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Exact solution of the heat equation with boundary condition of the fourth kind by He's variational iteration method

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ABSTRACT

In this paper, solutions of the heat equation with the boundary condition of the fourth kind are presented. The proposed solution is based on He's variational iteration method, after the application of which the exact solution of the problem is obtained.

1. Introduction

The variational iteration method was developed by Ji-Huan He [1–3]. This method provides an effective and efficient way of solving a wide range of nonlinear operator equations [4–9]. For example, Momani and his colleagues [10] applied the variational iteration method in solving the ordinary differential equations with boundary conditions. Likewise, Dehghan and Shakeri [11] used the method in the approximate solution of a differential equation arising in astrophysics. There are other publications, where the variational iteration method was utilized for solving the exact or approximate solution of partial differential equations. Momani and Abuasad [9] used the variational iteration method for solving Helmholtz's equation. Similarly, Wazwaz [12,13] employed the method in exact solutions of Laplace and wave equations. In Refs. [14,15] the heat-like and wave-like equations were solved, whereas the use of the method for the heat transfer or diffusion equations was described in Refs. [16–19]. The solution of the system of partial differential equations was described in Ref. [20]. Słota [21] applied the variational iteration method combined with an optimization for an approximate solution of one-phase direct and inverse Stefan problems with a Dirichlet boundary condition. Tatari and Dehghan [22] used the variational iteration method for computing a parameter in a semi-linear inverse parabolic equation. The convergence of the method was discussed by Tatari and Dehghan [23]. He in papers [24,25] described some new interpretations and applications of the variational iteration method.

In this paper, the author made an attempt at solving the heat equation with the boundary condition of the fourth kind. The proposed solution is based on He's variational iteration method, after the application of which, the sequence convergent to the exact solution of the problem is derived.

2. Statement of the problem

Let $D_1 = \{(x,t);\, x \in [x_1, 0],\, t \in [0, t^*)\}$ and $D_2 = \{(x,t);\, x \in [0, x_2],\, t \in [0, t^*)\}$ (Fig. 1). On the boundary of these domains five components are distributed:

$$\Gamma_1 = \{(x,0);\, x \in [x_1, 0]\}\,, \tag{1}$$

$$\Gamma_2 = \{(x,0);\, x \in [0, x_2]\}\,, \tag{2}$$

$$\Gamma_3 = \left\{(x_1,t);\, t \in [0, t^*)\right\}\,, \tag{3}$$

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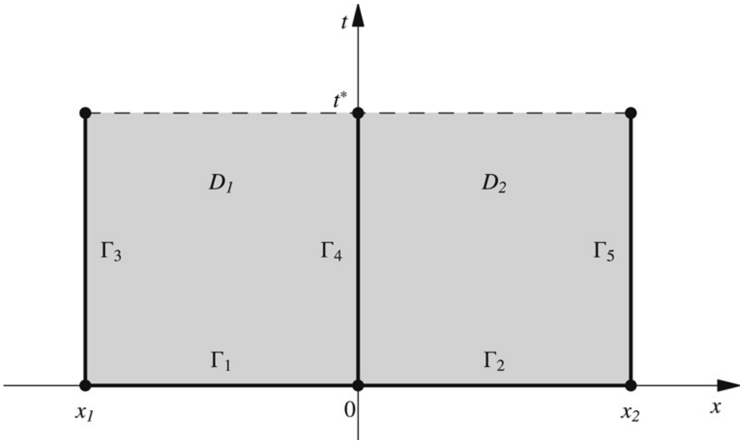

Fig. 1. Domain formulation of the problem.

$$\Gamma_4 = \left\{(0, t); t \in [0, t^*)\right\}, \tag{4}$$

$$\Gamma_5 = \left\{(x_2, t); t \in [0, t^*)\right\}, \tag{5}$$

where the initial and boundary conditions are given.

In domains D_1 and D_2 we consider the heat conduction equations:

$$\alpha_u \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u}{\partial t}(x,t), \quad (x,t) \in D_1, \tag{6}$$

$$\alpha_v \frac{\partial^2 v(x,t)}{\partial x^2} = \frac{\partial v}{\partial t}(x,t), \quad (x,t) \in D_2. \tag{7}$$

With initial conditions on boundaries Γ_1 and Γ_2:

$$u(x, 0) = \varphi_u(x), \quad x \in [x_1, 0], \tag{8}$$

$$v(x, 0) = \varphi_v(x), \quad x \in [0, x_2], \tag{9}$$

and Dirichlet's conditions on boundaries Γ_3 and Γ_5:

$$u(x_1, t) = \psi_u(t), \quad t \in [0, t^*), \tag{10}$$

$$v(x_2, t) = \psi_v(t), \quad t \in [0, t^*). \tag{11}$$

On common boundary Γ_4 the boundary conditions of the fourth kind are given (the condition of temperature continuity and the condition of heat flux continuity):

$$u(0, t) = v(0, t), \quad t \in [0, t^*), \tag{12}$$

$$-k_u \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = -k_v \left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0}, \quad t \in [0, t^*), \tag{13}$$

where α_u and α_v are the thermal diffusivity, k_u and k_v are the thermal conductivity, u and v are temperature, and t and x refer to time and spatial location, respectively. Let us assume that the functions that describe the considered task comply with the following compatibility conditions:

$$\varphi_u(x_1) = \psi_u(0), \qquad \varphi_u(0) = \varphi_v(0),$$

$$\varphi_v(x_2) = \psi_v(0), \qquad -k_u \frac{\partial \varphi_u}{\partial x}(0,0) = -k_v \frac{\partial \varphi_v}{\partial x}(0,0).$$

We shall seek functions $u(x, t)$ and $v(x, t)$, determined in domains D_1 and D_2 respectively, which meet the conditions of heat transfer and the above specified conditions.

3. He's variational iteration method

Using variational iteration method we are able to solve the nonlinear equation:

$$L(u(t)) + N(u(t)) = f(t), \tag{14}$$

where L is the linear operator, N is the nonlinear operator, f is a known function and u is a sought function. At first, we construct a correction functional:

$$u_n(t) = u_{n-1}(t) + \int_0^t \lambda(s)\left(L(u_{n-1}(s)) + N(\tilde{u}_{n-1}(s)) - f(s)\right) \mathrm{d}s \tag{15}$$

where $\tilde{u}_{n-1}$ is a restricted variation [1–5,26,6], $\lambda(s)$ is a general Lagrange multiplier [27,1,2], which can be identified optimally by the variational theory [28,1–3] and $u_0(s)$ is an initial approximation. Next, we determine the general Lagrange multiplier and identify it as a function of $\lambda = \lambda(s)$. Finally, we obtain the iteration formula:

$$u_n(t) = u_{n-1}(t) + \int_0^t \lambda(s)\left(L(u_{n-1}(s)) + N(u_{n-1}(s)) - f(s)\right) \mathrm{d}s, \tag{16}$$

from which an approximate solution (and frequently, an exact solution) of Eq. (14) may be derived.

4. Solution of the problem

To solve the problem of the heat transfer at ideal contact between the two objects, the variational iteration method is used. The author would like to present two solutions based on the use of correction functional in t-direction and in x-direction, respectively.

Method 1. The correction functionals in t-direction for Eqs. (6) and (7) can be expressed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(s)\left(\frac{\partial u_n(x,s)}{\partial s} - \alpha_u \frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2}\right) \mathrm{d}s, \tag{17}$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(s)\left(\frac{\partial v_n(x,s)}{\partial s} - \alpha_v \frac{\partial^2 \tilde{v}_n(x,s)}{\partial x^2}\right) \mathrm{d}s, \tag{18}$$

where $\tilde{u}_n$ and $\tilde{v}_n$ are restricted variation and λ_1 and λ_2 are the general Lagrange multiplier, which can be optimally be identified by the variational theory. The stationary conditions are given by:

$$\lambda_1'(s) = 0, \quad \left(1 + \lambda_1(s)\right)_{s=t} = 0, \tag{19}$$

$$\lambda_2'(s) = 0, \quad \left(1 + \lambda_2(s)\right)_{s=t} = 0, \tag{20}$$

so that

$$\lambda_1(s) = -1, \tag{21}$$

$$\lambda_2(s) = -1. \tag{22}$$

Hence, we obtain the following iteration formulas:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,s)}{\partial s} - \alpha_u \frac{\partial^2 u_n(x,s)}{\partial x^2}\right) \mathrm{d}s, \tag{23}$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,s)}{\partial s} - \alpha_v \frac{\partial^2 v_n(x,s)}{\partial x^2}\right) \mathrm{d}s. \tag{24}$$

As initial approximations $u_0(x,t)$ and $v_0(x,t)$ the functions describing the initial conditions may be selected. Because the compatibility conditions are fulfilled, the initial approximations meet conditions (12) and (13) given at the interface between the two domains.

Method 2. Now, correction functionals in x-direction will be applied for Eqs. (6) and (7):

$$u_{n+1}(x,t) = u_n(x,t) + \int_x^0 \lambda_1(s)\left(\frac{\partial^2 u_n(s,t)}{\partial s^2} - \frac{1}{\alpha_u}\frac{\partial \tilde{u}_n(s,t)}{\partial t}\right) \mathrm{d}s, \tag{25}$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^x \lambda_2(s)\left(\frac{\partial^2 v_n(s,t)}{\partial s^2} - \frac{1}{\alpha_v}\frac{\partial \tilde{v}_n(s,t)}{\partial t}\right) \mathrm{d}s, \tag{26}$$

where $\tilde{u}_n$ and $\tilde{v}_n$ are restricted variation and λ_1 and λ_2 ares the general Lagrange multiplier. From Eqs. (25) and (26), the general Lagrange multipliers can be identified as follows:

$$\lambda_1(s) = x - s, \tag{27}$$

$$\lambda_2(s) = s - x. \tag{28}$$

Substituting values of the general Lagrange multipliers into Eqs. (25) and (26) we obtain the following iteration formulas:

$$u_{n+1}(x,t) = u_n(x,t) + \int_x^0 (x-s)\left(\frac{\partial^2 u_n(s,t)}{\partial s^2} - \frac{1}{\alpha_u}\frac{\partial u_n(s,t)}{\partial t}\right)\mathrm{d}s, \tag{29}$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^x (s-x)\left(\frac{\partial^2 v_n(s,t)}{\partial s^2} - \frac{1}{\alpha_v}\frac{\partial v_n(s,t)}{\partial t}\right)\mathrm{d}s. \tag{30}$$

Next, the initial approximations may be selected in the following form:

$$u_0(x,t) = A_u + B_u\, x, \tag{31}$$

$$v_0(x,t) = A_v + B_v\, x, \tag{32}$$

where A_u, A_v, B_u and B_v are parameters independent from variable x. By employing conditions (12) and (13) we establish that parameters A_u, A_v, B_u and B_v must comply with the following relations:

$$A_v = A_u, \tag{33}$$

$$B_v = \frac{k_u}{k_v}\, B_u. \tag{34}$$

5. Examples

To illustrate the solution procedure and show the capability of the method, some examples are considered. All results are calculated by using the symbolic calculus software Mathematica.

Example 1. At first, we consider an example in which: $x_1 = -1, x_2 = 1, \alpha_u = \frac{1}{4}, \alpha_v = 1, k_u = 1, k_v = 2$ and

$$\varphi_u(x) = \frac{1}{6}\,\mathrm{e}^{-2x-2}\left(3\,\mathrm{e}^{4x+3} + \mathrm{e}^3 + 2\right), \tag{35}$$

$$\varphi_v(x) = \frac{1}{6}\,\mathrm{e}^{-x-2}\left(3\,\mathrm{e}^{2x+3} + \mathrm{e}^3 + 2\right), \tag{36}$$

$$\psi_u(t) = \frac{1}{6}\,\mathrm{e}^{t-1}\left(\mathrm{e}^4 + 2\,e + 3\right), \tag{37}$$

$$\psi_v(t) = \frac{1}{6}\,\mathrm{e}^{t-3}\left(3\,\mathrm{e}^5 + \mathrm{e}^3 + 2\right). \tag{38}$$

Starting with initial approximations in the form:

$$u_0(x,t) = \frac{1}{6}\,\mathrm{e}^{-2x-2}\left(3\,\mathrm{e}^{4x+3} + \mathrm{e}^3 + 2\right), \tag{39}$$

$$v_0(x,t) = \frac{1}{6}\,\mathrm{e}^{-x-2}\left(3\,\mathrm{e}^{2x+3} + \mathrm{e}^3 + 2\right), \tag{40}$$

and using iterative formulas (23) and (24) (the first method) we can obtain the following results:

$$u_1(x,t) = \frac{1}{6}\,\mathrm{e}^{-2x-2}\left(3\,\mathrm{e}^{4x+3} + \mathrm{e}^3 + 2\right)\left(1+t\right),$$

$$u_2(x,t) = \frac{1}{6}\,\mathrm{e}^{-2x-2}\left(3\,\mathrm{e}^{4x+3} + \mathrm{e}^3 + 2\right)\left(1+t+\frac{t^2}{2!}\right),$$

$$u_3(x,t) = \frac{1}{6}\,\mathrm{e}^{-2x-2}\left(3\,\mathrm{e}^{4x+3} + \mathrm{e}^3 + 2\right)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\right),$$

$$u_4(x,t) = \frac{1}{6}\,\mathrm{e}^{-2x-2}\left(3\,\mathrm{e}^{4x+3} + \mathrm{e}^3 + 2\right)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}\right),$$

$$u_5(x,t) = \frac{1}{6}\,\mathrm{e}^{-2x-2}\left(3\,\mathrm{e}^{4x+3} + \mathrm{e}^3 + 2\right)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}\right),$$

$$\vdots$$

and

$$v_1(x,t) = \frac{1}{6}\,\mathrm{e}^{-x-2}\left(3\,\mathrm{e}^{x+3} + \mathrm{e}^3 + 2\right)\left(1+t\right),$$
$$v_2(x,t) = \frac{1}{6}\,\mathrm{e}^{-x-2}\left(3\,\mathrm{e}^{x+3} + \mathrm{e}^3 + 2\right)\left(1+t+\frac{t^2}{2!}\right),$$
$$v_3(x,t) = \frac{1}{6}\,\mathrm{e}^{-x-2}\left(3\,\mathrm{e}^{x+3} + \mathrm{e}^3 + 2\right)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\right),$$
$$v_4(x,t) = \frac{1}{6}\,\mathrm{e}^{-x-2}\left(3\,\mathrm{e}^{x+3} + \mathrm{e}^3 + 2\right)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}\right),$$
$$v_5(x,t) = \frac{1}{6}\,\mathrm{e}^{-x-2}\left(3\,\mathrm{e}^{x+3} + \mathrm{e}^3 + 2\right)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}\right),$$
$$\vdots$$

Therefore,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \frac{1}{6}\,\mathrm{e}^{t-2x-2}\left(3\,\mathrm{e}^{4x+3} + \mathrm{e}^3 + 2\right), \tag{41}$$
$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = \frac{1}{6}\,\mathrm{e}^{t-2x-2}\left(3\,\mathrm{e}^{2x+3} + \mathrm{e}^3 + 2\right), \tag{42}$$

which are the exact solutions of the considered example.

Example 2. Now, we consider an example in which: $x_1 = -1, x_2 = 1, \alpha_u = \frac{1}{4}, \alpha_v = 1, k_u = 1, k_v = 2$ and

$$\varphi_u(x) = \mathrm{e}^{2x}, \tag{43}$$
$$\varphi_v(x) = \mathrm{e}^{x}, \tag{44}$$
$$\psi_u(t) = \mathrm{e}^{t-2}, \tag{45}$$
$$\psi_v(t) = \mathrm{e}^{t+1}. \tag{46}$$

Starting with initial approximations in the form:

$$u_0(x,t) = \mathrm{e}^{2x}, \tag{47}$$
$$v_0(x,t) = \mathrm{e}^{x}, \tag{48}$$

and using iterative formulas (23) and (24) (the first method) we can obtain the following results:

$$u_1(x,t) = \mathrm{e}^{2x}\left(1+t\right),$$
$$u_2(x,t) = \mathrm{e}^{2x}\left(1+t+\frac{t^2}{2!}\right),$$
$$u_3(x,t) = \mathrm{e}^{2x}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\right),$$
$$u_4(x,t) = \mathrm{e}^{2x}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}\right),$$
$$u_5(x,t) = \mathrm{e}^{2x}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}\right),$$
$$\vdots$$

and

$$v_1(x,t) = \mathrm{e}^{x}\left(1+t\right),$$
$$v_2(x,t) = \mathrm{e}^{x}\left(1+t+\frac{t^2}{2!}\right),$$
$$v_3(x,t) = \mathrm{e}^{x}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\right),$$

$$v_4(x,t) = e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right),$$

$$v_5(x,t) = e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}\right),$$

$$\vdots$$

Therefore,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = e^{2x+t}, \tag{49}$$

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = e^{x+t}, \tag{50}$$

which are the exact solutions.

Example 3. In the next example we assume: $x_1 = -1, x_2 = 1, \alpha_u = 2, \alpha_v = 3, k_u = 5, k_v = 7$ and

$$\varphi_u(x) = 5\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\sqrt{2}\sinh\left(\frac{x}{\sqrt{2}}\right), \tag{51}$$

$$\varphi_v(x) = 5\cosh\left(\frac{x}{\sqrt{3}}\right) + 5\sqrt{3}\sinh\left(\frac{x}{\sqrt{3}}\right), \tag{52}$$

$$\psi_u(t) = \frac{5+7\sqrt{2}}{2} e^{t-\frac{\sqrt{2}}{2}} + \frac{5-7\sqrt{2}}{2} e^{t+\frac{\sqrt{2}}{2}}, \tag{53}$$

$$\psi_v(t) = \frac{5}{2}(1-\sqrt{3})\, e^{t-\frac{\sqrt{3}}{3}} + \frac{5}{2}(1+\sqrt{3})\, e^{t+\frac{\sqrt{3}}{3}}. \tag{54}$$

Starting with initial approximations in the form:

$$u_0(x,t) = 5\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\sqrt{2}\sinh\left(\frac{x}{\sqrt{2}}\right), \tag{55}$$

$$v_0(x,t) = 5\cosh\left(\frac{x}{\sqrt{3}}\right) + 5\sqrt{3}\sinh\left(\frac{x}{\sqrt{3}}\right), \tag{56}$$

and using iterative formulas (23) and (24) (the first method) we can obtain the following results:

$$u_1(x,t) = \left(5\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\sqrt{2}\sinh\left(\frac{x}{\sqrt{2}}\right)\right)(1+t),$$

$$u_2(x,t) = \left(5\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\sqrt{2}\sinh\left(\frac{x}{\sqrt{2}}\right)\right)\left(1 + t + \frac{t^2}{2!}\right),$$

$$u_3(x,t) = \left(5\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\sqrt{2}\sinh\left(\frac{x}{\sqrt{2}}\right)\right)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right),$$

$$u_4(x,t) = \left(5\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\sqrt{2}\sinh\left(\frac{x}{\sqrt{2}}\right)\right)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right),$$

$$u_5(x,t) = \left(5\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\sqrt{2}\sinh\left(\frac{x}{\sqrt{2}}\right)\right)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}\right),$$

$$\vdots$$

and

$$v_1(x,t) = \left(5\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\sqrt{2}\sinh\left(\frac{x}{\sqrt{2}}\right)\right)(1+t),$$

$$v_2(x,t) = \left(5\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\sqrt{2}\sinh\left(\frac{x}{\sqrt{2}}\right)\right)\left(1 + t + \frac{t^2}{2!}\right),$$

$$v_3(x,t) = \left(5\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\sqrt{2}\sinh\left(\frac{x}{\sqrt{2}}\right)\right)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right),$$

$$v_4(x,t) = \left(5\,\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\,\sqrt{2}\,\sinh\left(\frac{x}{\sqrt{2}}\right)\right)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right),$$
$$v_5(x,t) = \left(5\,\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\,\sqrt{2}\,\sinh\left(\frac{x}{\sqrt{2}}\right)\right)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}\right),$$
$$\vdots$$

which leads to the exact solution

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = e^t\left(5\,\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\,\sqrt{2}\,\sinh\left(\frac{x}{\sqrt{2}}\right)\right), \tag{57}$$
$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = e^t\left(5\,\cosh\left(\frac{x}{\sqrt{2}}\right) + 7\,\sqrt{2}\,\sinh\left(\frac{x}{\sqrt{2}}\right)\right). \tag{58}$$

Example 4. In this example the assumed data are the same as in Example 1, but unlike previously, the second method described by iterative formulas (29) and (30) is used to arrive at a solution.

By considering conditions (33) and (34), the following initial approximations may be assumed:

$$u_0(x,t) = \frac{1}{3}\,e^{t-2}\left(1 + 2e^3 + 2\,x\,(e^3 - 1)\right), \tag{59}$$
$$v_0(x,t) = \frac{1}{3}\,e^{t-2}\left(1 + 2e^3 + x\,(e^3 - 1)\right). \tag{60}$$

Using iterative formulas (29) and (30) (the second method) we can obtain the following results:

$$u_1(x,t) = e^{t-2}\left(\frac{1}{3} - \frac{2\,x}{3} + \frac{2\,x^2}{3} - \frac{4\,x^3}{9}\right) + e^{t+1}\left(\frac{2}{3} + \frac{2\,x}{3} + \frac{4\,x^2}{3} + \frac{4\,x^3}{9}\right),$$
$$u_2(x,t) = e^{t-2}\left(\frac{1}{3} - \frac{2\,x}{3} + \frac{2\,x^2}{3} - \frac{4\,x^3}{9} + \frac{2\,x^4}{9} - \frac{4\,x^5}{45}\right) + e^{t+1}\left(\frac{2}{3} + \frac{2\,x}{3} + \frac{4\,x^2}{3} + \frac{4\,x^3}{9} + \frac{4\,x^4}{9} + \frac{4\,x^5}{45}\right),$$
$$\begin{aligned} u_3(x,t) &= e^{t-2}\left(\frac{1}{3} - \frac{2\,x}{3} + \frac{2\,x^2}{3} - \frac{4\,x^3}{9} + \frac{2\,x^4}{9} - \frac{4\,x^5}{45} + \frac{4\,x^6}{135} - \frac{8\,x^7}{945}\right) \\ &\quad + e^{t+1}\left(\frac{2}{3} + \frac{2\,x}{3} + \frac{4\,x^2}{3} + \frac{4\,x^3}{9} + \frac{4\,x^4}{9} + \frac{4\,x^5}{45} + \frac{8\,x^6}{135} + \frac{8\,x^7}{945}\right), \end{aligned}$$
$$\vdots$$

and

$$v_1(x,t) = e^{t-2}\left(\frac{1}{3} - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{18}\right) + e^{t+1}\left(\frac{2}{3} + \frac{x}{3} + \frac{x^2}{3} + \frac{x^3}{18}\right),$$
$$v_2(x,t) = e^{t-2}\left(\frac{1}{3} - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{18} + \frac{x^4}{72} - \frac{x^5}{360}\right) + e^{t+1}\left(\frac{2}{3} + \frac{x}{3} + \frac{x^2}{3} + \frac{x^3}{18} + \frac{x^4}{36} + \frac{x^5}{360}\right),$$
$$\begin{aligned} v_3(x,t) &= e^{t-2}\left(\frac{1}{3} - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{18} + \frac{x^4}{72} - \frac{x^5}{360} + \frac{x^6}{2160} - \frac{x^7}{15120}\right) \\ &\quad + e^{t+1}\left(\frac{2}{3} + \frac{x}{3} + \frac{x^2}{3} + \frac{x^3}{18} + \frac{x^4}{36} + \frac{x^5}{360} + \frac{x^6}{1080} + \frac{x^7}{15120}\right), \end{aligned}$$
$$\vdots.$$

The first sequence (component if e^{t-2}) contained in approximations $u_n(x,t)$ is convergent to following function:

$$f_{1u}(x) = \frac{1}{3}\left(\cosh(2\,x) - \sinh(2\,x)\right),$$

whereas the second sequence (component if e^{t+1}) is convergent to the following function:

$$f_{2u}(x) = \frac{1}{3}\left(2\,\cosh(2\,x) + \sinh(2\,x)\right).$$

The sequences in approximations $v_n(x,t)$ are correspondingly convergent to functions:

$$f_{1v}(x) = \frac{1}{3}\big(\cosh(x) - \sinh(x)\big),$$
$$f_{2v}(x) = \frac{1}{3}\big(2\,\cosh(x) - \sinh(x)\big).$$

Hence, the following exact solutions are derived after some transformations:

$$u(x,t) = \frac{1}{6}\,e^{t-2x-2}\big(3\,e^{4x+3} + e^3 + 2\big), \tag{61}$$
$$v(x,t) = \frac{1}{6}\,e^{t-2x-2}\big(3\,e^{2x+3} + e^3 + 2\big). \tag{62}$$

Example 5. In this example, the second method is used again, but this time, with the following input data: $x_1 = -1, x_2 = 1$, $\alpha_u = 1, \alpha_v = 1, k_u = 4, k_v = 3$ and

$$\varphi_u(x) = -\frac{3}{7}\,e^{-x}, \tag{63}$$
$$\varphi_v(x) = \frac{1}{14}\big(e^x - 7\,e^{-x}\big), \tag{64}$$
$$\psi_u(t) = \frac{3}{7}\,e^{t+1}, \tag{65}$$
$$\psi_v(t) = \frac{1}{14}\big(e^2 - 7\big)\,e^{t+1}. \tag{66}$$

By considering conditions (33) and (34), the following form of initial approximations may be assumed:

$$u_0(x,t) = \frac{3}{7}\,(x-1)\,e^t, \tag{67}$$
$$v_0(x,t) = \frac{1}{7}\,(4x-3)\,e^t. \tag{68}$$

Using iterative formulas (29) and (30) (the second method), we can obtain the following results:

$$u_1(x,t) = e^t\left(-\frac{3}{7} + \frac{3x}{7} - \frac{3x^2}{14} + \frac{x^3}{14}\right),$$
$$u_2(x,t) = e^t\left(-\frac{3}{7} + \frac{3x}{7} - \frac{3x^2}{14} + \frac{x^3}{14} - \frac{x^4}{56} + \frac{x^5}{280}\right),$$
$$u_3(x,t) = e^t\left(-\frac{3}{7} + \frac{3x}{7} - \frac{3x^2}{14} + \frac{x^3}{14} - \frac{x^4}{56} + \frac{x^5}{280} - \frac{x^6}{1680} + \frac{x^7}{11760}\right),$$
$$\vdots$$

and

$$v_1(x,t) = e^t\left(-\frac{3}{7} + \frac{4x}{7} - \frac{3x^2}{14} + \frac{2x^3}{21}\right),$$
$$v_2(x,t) = e^t\left(-\frac{3}{7} + \frac{4x}{7} - \frac{3x^2}{14} + \frac{2x^3}{21} - \frac{x^4}{56} + \frac{x^5}{210}\right),$$
$$v_3(x,t) = e^t\left(-\frac{3}{7} + \frac{4x}{7} - \frac{3x^2}{14} + \frac{2x^3}{21} - \frac{x^4}{56} + \frac{x^5}{210} - \frac{x^6}{1680} + \frac{x^7}{8820}\right),$$
$$\vdots.$$

The sequence contained in approximations $u_n(x,t)$ is convergent to function:

$$f_u(x) = \frac{3}{7}\big(\sinh(x) - \cosh(x)\big),$$

whereas the sequence contained in approximations $v_n(x, t)$ is convergent to function:

$$f_v(x) = \frac{1}{7}\left(4\,\sinh(x) - 3\,\cosh(x)\right).$$

Finally, after few simple transformations, the exact solution is derived:

$$u(x, t) = -\frac{3}{7}\,e^{t-x}, \tag{69}$$

$$v(x, t) = \frac{1}{14}\left(e^{t+x} - 7\,e^{t-x}\right). \tag{70}$$

6. Conclusions

In this paper, the solutions of the heat equation with the boundary condition of the fourth kind are presented. The proposed solutions are based on the variational iteration method. The calculations show that this method is effective for solving the problems under consideration.

Unlike classic methods based on the finite difference or final elements principles, the proposed method does not require the digitization of the domain. The variational iteration method renders a sequence of successive approximations, which is compatible with the exact solution, if such a solution exists.

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