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Laguerre-Type Exponentials and Generalized Appell Polynomials

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Abstract—General classes of two variables Appell polynomials are introduced by exploiting properties of an iterated isomorphism, related to the so-called Laguerre-type exponentials. Further extensions to the multi-index and multivariable cases are mentioned.

Keywords—Laguerre-type exponentials, Generating functions, Appell polynomials.

1. INTRODUCTION

In recent articles, in the framework of the monomiality principle [1,2], a class of generalized exponential functions, the so-called Laguerre-type exponentials (shortly L-exponentials), was introduced [3].

These functions are determined by using a differential isomorphism, denoted by the symbol $\mathcal{T} := \mathcal{T}_x$, acting onto the space $\mathcal{A} := \mathcal{A}_x$ of analytic functions of the x variable by means of the correspondence

$$D := D_x \to D_L := DxD; \qquad x\cdot \to D_x^{-1}, \tag{1.1}$$

where

$$D_x^{-n}(1) := \frac{x^n}{n!}, \tag{1.2}$$

so that

$$\mathcal{T}(x^n) = \frac{x^n}{n!}. \tag{1.3}$$

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The operator $\hat{D}_L = DxD$, is called in literature the Laguerre derivative and appears quite frequently in mathematical modelling relevant to vibrating phenomena in viscous fluids and even in mechanical problems such as the oscillating chain (see [4, pp. 282–284]).

According to the above isomorphism, substituting the derivative operator D with the Laguerrian derivative $\hat{D}_L$ and the multiplicative operator $x\cdot$ with the antiderivative $\hat{D}_x^{-1}$, the solutions of all linear differential equations are preserved.

This property allowed to construct in a straightforward way solutions to ordinary [5,6] or partial differential equations [3,7]).

A first example of the above-mentioned isomorphism was found proving the connection between the Hermite-Kampé de Fériet (or Gould-Hopper) polynomials [8–11], and the two variable Laguerre polynomials.

As it is well known, the Hermite-Kampé de Fériet polynomials are called *heat polynomials* [12], since they are elementary solutions of the heat equation, but they enter even in the explicit solutions of many classical or pseudo-classical BVP in the half-plane (see [13,14]). The relevant results were generalized to many variables problems in [15,16].

The so-called two variable Laguerre polynomials are defined by

$$\mathcal{L}_n(x,y) := n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(n-r)!(r!)^2}.$$

It was shown (see [3,17]) that the polynomials $\mathcal{L}_n(-x,y)$ are connected to the H-KdF polynomials $H_n^{(1)}(x,y) := (x+y)^n$ throughout the above-mentioned linear differential isomorphism, where the variable y is considered as a parameter.

In order to avoid the change of sign of the x variable, we use the more simple notation

$$L_n(x,y) := \mathcal{L}_n(-x,y) = n! \sum_{r=0}^{n} \frac{y^{n-r} x^r}{(n-r)!(r!)^2}. \tag{1.4}$$

Using this notation, we can write

$$L_n(x,y) := L_n^{(1)}(x,y) = \mathcal{T}_x \left(H_n^{(1)}(x,y) \right). \tag{1.5}$$

Note that the exponential function is transformed by $\mathcal{T}$ into the Laguerrian exponential $e_1(x)$

$$\mathcal{T} e^x = e_1(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}. \tag{1.6}$$

The use of the above isomorphism already permitted the definition of higher-order Laguerre polynomials which are the Laguerrian counterpart of the Gould-Hopper ones [17], and Laguerre-type Bessel functions [18].

Particular cases of higher-order type Laguerre polynomials were used for the computation of moments of chaotic radiations (see [19]).

It is convenient, in the following, to introduce a suitable notation regarding the isomorphism $\mathcal{T}_x^s$ and its iterations. According to the above definition we can write

$$\mathcal{T}_x = D_x^{-1} = D_x^{-1}(1), \tag{1.7}$$

$$\mathcal{T}_x^2 = \mathcal{T}_x D_x^{-1}(1) = D_{\mathcal{T}_x}^{-1}(1), \qquad \text{so that } D_{\mathcal{T}_x}^{-n}(1) = \frac{x^n}{(n!)^2}, \tag{1.8}$$

and, by induction

$$\mathcal{T}_x^s = \mathcal{T}_x^{s-1} D_x^{-1}(1) = D_{\mathcal{T}_x^{s-1}}^{-1}(1), \qquad \text{so that } D_{\mathcal{T}_x^{s-1}}^{-n}(1) = \frac{x^n}{(n!)^s}. \tag{1.9}$$

It is easily seen that, $\forall\, k \in \mathbf{N}$, $\forall\, s \in \mathbf{N}$,

$$\hat{D}_{\mathcal{T}_x}^{-1}\left(x^k\right) = \frac{k!x^{k+1}}{[(k+1)!]^2}, \dots, \hat{D}_{\mathcal{T}_x^s}^{-1}\left(x^k\right) = \frac{k!x^{k+1}}{[(k+1)!]^{s+1}}, \tag{1.10}$$

and, $\forall\, h \in \mathbf{N}$,

$$\hat{D}_{\mathcal{T}_x}^{-h}\left(x^k\right) = \frac{k!x^{k+h}}{[(k+h)!]^2}, \dots, \hat{D}_{\mathcal{T}_x^s}^{-h}\left(x^k\right) = \frac{k!x^{k+h}}{[(k+h)!]^{s+1}}.$$

This is in accordance with the results of a paper by Dattoli and Ricci [3] about the definition of the higher-order Laguerre-type exponentials, which are defined in such a way that

$$\mathcal{T}_x^s\left(e^x\right) = e_s(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{s+1}}. \tag{1.11}$$

Working with the iterated isomorphism $\mathcal{T}^s$, derivative operator $D := D_x$ must be substituted with the Laguerrian derivative

$$D_{sL} := \left(D_{sL}\right)_x := Dx\,D \dots x\,D \tag{1.12}$$

(containing $(s+1)$ ordinary derivatives with respect to the x variable).

General classes of higher-order Laguerre polynomials were defined in [17], by putting

$$L_m^{(j;s;\sigma)}(x,y) = \mathcal{T}_x^s \mathcal{T}_y^\sigma \left(H_m^{(j)}(x,y)\right), \tag{1.13}$$

which are explicitly expressed by

$$L_m^{(j;s;\sigma)}(x,y) = H_m^{(j)}\left(\mathcal{T}_x^s(x), \mathcal{T}_y^\sigma(y)\right) = m! \sum_{k=0}^{[m/j]} \frac{y^k x^{m-jk}}{[k!]^{\sigma+1}[(m-jk)!]^{s+1}} \tag{1.14}$$

and are given by the generating function

$$\sum_{m=0}^{\infty} L_m^{(j;s;\sigma)}(x,y) \frac{t^m}{m!} = \mathcal{T}_x^s \mathcal{T}_y^\sigma \exp\left\{xt + yt^j\right\} = e_s(xt) e_\sigma\left(yt^j\right).$$

The same procedure is used in the present article in order to generalize the Appell polynomials, considering Laguerre-type Appell polynomials of higher order, and extending in such a way the preceding definitions of the papers [20,21].

In this way, we are able to obtain general classes of Appell polynomials, including the Bernoulli and Euler ones, which can be defined throughout generating functions which include L-exponentials instead of the ordinary one. This can be done acting separately with respect to each independent variable, and in this case the relevant variable will be used, as an index of the considered isomorphism, in order to avoid confusion (e.g., $\mathcal{T}_x^s$ will denote the s-times iterated isomorphism acting with respect to the x variable, and so on).

Of course, we consider mainly the polynomials in two variables, in order to write down explicitly in a more friendly way the relevant properties, however all the formulas can be easily extended to the general case, by using a vectorial and multi-index approach which is mentioned in the concluding section.

2. 2D APPELL POLYNOMIALS

For any $j \geq 2$, the 2D Appell polynomials $R_n^{(j)}(x,y)$ are defined by means of the generating function

$$G_A^{(j)}(x,y;t) := A(t) e^{xt+yt^j} = \sum_{n=0}^{\infty} R_n^{(j)}(x,y) \frac{t^n}{n!}. \tag{2.1}$$

Even in this general case, the polynomial $R_n^{(j)}(x,y)$, is *isobaric of weight n*, so that it does not contain the variable y, for every $n = 0, 1, \dots, j-1$.

- Explicit forms of the polynomials $R_n^{(j)}$ in terms of the Hermite-Kampé de Fériet polynomials $H_n^{(j)}$ and vice-versa.

  The following representation formula holds:

$$
\begin{aligned}
R_n^{(j)}(x,y) &= \sum_{h=0}^{n} \binom{n}{h} \mathcal{R}_{n-h} H_h^{(j)}(x,y) \\
&= n! \sum_{h=0}^{n} \frac{\mathcal{R}_{n-h}}{(n-h)!} \sum_{r=0}^{[h/j]} \frac{x^{h-jr} y^r}{(h-jr)!r!},
\end{aligned} \tag{2.2}
$$

where the $\mathcal{R}_k$ are the "Appell numbers" appearing in the definition: $A(t) = \sum_{k=0}^{\infty} (\mathcal{R}_k/k!) t^k$, $(A(0) \neq 0)$;

$$
H_n^{(j)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} Q_{n-k} R_k^{(j)}(x,y),
$$

where the Q_k are the coefficients of the Taylor expansion in a neighborhood of the origin of the reciprocal function $1/A(t)$.

- Recurrence relation.

  It is useful to introduce the coefficients of the Taylor expansion:

$$
\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}. \tag{2.3}
$$

The following linear homogeneous recurrence relation for the generalized Appell polynomials $R_n^{(j)}(x,y)$ holds:

$$
\begin{aligned}
R_0^{(j)}(x,y) &= 1, \\
R_n^{(j)}(x,y) &= (x+\alpha_0) R_{n-1}^{(j)}(x,y) + \binom{n-1}{j-1} jy R_{n-j}^{(j)}(x,y) \\
&\quad + \sum_{k=0}^{n-2} \binom{n-1}{k} \alpha_{n-k-1} R_k^{(j)}(x,y).
\end{aligned} \tag{2.4}
$$

- Shift operators.

$$
\begin{aligned}
L_n^- &:= \frac{1}{n} D_x, \\
L_n^+ &:= (x+\alpha_0) + \frac{j}{(j-1)!}\, y\, D_x^{j-1} + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{n-k},
\end{aligned} \tag{2.5}
$$

- Differential equation.

$$
\begin{aligned}
&\left[\frac{\alpha_{n-1}}{(n-1)!} D_x^n + \cdots + \frac{\alpha_j}{j!} D_x^{j+1} + \left(\frac{\alpha_{j-1} + jy}{(j-1)!} \right) D_x^j \right. \\
&\left. + \frac{\alpha_{j-2}}{(j-2)!} D_x^{j-1} + \cdots + (x+\alpha_0)\, D_x - n \right] R_n^{(j)}(x,y) = 0,
\end{aligned} \tag{2.6}
$$

3. HIGHER-ORDER APPELL POLYNOMIALS

According to the above-mentioned properties, we can define a more general class of higher-order Appell polynomials. Namely, the following result holds.

THEOREM 3.1. *The polynomials $R_n^{(j;s;\sigma)}(x,y)$, defined by the generating function*

$$A(t)e_s(xt)e_\sigma\left(yt^j\right) = \sum_{n=0}^{\infty} R_n^{(j;s;\sigma)}(x,y)\,\frac{t^n}{n!}, \tag{3.1}$$

where

$$R_n^{(j;s;\sigma)}(x,y) := R_n^{(j)}\left(\mathcal{T}_x^s(x), \mathcal{T}_y^\sigma(y)\right), \tag{3.2}$$

are explicitly expressed by

$$R_n^{(j;s;\sigma)}(x,y) = \sum_{h=0}^{n} \binom{n}{h} R_{n-h} L_h^{(j;s;\sigma)}(x,y), \tag{3.3}$$

where the coefficients R_k are the Appell numbers associated with the function $A(t)$ (see equation (2.2)), and the higher-order Laguerre polynomials $L_h^{(j;s;\sigma)}(x,y)$, defined by equation (1.14), come into play

$$L_h^{(j;s;\sigma)}(x,y) = \mathcal{T}_x^s\mathcal{T}_y^\sigma H_h^{(j)}(x,y). \tag{3.4}$$

PROOF. Applying the isomorphisms $\mathcal{T}_x^s$ and $\mathcal{T}_y^\sigma$ to both sides of the generating function of the polynomials $R_n^{(j)}(x,y)$, yields

$$\mathcal{T}_x^s\mathcal{T}_y^\sigma A(t)e^{xt+yt^j} = \sum_{n=0}^{\infty} \mathcal{T}_x^s\mathcal{T}_y^\sigma R_n^{(j)}(x,y)\,\frac{t^n}{n!},$$

and therefore,

$$A(t)e_s(xt)e_\sigma(yt^j) = \sum_{n=0}^{\infty} R_n^{(j)}\left(\mathcal{T}_x^s(x), \mathcal{T}_y^\sigma(y)\right)\frac{t^n}{n!},$$

so that equations (3.1),(3.2) hold. Equation (3.4) is a consequence of (2.2) and (3.3).

Further properties are obtained by using the same procedure as above, and can be summarized as follows.

THEOREM 3.2. *The polynomials $R_n^{(j;s;\sigma)}(x,y)$, verify the recurrence relation*

$$\begin{aligned} &R_0^{(j;s;\sigma)}(x,y) = 1, \\ &R_n^{(j;s;\sigma)}(x,y) = \left(D^{-1}_{\mathcal{T}_x^{s-1}(x)} + \alpha_0\right) R_{n-1}^{(j;s;\sigma)}(x,y) + \binom{n-1}{j-1} jyR_{n-j}^{(j;s;\sigma)}(x,y) \\ &\qquad + \sum_{k=0}^{n-2} \binom{n-1}{k} \alpha_{n-k-1} R_k^{(j;s;\sigma)}(x,y), \end{aligned} \tag{3.5}$$

where the operator $D^{-1}_{\mathcal{T}_x^{s-1}(x)}$, acting on $R_{n-1}^{(j;s;\sigma)}(x,y)$, is defined by equations (1.9),(1.10), and the α_k are the coefficients of the expansion (2.3).

THEOREM 3.3. *Shift operators for the polynomials $R_n^{(j;s;\sigma)}(x,y)$, are given by*

$$\begin{aligned} &L_n^- := \frac{1}{n}\hat{D}_{sL}, \\ &L_n^+ := \left(D^{-1}_{\mathcal{T}_x^{s-1}(x)} + \alpha_0\right) + \frac{j}{(j-1)!}\,y\hat{D}_{sL}^{j-1} + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!}\,\hat{D}_{sL}^{n-k}. \end{aligned} \tag{3.6}$$

Theorem 3.4. *A differential equation satisfied by the polynomials $R_n^{(j;s;\sigma)}(x,y)$, is given by*

$$\begin{aligned}&\left[\frac{\alpha_{n-1}}{(n-1)!}\hat{D}_{sL}^{n}+\cdots+\frac{\alpha_j}{j!}\hat{D}_{sL}^{j+1}+\left(\frac{\alpha_{j-1}+jy}{(j-1)!}\right)\hat{D}_{sL}^{j}\right.\\&\left.+\frac{\alpha_{j-2}}{(j-2)!}\hat{D}_{sL}^{j-1}+\cdots+\left(D_{\mathcal{T}_x^{s-1}(x)}^{-1}+\alpha_0\right)\hat{D}_{sL}-n\right]R_n^{(j;s;\sigma)}(x,y)=0.\end{aligned} \tag{3.7}$$

4. CONCLUDING REMARKS

The results obtained in the preceding section could be generalized by using the multivariable Hermite polynomials $H_{k_1,\dots,k_N}^{(j_1,\dots,j_N)}$ defined throughout the generating function

$$e^{x_1t^{j_1}+\cdots+x_Nt^{j_N}}=\sum_{k=0}^{\infty}H_k^{(j_1,\dots,j_N)}(x_1,\dots,x_N)\frac{t^k}{k!}, \tag{4.1}$$

and noting that the corresponding multivariable Laguerre polynomials are obtained acting with linear isomorphisms operating separately with respect to each independent variable, namely

$$L_k^{(j_1,\dots,j_N;s_1,\dots,s_N)}(x_1,\dots,x_N):=\mathcal{T}_{x_1}^{s_1}\dots\mathcal{T}_{x_N}^{s_N}\left(H_k^{(j_1,\dots,j_N)}(x_1,\dots,x_N)\right). \tag{4.2}$$

By using the iterated isomorphisms acting with respect to the different independent variables, the following generating function can be easily obtained:

$$\begin{aligned}&\sum_{k=0}^{\infty}L_k^{(j_1,\dots,j_N;s_1,\dots,s_N)}(x_1,\dots,x_N)\frac{t^k}{k!}\\&=\mathcal{T}_{x_1}^{s_1}\dots\mathcal{T}_{x_N}^{s_N}\sum_{k=0}^{\infty}H_k^{(j_1,\dots,j_N)}(x_1,\dots,x_N)\frac{t^k}{k!}\\&=\mathcal{T}_{x_1}^{s_1}\dots\mathcal{T}_{x_N}^{s_N}\exp\left(x_1t^{j_1}+\cdots+x_Nt^{j_N}\right)\\&=\mathcal{T}_{x_1}^{s_1}x_1e^{t^{j_1}}\dots\mathcal{T}_{x_N}^{s_N}x_Ne^{t^{j_N}}=e_{s_1}\left(x_1t^{j_1}\right)\dots e_{s_N}\left(x_Nt^{j_N}\right),\end{aligned} \tag{4.3}$$

which generalizes equation (1.15).

Therefore, a general class of multidimensional and multivariable Appell polynomials can be introduced by means of the generating function

$$A(t)e_{s_1}\left(x_1t^{j_1}\right)\dots e_{s_N}\left(x_Nt^{j_N}\right)=\sum_{n=0}^{\infty}R_n^{(j_1,\dots,j_N;s_1,\dots,s_N)}(x_1,\dots,x_N)\frac{t^n}{n!}. \tag{4.4}$$

The relevant properties can be derived by using methods similar to those described in Section 3.

Further extensions could be obtained by introducing the multivariable and multi-index polynomials mentioned in the last section of [17].

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