A companion matrix resultant for Bernstein polynomials

Joab R. Winkler

Department of Computer Science, The University of Sheffield, Regent Court, 211 Portobello Street, Sheffield S1 4DP, UK

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Abstract

A closed form expression for a companion matrix $M$ of a Bernstein polynomial is obtained, and this is used to derive an expression for a resultant matrix of two Bernstein polynomials. It is shown that $M$ differs from its equivalent form for a power basis polynomial because an upper triangular Hankel matrix does not define a similarity transformation between $M$ and $M^T$. A measure of the numerical condition of a resultant matrix, for polynomials in an arbitrary basis, is reviewed and this is used to compare the stability of two resultant matrices. In particular, computational tests are performed and it is shown that the resultant matrix of two Bernstein polynomials is numerically better conditioned than the resultant matrix that is obtained when a simple parameter substitution is used to transform the polynomials to the power basis.

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1. Introduction

A resultant matrix of two polynomials is a matrix whose entries are functions of their coefficients, such that a necessary and sufficient condition for the polynomials to have a common root is that the determinant of this matrix, called the resultant of the polynomials, is exactly zero. Resultants are an established part of algebraic
geometry and they have applications in robotics [3] and computer-aided geometric design (CAGD) [7,15]. There has been a lot of research into their theoretical properties, but relatively little work that addresses the important issue of their implementation in a floating point environment. This is not a trivial problem because the degree of the greatest common divisor (GCD) of the polynomials is equal to the rank deficiency of the resultant matrix, but the rank of a matrix is not defined in a floating point environment. For example, the resultant of two power basis polynomials, one of degree 9 and the other of degree 19, is considered in [20] and even in the absence of noise, the incorrect numerical rank of the resultant matrix is obtained. Resultants must yield computationally reliable results in a floating point environment, and advanced methods from computational linear algebra are required because it is necessary to consider, for example, the proximity of a matrix to loss of rank and the perturbations in the matrix coefficients that cause this loss of rank. Ideally, this distance should be considered in the componentwise manner rather than the normwise manner because more refined estimates are obtained, but the determination of these componentwise measures is NP-hard [10, p. 140]. It is also necessary to consider the accuracy with which the GCD of the polynomials is computed.

In this paper, a companion matrix of a Bernstein polynomial is derived, and this is used to construct a resultant matrix for two Bernstein polynomials. It is recalled that the basis functions of the Bernstein basis are

\[ \phi_i(x) = \binom{n}{i}(1-x)^{n-i}x^i, \quad i = 0, \ldots, n. \]

This basis is chosen because it is the most stable basis in the interval \( I = [0, \ldots, 1] \), that is, the condition numbers of the roots, in this interval, of an arbitrary polynomial that is expressed in an arbitrary basis assume their minimum values when the polynomial is expressed in the Bernstein basis [5]. This theoretical result on the optimal representation of one polynomial, and the computational results on the resultant of two polynomials that are obtained in this paper, will provide strong evidence to suggest that the Bernstein basis is a numerically stable form in which to compute resultants in a floating point environment. Also, this basis is the preferred representation of curves and surfaces in CAGD, and thus there are compelling numerical and geometric reasons for the development of resultants for Bernstein polynomials. The work that is described in this paper is therefore an improvement on the work in [19,20], in which a resultant matrix for two scaled Bernstein basis polynomials is derived and tested computationally.

There are three commonly used methods—the Sylvester, Bézout and companion matrix—that are used to compute the resultant of two univariate polynomials. The Sylvester and Bézout forms are the most frequently used forms in CAGD, but the companion matrix resultant has been used to compute the intersection points and curves of, respectively, algebraic curves and surfaces [11,14], and the solutions of multipolynomial equations [12,13]. Preliminary computational results show that the Bernstein basis Bézout resultant matrix is numerically superior to its power basis
equivalent, and this observation is consistent with the results for the companion matrix resultant that are obtained in this paper. This work is therefore part of a more general investigation into resultants for Bernstein polynomials. For example, the Sylvester resultant matrix for these polynomials has been developed and its properties considered [22].

Reference is made in the paper to the scaled Bernstein basis, which is defined by the basis functions

\[ \phi_i(x) = (1 - x)^{n-i} x^i, \quad i = 0, \ldots, n, \]

and it is seen that a polynomial in the Bernstein basis may be transformed to the scaled Bernstein basis by moving the combinatorial factor from the basis functions to the coefficients,

\[ p(x) = \sum_{i=0}^{n} b_i \binom{n}{i} (1 - x)^{n-i} x^i = \sum_{i=0}^{n} \left( b_i \binom{n}{i} \right) (1 - x)^{n-i} x^i. \quad (1) \]

A review of previous work is considered in Section 2, and a companion matrix \( M \) of a Bernstein polynomial \( r(x) \) is derived in Section 3. It is shown in Section 4 that this enables a resultant matrix \( s(M) \) of \( r(x) \) and another Bernstein polynomial \( s(x) \) to be developed, and some theoretical properties of \( M \) are considered in Section 5. A normwise condition number of a resultant matrix is reviewed in Section 6, and this is used in Section 7 to compare the stability of a companion matrix resultant of two Bernstein polynomials, and the resultant matrix that is obtained when a simple parameter substitution is used to transform the polynomials to the power basis. These results are discussed in Section 8, and Section 9 contains the conclusions.

2. Previous work

It is nearly always assumed in the theoretical development of resultants that the polynomials are expressed in the power basis, but resultant matrices for other polynomial bases have been developed. For example, a Bézoutian matrix for two Chebyshev polynomials is derived in [2], and a companion matrix and resultant matrix for two scaled Bernstein polynomials are derived in [19] and tested computationally in [20].

A procedure that is widely used to compute the resultant of two Bernstein polynomials requires a simple parameter substitution [19], and this method reduces to the calculation of the resultant of two power basis polynomials. Specifically, it is readily verified that the parameter substitution

\[ t = \frac{x}{1 - x}, \quad x \neq 1, \quad (2) \]

transforms the Bernstein basis polynomial

\[ p(x) = \sum_{i=0}^{n} a_i \binom{n}{i} (1 - x)^{n-i} x^i. \]
to the power basis polynomial
\[ q(t) = (1 + t)^n p \left( \frac{t}{1 + t} \right) = \sum_{i=0}^{n} c_i t^i, \quad c_i = a_i \binom{n}{i}, \quad t \neq -1, \]
and thus if \( x_0 \) is a root of \( p(x) \), then \( t_0 = x_0/(1 - x_0) \) is a root of \( q(t) \). The coefficients \( c_i \) of the power basis polynomial \( q(t) \) are the scaled Bernstein coefficients of \( p(x) \). Although the parameter substitution (2) enables the established theory of resultants for power basis polynomials to be employed, it necessarily implies that one of the advantages of the Bernstein basis with respect to the power basis — its enhanced numerical stability — is lost, and thus this method cannot be recommended for the numerical computation of resultants. It follows that it is desirable to retain the Bernstein basis throughout the computations, and thus it is advantageous to construct a resultant matrix for two Bernstein polynomials, such that the power basis is not used.

Several researchers have addressed the issue of the computation of the GCD of two polynomials whose coefficients are floating point numbers. For example, Noda and Sasaki [16] consider the approximate GCD of these polynomials and extend the Euclidean algorithm to include this situation. Similarly, the quasi-GCD of two polynomials is introduced in [17] in order to consider the situation that arises when the coefficients of the polynomials are subject to error, and Euclid’s algorithm is modified to include this scenario. Sederberg and Chang [18] consider the minimum polynomials \( \varepsilon_1(t) \) and \( \varepsilon_2(t) \) that must be added to, respectively, the polynomials \( f(t) \) and \( g(t) \), such that the perturbed polynomials \( f(t) + \varepsilon_1(t) \) and \( g(t) + \varepsilon_2(t) \) have a common divisor. These three papers only consider power basis polynomials, but it is shown in this paper that the numerical superiority of the Bernstein basis suggests that all computations of the GCD in a floating point environment should be performed in this basis, and not in the power basis.

It is assumed in [16–18] that the coefficients of the polynomials are defined within a tolerance, such that the conventional definition of the GCD of two (or more) polynomials is not valid because its uniqueness (up to an arbitrary scalar multiplier) is not guaranteed. By contrast, it is assumed in this paper that the coefficients of the polynomials are not subject to uncertainty, and interest is restricted to studying the effect of roundoff error due to floating point arithmetic on the computation of the GCD. It is therefore assumed that the GCD is unique, to within an arbitrary scalar multiplier, and it is shown that the polynomial basis has a critical effect on the accuracy of the computation of the degree of the GCD.

The difficulties of computing resultants in a floating point environment arise from the requirement that the determinant of a resultant matrix be exactly zero for the polynomials to have a non-constant divisor; since the coefficients of a polynomial may be multiplied by an arbitrary non-zero constant without changing its roots, the determinant of the resultant matrix may be scaled arbitrarily, and thus a non-zero determinant does not yield any information on the proximity of the roots of the polynomials. This observation will be important in Section 6, where the development of a condition number of a resultant matrix is reviewed.
3. A companion matrix of a Bernstein polynomial

A companion matrix $C_p$ of a polynomial $p(\lambda)$, expressed in an arbitrary basis $\phi(\lambda) = \{\phi_i(\lambda)\}_{i=0}^{n}$, is defined by

$$p(\lambda) = \det (C_p - \lambda I) = \sum_{i=0}^{n} a_i \phi_i(\lambda),$$

where the structure of $C_p$ is defined for each basis. The matrix $C_p$ takes on a particularly simple form for the power basis, but it is shown in Section 3.1 that it assumes a more complex form for the Bernstein basis. It is clear that the eigenvalues of $C_p$ are identically equal to the roots of $p(\lambda)$. Numerical aspects of the determination of the roots of $p(\lambda)$, in its power basis form, by computing the eigenvalues of $C_p$ are in [4].

An expression for a companion matrix $M$ of a Bernstein polynomial is developed in Section 3.1 and some computational aspects are considered in Section 3.2. The matrix $M$ is used in Section 4 to construct a resultant matrix for two Bernstein polynomials. The development is similar to that of a companion matrix of a scaled Bernstein basis polynomial [19], but it is shown that the combinatorial factors introduce complications. In particular, the expression for $M$ includes a diagonal matrix $F$ (which reduces to $I$ for the scaled Bernstein basis), and it will be shown in Section 5 that $F \neq I$ has an implication for an important theoretical property of $M$.

3.1. Theoretical development

Consider the square matrices $A$ and $E$, both of order $n$,

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
-b_0 & -b_1 & -b_2 & -b_3 & \cdots & -b_{n-2} & -b_{n-1}
\end{bmatrix}, \quad (3)$$

and

$$E = \begin{bmatrix}
{\binom{\lambda}{0}} & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & {\binom{\lambda}{1}} & 1 & 0 & \cdots & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & {\binom{\lambda}{n-2}} & 1 \\
-b_0 & -b_1 & -b_2 & -b_3 & \cdots & -b_{n-2} & -b_{n-1} + {\binom{\lambda}{n-1}}
\end{bmatrix}. \quad (4)$$

It follows from (3) and (4) that

$$E = F + A,$$
\[
F = \text{diag} \left[ \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-3}, \binom{n}{n-2}, \binom{n}{n-1} \right], \quad (5)
\]

and that \( A - \lambda E \) is equal to
\[
\begin{bmatrix}
-\lambda \binom{n}{0} & \delta & 0 & \cdots & 0 & 0 \\
0 & -\lambda \binom{n}{1} & \delta & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\lambda \binom{n}{n-1} & \delta \\
-b_0\delta & -b_1\delta & -b_2\delta & \cdots & -b_{n-2}\delta & -b_{n-1}\delta - \lambda \binom{n}{n-1} \\
\end{bmatrix}, \quad (6)
\]

where \( \delta = 1 - \lambda \). Comparison of (5) with the definitions of the matrices \( A \) and \( E \) in [19] shows that a companion matrix of the scaled Bernstein basis is obtained by replacing \( F \) by the identity matrix and redefining the coefficient \( b_i \) to include the combinatorial factor \( \binom{n}{i} \), as shown in (1).

It is necessary to calculate the characteristic polynomial of \( A - \lambda E \) in order to establish that
\[
\det(A - \lambda E) = (-1)^n \sum_{i=0}^{n} b_i \binom{n}{i} (1 - \lambda)^{n-i} \lambda^i, \quad b_n = 1. \quad (7)
\]

This result is obtained by considering the determinant of (6). Let \( D_0 = \det(A - \lambda E) \) and let \( D_i \) be the determinant of the matrix that is formed when rows \( k = 1, \ldots, i \), and columns \( k = 1, \ldots, i \), are deleted from \( A - \lambda E \). It follows from (6) that
\[
D_0 = (-1)^n b_0 (1 - \lambda)^n - \lambda n D_1, \quad (8)
\]

and
\[
D_1 = -(-1)^n b_1 (1 - \lambda)^{n-1} - \lambda \binom{n}{2} D_2. \quad (9)
\]
The substitution of (9) into (8) yields
\[
D_0 = (-1)^n \left[ b_0 (1 - \lambda)^n + b_1 \binom{n}{1} (1 - \lambda)^{n-1} \lambda \right] + \lambda^2 \binom{n}{2} D_2. \quad (10)
\]

Similarly, it follows from (6) that
\[
D_2 = (-1)^n b_2 (1 - \lambda)^{n-2} - \lambda \binom{n}{3} D_3, \quad (11)
\]

and the substitution of this equation into (10) yields
\[
D_0 = (-1)^n \sum_{i=0}^{2} b_i \binom{n}{i} (1 - \lambda)^{n-i} \lambda^i - \lambda^3 \binom{n}{3} D_3.
\]

The formulae (8), (9) and (11) that relate the determinants \( D_i \) and \( D_{i+1} \) are of the form of the method of Horner for the nested multiplication of polynomials. The process is continued until
\[ D_0 = (-1)^n \sum_{i=0}^{n-2} b_i \binom{n}{i} (1 - \lambda)^{n-i} \lambda^i + (-1)^{n-1} \lambda^{n-1} \binom{n}{n-1} D_{n-1}, \quad (12) \]

where
\[ D_{n-1} = -b_{n-1} (1 - \lambda) - \lambda \frac{n}{(n-1)}. \quad (13) \]
is obtained. The substitution of (13) into (12) yields
\[ D_0 = (-1)^n \sum_{i=0}^{n} b_i \binom{n}{i} (1 - \lambda)^{n-i} \lambda^i, \quad b_n = 1, \]
and thus (7) is established. It follows that the Bernstein polynomial
\[ p(\lambda) = (-1)^n \sum_{i=0}^{n} b_i \binom{n}{i} (1 - \lambda)^{n-i} \lambda^i, \quad b_n = 1, \quad (14) \]
is the characteristic polynomial of the pair of matrices \((A, E)\) if \(x_0 = 1\) is not a root of this polynomial. If, however, the polynomial is such that \(b_n = 0\), then a polynomial of degree \((n - 1)\) is considered by removing the factor \((1 - \lambda)\).

It follows from (7) that
\[ \det(-\lambda E) = \lim_{\lambda \to \infty} \det(A - \lambda E) = \lim_{\lambda \to \infty} (-1)^n \sum_{i=0}^{n} b_i \binom{n}{i} (1 - \lambda)^{n-i} \lambda^i, \]
and thus
\[ \det E = \sum_{i=0}^{n} (-1)^{n-i} b_i \binom{n}{i}, \quad b_n = 1. \quad (15) \]
The companion matrix \(M\) of \(p(\lambda)\) is given by \(E^{-1} A\), and a closed form expression for \(E^{-1}\) is obtained by using the Sherman–Morrison formula [8]. This formula requires that \(E\) be written in the form
\[ E = C - e_n b^T, \]
where
\[ C = \begin{bmatrix} \binom{n}{0} & 1 & 0 & 0 & \cdots & 0 & 0 \\ \binom{n}{1} & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{i} & 1 \\ 0 & 0 & 0 & \cdots & 0 & \binom{n}{n-i} \end{bmatrix}, \quad b = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix}, \]
and $e_n$ is the $n$th standard basis vector. The Sherman–Morrison formula states that the inverse of $E$ is given by

$$E^{-1} = C^{-1} + \frac{1}{\tau} C^{-1} e_n b^T C^{-1}, \quad \tau = 1 - b^T C^{-1} e_n,$$

(16)

where the elements $c_{ij}$ of $C$ are

$$c_{ij} = \begin{cases} \binom{n}{i} & \text{if } i = j, \\ \binom{n-1}{i-1} & \text{if } i + 1 = j, \\ 0 & \text{otherwise}. \end{cases} \quad (17)$$

It is required to derive an expression for the elements $d_{ij}$ of $D = C^{-1}$, and this is established in the following theorem.

**Theorem 3.1.** The elements $d_{ij}$ of $D = C^{-1}$ are given by

$$d_{i,k+i} = \begin{cases} (-1)^{k-j} \prod_{m=j}^{k} c_{mm} & 1 \leq i \leq n, \ 0 \leq k \leq n-i, \\ 0 & \text{otherwise}. \end{cases} \quad (18)$$

**Proof.** By definition, the elements of $D$ satisfy

$$\sum_{j=i}^{i+1} c_{ij} d_{jk} = \delta_{ik}. \quad (19)$$

It follows from (18) that the elements of $D$ on and above the diagonal are given by

$$d_{jk} = \frac{(-1)^{k-j}}{\prod_{m=j}^{k} c_{mm}}, \quad 1 \leq j \leq n, \ j \leq k \leq n, \quad (20)$$

and thus

$$\sum_{j=i}^{i+1} c_{ij} d_{jk} = \sum_{j=i}^{i+1} c_{ij} \frac{(-1)^{k-j}}{\prod_{m=j}^{k} c_{mm}}, \quad 1 \leq j \leq n, \ j \leq k \leq n.$$

Consider the situations $k = i$ and $k \neq i$.

If $k = i$,

$$\sum_{j=i}^{i+1} c_{ij} d_{ji} = \sum_{j=i}^{i+1} c_{ij} \frac{(-1)^{i-j}}{\prod_{m=j}^{i} c_{mm}} = \sum_{j=i}^{i} c_{ij} \frac{(-1)^{i-j}}{\prod_{m=j}^{i} c_{mm}} = 1,$$

since it follows from (20) that $j \leq k$ implies $j \leq i$, and thus only the term corresponding to $j = i$ is considered in the summation.
If \( k \neq i \),
\[
\sum_{j=i}^{i+1} c_{ij} d_{jk} = \sum_{j=i}^{i+1} \frac{(-1)^{k-j} c_{ij}}{\prod_{m=j}^{i} c_{mm}}
\]
\[
= \frac{(-1)^{k-i} c_{ii}}{\prod_{m=i}^{i} c_{mm}} + \frac{(-1)^{k-i-1} c_{i,i+1}}{\prod_{m=i+1}^{i} c_{mm}}
\]
\[
= \frac{(-1)^{k-i} c_{ii}}{\prod_{m=i}^{i} c_{mm}} + \frac{(-1)^{k-i-1} c_{i,i+1}}{\prod_{m=i+1}^{i} c_{mm}} \quad \text{since} \quad c_{i,i+1} = 1
\]
\[
= 0,
\]
and thus (19) is satisfied, and hence (18) is established. \(\square\)

This result enables the expression (16) for \( E^{-1} \) to be evaluated. Consider first the term \( C^{-1} e_n = D e_n \), the \( n \)th column of \( D \). It follows from (17) that
\[
\prod_{m=1}^{n} c_{mm} = 1,
\]
and thus
\[
d_{in} = \frac{(-1)^{n-i} \prod_{m=1}^{n} c_{mm}}{\prod_{m=i}^{n} c_{mm}} = (-1)^{n-i} \prod_{m=1}^{n} c_{mm} = (-1)^{n-i} \prod_{m=1}^{n} \left(\frac{n}{m-1}\right)
\]
\[
= (-1)^{n-i} \left(\begin{array}{c}
n \\ i-1\end{array}\right).
\]

This enables a simple expression for \( \tau \), which is defined in (16), to be developed. In particular,
\[
\tau = 1 - b^T C^{-1} e_n
\]
\[
= 1 - \left[ b_0 \quad b_1 \quad \cdots \quad b_{n-1} \right] \left[ \begin{array}{c}
(-1)^{n-1} \binom{n}{0} \\
(-1)^{n-2} \binom{n}{1} \\
\vdots \\
(-1)^{n-n} \binom{n}{n-1}
\end{array}\right]
\]
\[
= 1 - \sum_{i=0}^{n-1} (-1)^{n-1-i} b_i \binom{n}{i}
\]
\[
= b_n + \sum_{i=0}^{n-1} (-1)^{n-i} b_i \binom{n}{i} \quad \text{since} \quad b_n = 1
\]
\[
= \sum_{i=0}^{n} (-1)^{n-i} b_i \binom{n}{i}, \quad b_n = 1.
\]
and (15) shows that $\tau = \det E$. It follows from (16) that

$$M = E^{-1}A = (F + A)^{-1}A = \left( I + \frac{De_n b^T}{\tau} \right) DA,$$

(21)

is a companion matrix of the polynomial (14), that is, the roots of this polynomial are identically equal to the eigenvalues of $M$.

3.2. Computational implementation

It is clear from (21) that the computational cost of constructing $M$ is higher than the cost of constructing its power basis equivalent. Since closed form expressions for the elements of $A$, $D$, $b$ and $e_n$ have been developed, the cost of constructing $M$ can be reduced. In particular, $D$ is an upper triangular matrix, and

$$\frac{e_n b^T}{\tau} = \frac{1}{\tau} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix}.$$  

It follows, therefore, that the elements of the matrix $I + (1/\tau)De_n b^T$ can be explicitly defined. Similarly, the matrix $DA$ is the product of an upper triangular matrix and a companion matrix in power basis form, and the elements of this product can also be defined. The computation of $M$ can therefore be reduced to two matrix multiplications, and the addition of the identity matrix.

4. A resultant matrix

A companion matrix of a scaled Bernstein polynomial $f(x)$ is developed in [19], and it is shown in Section 3 of this reference that it may be used to construct a resultant matrix for $f(x)$ and another scaled Bernstein polynomial. The same method is used to construct a resultant matrix for two Bernstein polynomials, and the result is stated in the following theorem.

**Theorem 4.1.** Let $r(x)$ and $s(x)$ be two Bernstein polynomials with coefficients $\{r_j\}_{j=0}^n$, $r_n = 1$, and $\{s_j\}_{j=0}^m$, respectively,

$$r(x) = \sum_{j=0}^n r_j\binom{n}{j}(1-x)^{n-j}x^j,$$

$$s(x) = \sum_{j=0}^m s_j\binom{m}{j}(1-x)^{m-j}x^j.$$

(22)

If $M$ is the companion matrix of the polynomial $r(x)$ and the eigenvalues of $M$ are $\{\lambda_i\}_{i=1}^n$, then
\[ \det(s(M)) = \prod_{i=1}^{n} s(\lambda_i), \]

and thus the determinant of \( s(M) \) is equal to zero if and only if \( \lambda_i \) is a root of \( s(x) \). Since the eigenvalues \( \{\lambda_i\}_{i=1}^{n} \) are the roots of \( r(x) \), it follows that \( s(M) \) is a resultant matrix for the polynomials \( r(x) \) and \( s(x) \).

The degree and coefficients of the GCD of \( r(x) \) and \( s(x) \) are obtained from \( s(M) \) using the following theorem [7,15].

**Theorem 4.2.** Let \( w(x) \) be the GCD of \( s(x) \) and \( r(x) \). Then

1. The degree of \( w(x) \) is equal to \( n - \text{rank}(s(M)) \).
2. The coefficients of \( w(x) \) are proportional to the last row of \( s(M) \) after it has been reduced to row echelon form.

These results are illustrated by a simple example, and it will be seen that a factor of the form \((1 - x)x^q\) must be deleted in order to determine the GCD. This factor arises because \( M \), which is of order \( n \times n \), is a companion matrix of a polynomial of order \( n \). This polynomial is defined by \((n+1)\) basis functions, and the coefficient \( b_n \) of \( x^n \) does not occur in \( M \) but arises from the term \(-\lambda I\) in the expression \( M - \lambda I \). It follows that \( M \) contains only the coefficients \( \{b_i\}_{i=0}^{n-1} \) of the basis functions \( \{(\frac{n}{i})(1-x)^{n-i}x^i\}_{i=0}^{n-1} \), and thus a factor of the form \((1 - x)x^q\) arises in the GCD.

**Example 4.1.** Consider the Bernstein polynomials

\[ r(x) = 3(1 - x)^3 - \frac{5}{6}(3(1 - x)^2x) - \frac{1}{2}(3(1 - x)x^2) + x^3, \]

and

\[ s(x) = 2(1 - x)^2 - \frac{3}{2}(2(1 - x)x) + x^2, \]

where \( r(\frac{1}{2}) = r(\frac{2}{3}) = r(3) = 0 \) and \( s(\frac{1}{2}) = s(\frac{2}{3}) = 0 \). The GCD of \( r(x) \) and \( s(x) \) is \( s(x) \).

The companion matrix of \( r(x) \) is

\[ M = -\frac{1}{3} \begin{bmatrix} -3 & \frac{5}{6} & -\frac{1}{2} \\ 9 & -\frac{11}{2} & 1 \\ -9 & \frac{17}{2} & -4 \end{bmatrix}, \]

and thus

\[ s(M) = 2(I - M)^2 - \frac{3}{2} (2(I - M) M) + M^2 = \begin{bmatrix} 8 & -4 & \frac{4}{3} \\ -36 & 18 & -6 \\ 54 & -27 & 9 \end{bmatrix}. \]
The reduction of \( s(M) \) to row echelon form yields
\[
\begin{bmatrix}
6 & -3 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]
and since the rank of this matrix is 1, the degree of the GCD of \( r(x) \) and \( s(x) \) is \( 3 - 1 = 2 \). The GCD is calculated from the first row of this matrix,
\[
6(1 - x)^3 - 3(3(1 - x)^2 x) + 1(3(1 - x)x^2)
= 3(1 - x)\left[2(1 - x)^2 - \frac{3}{2} (2(1 - x)x) + x^2\right].
\]
The factor \((1 - x)\) is ignored, and thus the GCD is proportional to \( s(x) \).

Consider now the matrix polynomial \( r(N) \) where \( N \) is the companion matrix of \( s(x) \),
\[
N = \begin{bmatrix}
\frac{1}{3} & \frac{17}{3} \\
-\frac{2}{3} & \frac{5}{6}
\end{bmatrix},
\]
and hence
\[
r(N) = 3(I - N)^3 - \frac{5}{6} (3(I - N)^2 N) - \frac{1}{2} (3(I - N)N^2) + N^3 = 0.
\]
Since the rank of this matrix is zero, the degree of the GCD of \( r(x) \) and \( s(x) \) is \( 2 - 0 = 2 \). Furthermore, since \( s(x) \) is of degree 2, the GCD is proportional to \( s(x) \).

It is readily verified that the polynomial \( s(M) \) can be constructed by the de Casteljau algorithm, and thus the computational cost of constructing \( s(M) \) is significantly higher than the cost of constructing the Bézout resultant matrix. The companion matrix resultant is, however, smaller than the Bézout resultant matrix. It must be recalled that degree elevation is required for the construction of the Bézout resultant matrix if the polynomials are of different degrees, but this procedure is not required for the companion matrix resultant. Also, if the polynomials have a non-constant common divisor, then it is cheaper to compute the GCD from the companion matrix resultant than it is from the Bézout resultant matrix because it is smaller.

5. Properties of the companion matrix

It is shown in [1,19] that a companion matrix and its transpose for, respectively, the power and scaled Bernstein bases satisfy the similarity transformation
\[
TN = N^TT,
\]
where \( N \) is a companion matrix in the specified basis, and \( T \) is the upper triangular Hankel matrix whose entries \( \{t_i\}_{i=1}^n, \ t_n = 1, \) are the coefficients in this basis,
It is shown that this similarity transformation is not shared by the companion matrix of a Bernstein polynomial, and that this is due to the diagonal matrix $F$, defined in (5), of binomial coefficients. It is interesting to note that the Bernstein basis Sylvester resultant matrix does not share the striped pattern of its power basis equivalent, and this is also due to a diagonal matrix of binomial coefficients [22].

It follows from (21) that

$$M^{-1} = I + A^{-1}F$$

where $A$ satisfies (23) since it is in the form of a companion matrix of a power basis polynomial. It follows from (23) that $A^{-1} = T^{-1}A^{-T}T$ and thus

$$M^{-1} = I + T^{-1}A^{-T}TF.$$

It also follows from (21) that

$$A^{-T} = F^{-1}(M^{-1} - I)^T,$$

and hence

$$M^{-1} = I + T^{-1}F^{-1}(M^{-1} - I)^TTF,$$

or

$$(FT)(M^{-1} - I) = (M^{-1} - I)^T(TF) = (M^{-1} - I)^T(FT)^T,$$

since $F$ and $T$ are symmetric. This equation reveals the effect of the diagonal matrix $F$ of binomial coefficients; if $F = I$, then (24) reduces to (23) and $M$ is now the companion matrix of the scaled Bernstein basis, and thus the coefficients $\{b_i(\binom{n}{i})\}_{i=0}^{n}$ are replaced by $\{b_i(\binom{n}{i})\}_{i=0}^{n}$, as shown in (1). It is noted that (24) is not a similarity transformation between $(M^{-1} - I)$ and $(M^{-1} - I)^T$ because $FT \neq (FT)^T$.

It is easy to show that if $V$ is the matrix of eigenvectors of $M$ and $A$ is the diagonal matrix of eigenvalues (assumed to be distinct) of $M$, then

$$V^{-1}MV = A \quad \text{and} \quad (VV^T)^{-1}M(VV^T)^{-1} = M^T,$$

and thus $VV^T$ defines a similarity transformation between $M$ and $M^T$. This similarity transformation is also satisfied by a power basis companion matrix and its transpose [1], and a companion matrix and its transpose of a polynomial in the scaled Bernstein basis [19].

6. The numerical condition of a resultant matrix

A comparison of the numerical stability of two resultant matrices requires that a condition number of a resultant matrix be defined. Since the degree of the GCD of
the two polynomials is defined by the rank deficiency of their resultant matrix, an appropriate measure of the condition is the reciprocal of the distance to the nearest matrix that has unit loss of rank. It was noted in Section 1 that the componentwise distance to singularity is NP-hard, but a less refined measure, the normwise measure, is easily computed [20] because it is based on the following theorem [8].

Theorem 6.1. Let \( \mathbf{X} \in \mathbb{R}^{n \times n} \) be of rank \( r \), and let \( \mathbf{USV}^T \) be its singular value decomposition. Let \( \mathbf{X}_k = \mathbf{US}_k\mathbf{V}^T \) where \( k < r \), and let \( \mathbf{S}_k \in \mathbb{R}^{n \times n} \) be the diagonal matrix with elements

\[
(\sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_k \quad 0 \quad 0 \quad \cdots \quad 0), \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0.
\]

Then the rank of \( \mathbf{X}_k \) is \( k \) and

\[
\sigma_{k+1} = \| \mathbf{X} - \mathbf{X}_k \|_2 = \min_{\text{rank}(\mathbf{Y}) = k} \| \mathbf{X} - \mathbf{Y} \|_2.
\]

This theorem states that \( \sigma_{k+1} \) is the minimum normwise distance between a given matrix \( \mathbf{X} \) of rank \( r > k \) and the set of all matrices of rank \( k \), and that this minimum distance occurs for the matrix \( \mathbf{X}_k \). Since a unit loss of rank is required for quantifying the numerical condition of a resultant matrix, it follows that \( k = r - 1 \), and thus the minimum normwise perturbation of \( s(\mathbf{M}) \) that is required to cause this loss of rank is

\[
\sigma_r = \left\| s(\mathbf{M})^+ \right\|^{-1}_2,
\]

where

\[
s(\mathbf{M})^+ = \mathbf{V} \operatorname{diag}(\sigma_1^{-1} \quad \sigma_2^{-1} \quad \cdots \quad \sigma_r^{-1} \quad 0 \quad 0 \quad \cdots \quad 0)\mathbf{U}^T,
\]

is the Moore–Penrose inverse of \( s(\mathbf{M}) \) [8].

It is shown in [20] that \( \sigma_r \) is not an adequate measure of the distance to singularity of a resultant matrix because a normalising constraint is not imposed on \( s(x) \), and thus the singular values of \( s(\mathbf{M}) \) can be made arbitrarily small or large by scaling the coefficients of \( s(x) \). It is noted that the polynomial \( r(x) \) is normalised since \( r_n = 1 \). This arbitrary scaling of \( s(x) \) and therefore \( s(\mathbf{M}) \), which implies that the distance to loss of unit rank of \( s(\mathbf{M}) \) can be made arbitrarily small or large, can be removed by normalising by \( \sigma_1 \), from which it follows that the normalised distance to singularity of \( s(\mathbf{M}) \) is

\[
d(s(\mathbf{M})) = \frac{\left\| s(\mathbf{M})^+ \right\|^{-1}_2}{\| s(\mathbf{M}) \|_2},
\]

which may be defined as the reciprocal of the condition number of a matrix of rank \( r \).

Let \( \tilde{s}(x) \) and \( \tilde{r}(x) \) be the polynomials that are obtained when the parameter substitution (2) is used to transform (in a weak sense) the polynomials (22) to the power basis.
\[
\tilde{r}(x) = \sum_{j=0}^{n} \tilde{r}_j x^j, \quad \tilde{r}_j = r_j \binom{n}{j},
\]

and
\[
\tilde{s}(x) = \sum_{i=0}^{m} \tilde{s}_i x^i, \quad \tilde{s}_i = s_i \binom{m}{i}.
\]

If \( P \) is a companion matrix of \( \tilde{r}(x) \), then
\[
\tilde{s}(P) = \sum_{i=0}^{m} \tilde{s}_i P^i,
\]

is a resultant matrix for \( \tilde{s}(x) \) and \( \tilde{r}(x) \), and it follows from (26) and its equivalent for \( \tilde{s}(P) \) that the ratio of the distance to singularity of \( \tilde{s}(P) \) to the distance to singularity of \( s(M) \) is
\[
d(\tilde{s}(P), s(M)) = \frac{\|\tilde{s}(P)^+\|_2^{-1}}{\|\tilde{s}(P)\|_2 \|s(M)^+\|_2} = \frac{\|s(M)\|_2 \|s(M)^+\|_2}{\|\tilde{s}(P)\|_2 \|\tilde{s}(P)^+\|_2}. \tag{27}
\]

This measure is used in Section 7 to compare the distance to singularity of \( \tilde{s}(P) \) and \( s(M) \). The combination of these computational experiments and the theoretical result in [5] will show that the Bernstein basis is a numerically stable representation in which to compute resultants.

It is shown in [22] that (27) cannot be used for the Sylvester resultant matrix \( S(u, v) \) of the polynomials \( u(x) \) and \( v(x) \), of degrees \( m \) and \( n \) respectively, because this measure is not scale invariant, that is, it can be made arbitrarily small or large by scaling the coefficients of \( u(x) \) and/or \( v(x) \). This property arises because the polynomials in \( S(u, v) \) are decoupled; the first \( n \) rows of this matrix are occupied by the coefficients of \( u(x) \), and the last \( m \) rows are occupied by the coefficients of \( v(x) \). Thus, despite the popularity of the Sylvester resultant matrix, it possesses a significant theoretical problem.

7. Examples

This section contains several examples that illustrate that the resultant matrix of two Bernstein polynomials is numerically superior to the resultant matrix that is obtained when the parameter substitution (2) is used to transform the polynomials \( r(x) \) and \( s(x) \) to a power basis form, and then using the resultant matrix that is based on a companion matrix of a polynomial in this basis. The use of the Bernstein basis necessarily implies that interest is restricted to the interval \( I \), and thus at least one of the roots is chosen to lie in this interval. A similar choice is made in [6], where the numerical stability of the Bernstein basis is considered and computational experiments are performed on two polynomials, all of whose roots lie in \( I \).
Example 7.1. Consider the truncated Wilkinson polynomial whose upper index is 19 and not 20 because, as noted in Sections 2 and 3.1, the root $x_0 = 1$ must be excluded,
\[ r(x) = \prod_{i=1}^{19} \left( x - \frac{i}{20} \right) = \sum_{i=0}^{19} r_i \binom{19}{i} (1-x)^{19-i}x^i. \] (28)

The matrices $s(M)$ and $\tilde{s}(P)$ were computed for several polynomials $s(x)$, and the ratio of the distances to singularity $d(\tilde{s}(P), s(M))$ was calculated. The results of the numerical experiments are shown in Table 1 and it is seen that $s(M)$ is further away from singularity than $\tilde{s}(P)$ in all the tests, and it is therefore better conditioned. Furthermore, the improvement in the numerical condition is several orders of magnitude, even for polynomials with high degree multiple roots. It is noted that these results for the Bernstein basis are better than the equivalent results for the scaled Bernstein basis in [20].

Example 7.2. The resultant matrices $s(M)$ and $\tilde{s}(P)$ of the polynomial (28) and the polynomial
\[ s(x) = \prod_{i=2}^{10} \left( x - \frac{i}{20} \right) = \sum_{i=0}^{9} s_i \binom{9}{i} (1-x)^{9-i}x^i, \]
were calculated. The polynomials have nine common roots and thus both $\tilde{s}(P)$ and $s(M)$ are of rank 10. Fig. 1 shows the variation of the logarithm of the normalised singular values of $s(M)$ and $\tilde{s}(P)$, $\log_{10} \sigma_i(s(M))/\sigma_1(s(M))$ and $\log_{10} \sigma_i(\tilde{s}(P))/\sigma_1(\tilde{s}(P))$.

Table 1
The ratio $d(\tilde{s}(P), s(M))$ for Example 7.1

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Polynomial $s(x)$</th>
<th>$d(\tilde{s}(P), s(M))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(x - 0.50)$</td>
<td>$1.01 \times 10^{-7}$</td>
</tr>
<tr>
<td>2</td>
<td>$(x - 0.50)^2$</td>
<td>$1.02 \times 10^{-8}$</td>
</tr>
<tr>
<td>3</td>
<td>$(x - 0.50)^3$</td>
<td>$5.31 \times 10^{-10}$</td>
</tr>
<tr>
<td>4</td>
<td>$(x - 0.50)^4$</td>
<td>$1.46 \times 10^{-10}$</td>
</tr>
<tr>
<td>5</td>
<td>$(x - 0.05) (x - 0.10) (x - 0.95)$</td>
<td>$4.44 \times 10^{-7}$</td>
</tr>
<tr>
<td>6</td>
<td>$(x - 0.0501) (x - 0.10) (x - 0.95)$</td>
<td>$1.11 \times 10^{-11}$</td>
</tr>
<tr>
<td>7</td>
<td>$(x - 0.85)^2 (x - 0.90)^2$</td>
<td>$1.21 \times 10^{-5}$</td>
</tr>
<tr>
<td>8</td>
<td>$(x - 0.90)^4$</td>
<td>$1.08 \times 10^{-5}$</td>
</tr>
<tr>
<td>9</td>
<td>$(x - 0.95)^2 (x - 0.99)^2$</td>
<td>$3.06 \times 10^{-3}$</td>
</tr>
<tr>
<td>10</td>
<td>$(x - 0.05)^3 (x - 0.95)$</td>
<td>$1.93 \times 10^{-12}$</td>
</tr>
<tr>
<td>11</td>
<td>$(x - 0.95)$</td>
<td>$1.73 \times 10^{-5}$</td>
</tr>
<tr>
<td>12</td>
<td>$(x - 0.95)^2$</td>
<td>$8.62 \times 10^{-6}$</td>
</tr>
<tr>
<td>13</td>
<td>$(x - 0.95)^3$</td>
<td>$9.98 \times 10^{-5}$</td>
</tr>
<tr>
<td>14</td>
<td>$(x - 0.95)^4$</td>
<td>$2.23 \times 10^{-3}$</td>
</tr>
<tr>
<td>15</td>
<td>$(x - 0.80)^2 (x - 0.85)^2 (x - 0.90)^2$</td>
<td>$7.24 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
Example 7.2. The variation of (a) $\log_{10} \frac{\sigma_i(s(M))}{\sigma_1(s(M))}$ and (b) $\log_{10} \frac{\sigma_i(\tilde{s}(P))}{\sigma_1(\tilde{s}(P))}$ against the index $i$ for Example 7.2.

$\sigma_1(\tilde{s}(P))$ respectively, against $i$, and it is seen that the numerical rank of $s(M)$ is well defined because of the sharp cutoff at $i = 10$. The singular values of $\tilde{s}(P)$ show, however, a steady decrease and the numerical rank of this matrix is not defined. It follows that only $s(M)$ yields the correct result.

Example 7.3. The ratio of the distance to singularity of the resultant matrices $s(M)$ and $\tilde{s}(P)$ of several pairs of polynomials $r(x)$ and $s(x)$ was calculated and the results are shown in Table 2. Comparison of the results in Tables 1 and 2 shows that the numerical superiority of $s(M)$ over $\tilde{s}(P)$ is less marked in Table 2 than in Table 1. Even in experiment 4, for which $d(\tilde{s}(P), s(M)) = 17.8$, the singular values of $\tilde{s}(P)$ are

\[
\begin{align*}
0.934, & \quad 0.696, \quad 0.443, \quad 1.32 \times 10^{-17}, \quad 2.39 \times 10^{-18}, \quad 0.00, \\
\end{align*}
\]

and the singular values of $s(M)$ are

\[
\begin{align*}
1.16, & \quad 0.156, \quad 3.09 \times 10^{-2}, \quad 2.23 \times 10^{-17}, \quad 2.12 \times 10^{-18}, \quad 0.00, \\
\end{align*}
\]

and thus the numerical rank of both $\tilde{s}(P)$ and $s(M)$ is well defined.
The focus of this paper has been the development of a numerically stable resultant matrix for Bernstein polynomials and a quantitative comparison between it and the resultant matrix that is obtained when a simple parameter substitution is used to transform the Bernstein polynomials to the power basis. The next example is slightly different because it compares the resultant matrix of two Bernstein polynomials \( r(x) \) and \( s(x) \), and the resultant matrix of the same polynomials, expressed in the power basis, \( f(x) \) and \( g(x) \) respectively, that is,

\[
\begin{align*}
\text{Experiment} & \quad \text{Polynomial } r(x) & \quad \text{Polynomial } s(x) & \quad d(\tilde{s}(P), s(M)) \\
1 & \quad (x - 0.10)(x - 0.20)^3 & \quad (x - 0.10)(x - 0.20)^3 & \quad 3.51 \\
2 & \quad (x - 0.90)(x - 0.97)^2 & \quad (x - 0.97)(x - 0.98) & \quad 5.13 \times 10^{-2} \\
3 & \quad (x - 0.10)^2(x - 0.15)^2 & \quad (x - 0.10)(x - 0.15) & \quad 1.39 \times 10^1 \\
4 & \quad (x - 0.10)^3(x - 0.15)^2 & \quad (x - 0.10)(x - 0.15) & \quad 1.78 \times 10^1 \\
5 & \quad (x - 0.10)(x - 0.20) & \quad (x - 0.105)(x - 0.205) & \quad 2.35 \times 10^{-2} \\
6 & \quad (x - 0.80)(x - 0.81) & \quad (x - 0.80)(x - 0.82) & \quad 4.67 \times 10^{-3} \\
\end{align*}
\]

Thus, a basis transformation in a weak sense is used in Examples 7.1, 7.2 and 7.3, but a basis transformation in the strong sense (the exact transformation) is used in Example 7.4.

**Example 7.4.** Consider the truncated Wilkinson polynomial (28) in the power and Bernstein forms, \( f(x) \) and \( r(x) \) respectively, and the quintic polynomial

\[
\begin{align*}
\end{align*}
\]
Fig. 2. The variation of (a) \( \log_{10} \sigma_i(s(M))/\sigma_1(s(M)) \) and (b) \( \log_{10} \sigma_i(g(P))/\sigma_1(g(P)) \) against the index \( i \) for Example 7.4.

\[ s(x) = (x - 0.80)^3(x - 0.95)^2 = \sum_{i=0}^{5} s_i \binom{5}{i} (1 - x)^{5-i} x^i, \]

and its power basis form \( g(x) \), where (31) is satisfied. The resultant matrix has rank 17 since the degree of the GCD is two. Fig. 2 shows the variation of the logarithm of the normalised singular values of \( s(M) \) and \( g(P) \), \( \log_{10} \sigma_i(s(M))/\sigma_1(s(M)) \) and \( \log_{10} \sigma_i(g(P))/\sigma_1(g(P)) \) respectively, against \( i \), where \( P \) is a companion matrix of \( f(x) \). It is seen from Fig. 2 that the numerical rank of \( s(M) \) is 17, the correct value, and that the numerical rank of \( g(P) \) is 14, but this value is not well defined, and it is incorrect.

The accuracy of the degree of the GCD of the polynomials is considered in the four examples in this section, but the accuracy of its computation must also be determined. A measure of the error with which an unknown GCD is computed is, in general, not trivial. If, however, interest is restricted to tests in which the GCD is known, then a simple \textit{a posteriori} test can be performed. In particular, if \( \hat{x}_0 \) is an approximation of the root \( x_0 \) of the polynomial with coefficients \( \{a_i\}_{i=0}^{n} \) with respect to the basis \( \{\phi_i(x)\}_{i=0}^{n} \)...

then the normalised residual $\gamma(x_0)$, which is defined by

$$
\gamma(x_0) = \frac{|p(\hat{x}_0)|}{\|a\|_2},
$$

quantifies the error in the root $x_0$. This measure was used in the examples in this section, and $\gamma(x_0)$ was small, such that the GCD derived from $s(M)$ was computed accurately in all the examples.

8. Discussion

The computational results in Section 7 show that the Bernstein basis companion matrix resultant is numerically superior to its power basis equivalent. The proof of this result requires that the transformation of this resultant matrix between the bases be considered, and it is shown in this section that this proof is not trivial.

Consider the polynomials (29) and (30) where (31) is satisfied. It is shown in [21] that if $P$ is a companion matrix of $f(x)$, then

$$
s(M) = B^{-1} g(P) B,
$$

where $g(P)$ is a companion matrix resultant for the polynomials $f(x)$ and $g(x)$. The matrices $B$ and $B^{-1}$ are upper triangular, and closed form expressions for their elements are in [21]. Repetition of the development of (27) leads to

$$
d(g(P), s(M)) = \|s(M)\|_2 \|s(M)^+\|_2 = \sigma_1(s(M))\sigma_r(g(P))
$$

using (25). The basis transformation (32) from the power basis form $g(P)$ to the Bernstein basis form $s(M)$ is computed if $d(g(P), s(M)) < 1$ because this implies that $g(P)$ is nearer singularity than is $s(M)$.

It is desirable to obtain a lower bound of $d(g(P), s(M))$, but this is not trivial because there does not exist a simple expression for the singular values of $s(M)$ in terms of the singular values of $g(P)$. The evaluation of (33) requires that $s(M)$ be computed from $g(P)$, but this imposes numerical problems because errors in $g(P)$ may be magnified. In particular, it is shown in [21] that (32) can be written as

$$
\text{vec } s(M) = (B^T \otimes B^{-1}) \text{ vec } g(P),
$$

where $X \otimes Y$ denotes the Kronecker product [9] of the rectangular matrices $X$ and $Y$, and the vector $\text{vec } Z$ is formed by stacking the columns of the matrix $Z$,

$$
\text{vec } Z = \begin{bmatrix}
Z_1 \\
Z_2 \\
\vdots \\
Z_n
\end{bmatrix}.
$$
The variation of $\log_{10} \kappa_2(B^T \otimes B^{-1})$ with the order of $B$.

where $Z_{p}$ is the $p$th column of $Z$. Clearly, vec $Z$ is of length $mn$ if $Z$ is of order $m \times n$.

The vectors vec $s(M)$ and vec $g(P)$ are of length $n^2$, and $(B^T \otimes B^{-1})$ is of order $n^2 \times n^2$. Eq. (34) is of the standard form $b = Ax$ where $b = \text{vec } s(M)$, $x = \text{vec } g(P)$ and $A = B^T \otimes B^{-1}$. The condition number of the computation of vec $s(M)$ from vec $g(P)$ (or equivalently $s(M)$ from $g(P)$) is

$$
\max_{\delta x, x} \frac{\Delta b}{\Delta x} = \|A\|\|A^{-1}\|, \quad \Delta b = \frac{\|\delta b\|}{\|b\|}, \quad \Delta x = \frac{\|\delta x\|}{\|x\|}.
$$

The variation of $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$ with the order $n$ of $B$ is shown in Fig. 3 and it is seen that the computation of $s(M)$ from $g(P)$ may be ill-conditioned, even for low values of $n$. It follows from this result that the resultant matrix should be evaluated either in the power basis or in the Bernstein basis, but a basis transformation should not be performed. Thus, since the determination of $d(g(P), s(M))$ in (33) requires the computation of $s(M)$ from $g(P)$, its computational reliability cannot be guaranteed. The combination of this result on the numerical condition of the transformation from $g(P)$ to $s(M)$, and the examples in Section 7, suggests that the resultant of two polynomials should be evaluated when the polynomials are expressed in the
Bernstein basis rather than the power basis. This topic is considered in more detail in [21], where (32) is derived, closed form expressions for the elements of $B$ and $B^{-1}$ are obtained, and the theoretical properties of $B$ are investigated.

9. Conclusions

A companion matrix $M$ of a Bernstein polynomial $r(x)$ was developed and used to construct a resultant matrix $s(M)$ of two Bernstein polynomials $r(x)$ and $s(x)$. The matrix $M$ has a more complex structure than its power basis equivalent, and is in general dense. It was shown that if the coefficients of the polynomials are free of errors such that only the effects of roundoff error due to floating point arithmetic need be considered, then the resultant matrix $s(M)$ is numerically superior to its power basis equivalent. This result is in accord with the improved numerical stability of the Bernstein basis with respect to the power basis. It was shown that the transformation between $g(P)$ and $s(M)$ may be ill-conditioned, and that it is desirable to compute the resultant of two polynomials when they are expressed in the Bernstein basis rather than the power basis.

A more difficult numerical problem arises when the data are subject to errors, in which case the entries of $s(M)$ are random variables and the usual definitions of the rank of a matrix and the GCD of two polynomials are not valid. Revised definitions of these terms must be developed, and it is necessary to consider a family of GCDs. If the tolerance in the data is small, this family of GCDs manifests itself in the ill-conditioned nature of the resultant matrix, and mathematical discrimination between these computed solutions is not possible because they are all correct, as specified by the data. The selection of a particular GCD from this family of solutions may be governed by the characteristics of the problem that motivated its computation.

References


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1 It is noted that Examples 7.1, 7.2 and 7.3 compare the resultant matrices in the power and Bernstein bases when the parameter substitution (2) is used to perform a basis transformation in a weak sense, but (32) and (33) assume the basis transformation is performed in the strong (exact) sense. It is assumed, however, that the Bernstein basis resultant matrix is numerically superior to its power basis form, and that this is independent of the method used to perform the basis transformation.


