \left( m^2(t) - k \int_0^1 v^2(t, x) \, dx \right) = (a + 1)m^3(t) - (a + 3)m(t) \int_0^1 v^2(t) \, dx - k(a+2) \int_0^1 v^3(t, x) \, dx
\]

\[
\geq (a + 1)m^3(t) - \left[ (a + 3) + k(a+2) \right] m(t) \int_0^1 v^2(t, x) \, dx.
\]

Thus,

\[
\left. \frac{d}{dt} \left( m^2(t) - k \int_0^1 v^2(t, x) \, dx \right) \right|_{t=t_0}
\]

\[
\geq (a + 1)m^3(t_0) - \left[ (a + 3) + k(a+2) \right] m(t_0) \int_0^1 v^2(t_0, x) \, dx
\]

\[
= \left[ (a + 1)k - (a + 3) - k(a+2) \right] m(t_0) \int_0^1 v^2(t_0, x) \, dx
\]

\[
= [-(a + 3) - k] m(t_0) \int_0^1 v^2(t_0, x) \, dx.
\]

Since \(-k - (a + 3) < 0\) and \(m(t_0) < 0\), we have

\[
\frac{d}{dt} \left( m^2(t_0) - k \int_0^1 v^2(t_0, x) \, dx \right) > 0.
\]
which implies that there exists \( t_2 > t_1 \) such that

\[
m^2(t) - k \int_0^1 v^2(t, x) \, dx \geq 0 \quad \text{for all } t \leq t_2.
\]

Then, by reiterating the same argument, we obtain

\[
m^2(t) - k \int_0^1 v^2(t, x) \, dx \geq 0 \quad \text{for all } t,
\]

as long as the solution exists. \( \square \)

**Proof of Theorem 9.** Observe that

\[
\frac{dm(t)}{dt} = \frac{a + 1}{2} m^2(t) - \frac{a + 3}{2} \int_0^1 v^2(t, x) \, dx
\]

\[
\leq \left( \frac{a + 1}{2} - \frac{a + 3}{2k} \right) m^2(t).
\]

As \( \frac{a+1}{2} - \frac{a+3}{2k} < -1 + 1/2 = -1/2 \), that is,

\[
\frac{dm(t)}{dt} \leq -\frac{1}{2} m^2(t),
\]

we have \( m(t) \to -\infty \) as \( t \to T < \infty \). \( \square \)

### 3.3. Exact blow-up rates

In this subsection, we consider the function \( k(t) \) satisfying

\[
m^2(t) - k(t) \int_0^1 v^2(t, x) \, dx = 0. \tag{40}
\]

Then \( k(t) > 0 \) is differentiable a.e. and absolutely continuous in \((0, T)\). Moreover, for \( a < -2 \), since

\[
0 = \frac{d}{dt} \left( m^2(t) - k(t) \int_0^1 v^2(t, x) \, dx \right)
\]

\[
\geq (-k(t) - (a + 3))m(t) \int_0^1 v^2(t, x) \, dx - k'(t) \int_0^1 v^2(t, x) \, dx,
\]

we have

\[
k'(t) \geq - (k(t) + a + 3)m(t). \tag{41}
\]
Lemma 5. Let $-2 \leq a < -1$. Assume that $m(0) < 0$. Then we have $k(t) \to \infty$ as $t \to T$.

This is evident since $m^2(t) \to \infty$ as $t \to T$, while $\int_0^1 v^2(t, x) \, dx$ remains bounded (cf. [27]).

Lemma 6. Let $a < -2$. Assume that $m(0) < 0$. Then we have $k(t) \to \infty$ as $t \to T$.

Remark 6. We remark that in case (i), $k(t) + a + 3$ is positive for all times, while in case (ii), we make use of Lemma 4 to conclude that $k(t) > -(a + 3) \forall t \geq 0$.

To demonstrate Lemma 6, we need two more lemmata.

Lemma 7. Assume that $-3 \leq a < -2$. Then we have

$$m(t) \leq -\frac{2r}{-(a+1)r + (a+3)} \frac{1}{T-t} \text{ for some } r \in \mathbb{R}.$$  

Proof. There exists an $r > 0$ such that

$$m^2(0) - r \int_0^1 \bar{v}(x)^2 \, dx > 0.$$  

By Lemma 4, this implies

$$m^2(t) - r \int_0^1 v^2(t, x) \, dx \geq 0 \text{ for all } t \in [0, T).$$  

By (43) and (42), we have

$$-\frac{1}{m^2(t)} \frac{dm(t)}{dt} \leq -\frac{a+1}{2} + \frac{a+3}{} \frac{\int_0^1 v^2(t, x) \, dx}{m^2(t)}.$$  

Recalling that $m$ is absolutely continuous, an integration over $[t, T)$ tells that

$$m(t) \leq -\frac{2r}{-(a+1)r + (a+3)} \frac{1}{T-t}.$$  

Lemma 8. Assume that $a < -3$ and $k > -(a + 3)$ and that the initial datum satisfies (39). Then we have

$$m(t) \leq -\frac{2}{-(a+1)(T-t)}.$$
Proof. By (43), we have
\[- \frac{1}{m^2(t)} \frac{dm(t)}{dt} \leq - \frac{a + 1}{2}.\]
An integration over \([t, T]\) yields
\[m(t) \leq - \frac{2}{-(a + 1)(T - t)}.\]

Remark 7. We summarize Lemma 7 and Lemma 8 by concluding that in both of the cases (i) and (ii), we have
\[-m(t) \geq \frac{C}{T - t} \quad \text{for some positive constant } C. \quad (44)\]

Proof of Lemma 6. From (41), Remark 6 and (44), we have
\[\frac{k'(t)}{k(t) + a + 3} \geq -m(t) \geq \frac{C}{T - t}.\]
Integrating over \([0, t]\), we have
\[\log \frac{k(t) + a + 3}{k(0) + a + 3} \geq -C \log \frac{T - t}{T},\]
which tells us that
\[k(t) + a + 3 = (k(0) + a + 3) \left( \frac{T}{T - t} \right)^C \to \infty \quad \text{as } t \to T.\]

Now we are in a position to prove the following theorem.

Theorem 10. Let \(a < -1\). Assume that either
(i) \(-3 \leq a < -1\) or
(ii) \(a < -3\) and condition (39)
holds. Let \(T\) be the maximal existence time. Then we have
\[\lim_{t \to T} (T - t)m(t) = \frac{2}{a + 1}. \quad (45)\]

Proof. For \(a = -3\) (which is exactly the Burgers equation differentiated twice in space), the result is trivial since \(m(t)\) then satisfies
\[\frac{dm(t)}{dt} = \frac{a + 1}{2} m^2(t).\]
Thus, we consider only the case that \(a \in (-\infty, -1) \setminus \{-3\}.\) By (43) and (40), we have
\[- \frac{1}{m^2(t)} \frac{dm(t)}{dt} = - \frac{a + 1}{2} + \frac{a + 3}{2k(t)}. \quad (46)\]
Observe that \( k(t) \) is continuous in \([0, T)\) and absolutely continuous in \((0, T)\). Let \( \varepsilon \in (0, -\frac{a+1}{2}) \) be arbitrary. Since, by Lemmas 5 and 6, \( k(t) \to \infty \) as \( t \to T \), there exists \( t_0 \in [0, T) \) such that

\[
|m(t_0)| \geq \frac{a+3}{2\varepsilon} \max_{t \in [0, T]} \int_0^1 v^2(t, x) \, dx
\]

in the case \(-2 \leq a < -1\), and

\[
k(t_0) > \frac{|a+3|}{2\varepsilon}.
\]

Moreover, we have

\[
k(t) > \frac{|a+3|}{2\varepsilon} \quad \text{for all } t \in [t_0, T).
\]

Indeed, since \( k(t) \) is monotonously increasing for \( a < -2 \) by (41), and (38) shows that \(|m(t)|\) is monotonously increasing for \(-2 \leq a < -1\), (47) follows.

Now

\[
\frac{m'(t)}{m(t)^2} - \frac{a+1}{2} = -\frac{a+3}{2k(t)}
\]

by (38), so that an application of (47) in the case \( a < -3 \) yields

\[
0 \leq \frac{m'(t)}{m(t)^2} - \frac{a+1}{2} \leq \varepsilon.
\]

while in the case \(-1 > a > -3\) one gets

\[
0 \geq \frac{m'(t)}{m(t)^2} - \frac{a+1}{2}\geq -\varepsilon.
\]

Thus we have

\[
-\frac{a+1}{2} - \varepsilon \leq -\frac{1}{m^2} \frac{dm}{dt} \leq -\frac{a+1}{2} + \varepsilon \quad \text{on } (t_0, T).
\]

Observing that \( m \) is absolutely continuous in \([0, T)\), an integration of the inequality given above over the time interval \((t, T)\), \( t \in (t_0, T) \) yields

\[
\left(-\frac{a+1}{2} - \varepsilon\right)(T-t) \leq -\frac{1}{m(t)} \leq \left(-\frac{a+1}{2} + \varepsilon\right)(T-t) \quad \text{on } (t, T).
\]

Thus we have

\[
\frac{1}{-\frac{a+1}{2} + \varepsilon} \leq m(t)(T-t) \leq \frac{1}{-\frac{a+1}{2} - \varepsilon}.
\]

Since \( \varepsilon > 0 \) is arbitrary, this gives the desired result. \( \square \)
4. Discussion and conclusion

We have adapted a technique of Bressan and Constantin [2] to generate a class of global solutions to the generalized Proudman–Johnson equation (1) on the real line for parameter values $a = \frac{n+3}{n+1}, n \in \mathbb{N}$. This assumption makes it possible to express the solutions by means of finite binomial series, which is crucial to our approach using modified characteristics. The demonstration that there actually are global solutions for any real $a \in [-2, -1)$ remains an open problem. We believe that this can indeed be shown. A strong indication supporting this conjecture is that in this parameter range, the energy of the solutions with finite initial energy $\| \tilde{u}_x \|_{L^2(\mathbb{R})}$ remains bounded for all times.

Furthermore, we computed exact blow-up rates for the periodic generalized Proudman–Johnson equation (3) for all $a \in (-\infty, -1)$. Because solutions for $a \in [-1, 1)$ are known to be global (cf. Theorem 1 [27]), we thus get a rather complete picture of the asymptotic behavior of solutions to the generalized Proudman–Johnson equation with negative parameters. As for the range $[1, +\infty)$, there is still much room for scrutiny.

Acknowledgments

The authors wish to thank Professor Hisashi Okamoto for introducing them to the generalized Proudman–Johnson equation, and for his beneficial advice and constant encouragement. They are also grateful to the referee for his highly constructive comments on an earlier version of this manuscript.

References