Schur-convexity and the Simpson formula

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ABSTRACT

The main objective of this work is to give a necessary and sufficient condition for the function defined as the difference of the Simpson quadrature rule and the arithmetic integral mean to be Schur-convex.

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1. Introduction

One of the most famous quadrature rules in numerical integration is the Simpson formula. Namely, for a function $f : [a, b] \to \mathbb{R}$ such that $f^{(4)}$ is continuous on $[a, b]$, the following identity is valid:

$$\frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{6} \left( f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right) = -\frac{(b-a)^4}{2880} f^{(4)}(\xi),$$

where $\xi$ is some number between $a$ and $b$ (see for example [1]).

Throughout this work, let $I$ be a non-empty open interval in $\mathbb{R}$. The aim of this work is to establish a necessary and sufficient condition for the function $S : I^2 \to \mathbb{R}$ defined as

$$S(x, y) = \begin{cases} \frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y) - \frac{1}{y-x} \int_x^y f(t)dt, & x, y \in I, x \neq y \\ 0, & x = y \in I \end{cases}$$

(2)

to be Schur-convex. Let us recall the definition of Schur-convexity (see for example [2] or [3]).

Definition 1. Function $F : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex on $A$ if

$$F(x_1, x_2, \ldots, x_n) \leq F(y_1, y_2, \ldots, y_n)$$

for every $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in A$ such that $x < y$, i.e. such that

$$\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]} \quad \text{and} \quad \sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]} \quad \text{for} \quad k = 1, 2, \ldots, n - 1$$

where $x_{[i]}$ denotes the $i$th-largest component in $x$. Function $F$ is said to be Schur-concave on $A$ if $-F$ is Schur-convex.

Note that every convex and symmetric function is Schur-convex.

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Schur-convexity has aroused the interest of many researchers, and numerous papers have been devoted to it. For example, the following theorems were given in [4,5], respectively. Also, in [6] some related results were given.

**Theorem 1.** Suppose $f : I \rightarrow \mathbb{R}$ is a continuous function. Function

$$F(x, y) = \begin{cases} \frac{1}{y - x} \int_x^y f(t) dt, & x, y \in I, x \neq y \\ f(x), & x = y \in I \end{cases}$$

is Schur-convex (Schur-concave) on $I^2$ if and only if $f$ is convex (concave) on $I$.

**Theorem 2.** Suppose $f : I \rightarrow \mathbb{R}$ is a continuous function. Function

$$M(x, y) = \begin{cases} \frac{1}{y - x} \int_x^y f(t) dt - f \left(\frac{x + y}{2}\right), & x, y \in I, x \neq y \\ 0, & x = y \in I \end{cases}$$

is Schur-convex (Schur-concave) on $I^2$ if and only if $f$ is convex (concave) on $I$. Furthermore, function

$$T(x, y) = \begin{cases} \frac{f(x) + f(y)}{2} - \frac{1}{y - x} \int_x^y f(t) dt, & x, y \in I, x \neq y \\ 0, & x = y \in I \end{cases}$$

is Schur-convex (Schur-concave) on $I^2$ if and only if $f$ is convex (concave) on $I$.

These results provided the motivation for the investigation in this work. In order to prove our result, we shall need the following lemma which gives a useful characterization of Schur-convexity (see [2] or [3]).

**Lemma 1.** Let $f : I^n \rightarrow \mathbb{R}$ be a continuous symmetric function. If $f$ is differentiable on $I^n$, then $f$ is Schur-convex on $I^n$ if and only if

$$(x_i - x_j) \left( \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j} \right) \geq 0$$

for all $x_i, x_j \in I, i \neq j, i, j = 1, 2, \ldots, n$. Function $f$ is Schur-concave if and only if the reversed inequality sign holds.

Another result vital in our proof was derived in [7].

**Theorem 3.** If $f : [a, b] \rightarrow \mathbb{R}$ is such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a, b]$ for some $n \geq 1$, then

$$\int_a^b f(t) dt - \frac{b - a}{2} [f(a) + f(b)] + \sum_{k=1}^{(n-1)/2} \frac{(b - a)^{2k}}{(2k)!} B_{2k}[f^{(2k-1)}(b) - f^{(2k-1)}(a)]$$

$$= \frac{(b - a)^n}{n!} \int_a^b \left( B_n \left( \frac{b - t}{b - a} \right) - B_n \right) df^{(n-1)}(t),$$

where $B_n(x)$ are the Bernoulli polynomials and $B_n = B_n(0)$ Bernoulli numbers.

For details on Bernoulli polynomials, one can see for example [1]. One of their properties which is going to be needed is the following:

$$(-1)^k (B_{2k}(x) - B_{2k}) \geq 0 \quad \text{for } 0 \leq x \leq 1 \text{ and } k \in \mathbb{N}. \quad (4)$$

Finally, we recall the definition of $n$-convexity.

**Definition 2.** Function $f : [a, b] \rightarrow \mathbb{R}$ is said to be $n$-convex on $[a, b]$ for some $n \geq 0$ if for any choice of $n + 1$ mutually different points $x_0, \ldots, x_n \in [a, b]$, we have $[x_0, \ldots, x_n] f \geq 0$, where $[x_0, \ldots, x_n] f$ is the $n$th-order divided difference of $f$.

If $f^{(n)}$ exists, then $f$ is $n$-convex if and only if $f^{(n)} \geq 0$. For more details see for example [3].
2. The main result

**Theorem 4.** If \( f \in C^4(I) \) then the following statements are equivalent:

(a) The function \( S \) defined by (2) is Schur-convex on \( I^2 \).

(b) For all \( x, y \in I, x < y \), we have

\[
\frac{1}{y-x} \int_x^y f(t) \, dt \leq \frac{1}{6} f(x) + \frac{2}{3} f \left( \frac{x+y}{2} \right) + \frac{1}{6} f(y).
\]

(c) The function \( f \) is 4-convex on \( I \).

**Proof.** First, we prove (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c). Assume \( S \) is Schur-convex on \( I^2 \). Since \( \left( \frac{x+y}{2} \right) \prec (x, y) \), we have

\[
0 \leq \frac{1}{6} f(x) + \frac{2}{3} f \left( \frac{x+y}{2} \right) + \frac{1}{6} f(y) - \frac{1}{y-x} \int_x^y f(t) \, dt,
\]

for all \( x, y \in I, x \neq y \), so (b) is valid. Now, applying (1) implies

\[
0 \leq \frac{(y-x)^4}{2880} f^{(4)}(\xi), \quad \xi \in (x, y)
\]

and so, since \( f \in C^4(I) \) and \( x \) and \( y \) are arbitrary, we conclude that \( f \) is 4-convex.

Note that the implication (c) \( \Rightarrow \) (b) also follows immediately from (1).

To prove (c) \( \Rightarrow \) (a), assume \( f \) is 4-convex. Simple calculation gives

\[
(y-x) \left( \frac{\partial S}{\partial y} - \frac{\partial S}{\partial x} \right) = \frac{2}{y-x} \int_x^y f(t) \, dt - [f(y) + f(x)] + \frac{y-x}{6} [f'(y) - f'(x)].
\]

For \( n = 4 \), identity (3) yields

\[
\frac{1}{b-a} \int_a^b f(t) \, dt = \frac{f(a) + f(b)}{2} - \frac{b-a}{12} [f'(b) - f'(a)]
\]

\[
+ \frac{(b-a)^3}{4!} \int_a^b \left( B_4 \left( \frac{b-t}{b-a} \right) - B_4 \right) f^{(4)}(t) \, dt,
\]

so we deduce

\[
(y-x) \left( \frac{\partial S}{\partial y} - \frac{\partial S}{\partial x} \right) = \frac{(y-x)^3}{12} \int_x^y \left( B_4 \left( \frac{y-t}{y-x} \right) - B_4 \right) f^{(4)}(t) \, dt.
\]

Now, applying (4) with \( k = 2 \) and Lemma 1 with \( n = 2 \), we reach the conclusion that \( S \) is Schur-convex.

Thus, the proof is complete. \( \square \)

**Remark 1.** Results for a Schur-concave function \( S \) and a 4-concave function \( f \) follow easily from Theorem 4 for \(-S\) and \(-f\).

**Remark 2.** Since \( n \)-convex functions are continuous, they can be represented as a uniform limit of a sequence of the corresponding Bernstein polynomials (see for example [3]). The Bernstein polynomials of \( n \)-convex functions are also \( n \)-convex. Also, if the corresponding Bernstein polynomials are \( n \)-convex, so is the function \( f \). Having this in mind, the implications (c) \( \Rightarrow \) (a) \( \Rightarrow \) (b) in Theorem 4 can be proved without the assumption that \( f \in C^4(I) \). The conjecture is that implications (a) \( \Rightarrow \) (c) and (b) \( \Rightarrow \) (c) also remain valid without the regularity condition \( f \in C^4(I) \).

**Remark 3.** Note that for \( n = 2 \) and \( n = 3 \), identity (3) gives

\[
(y-x) \left( \frac{\partial S}{\partial y} - \frac{\partial S}{\partial x} \right) = \frac{2}{y-x} \int_x^y f(t) \, dt - [f(x) + f(y)] + \frac{y-x}{6} [f'(y) - f'(x)]
\]

\[
= (y-x) \int_x^y B_2 \left( \frac{y-t}{y-x} \right) f''(t) \, dt
\]

\[
= \frac{(y-x)^2}{3} \int_x^y B_3 \left( \frac{y-t}{y-x} \right) f'''(t) \, dt,
\]

since \( B_2 = 1/6 \) and \( B_3 = 0 \). Furthermore, since \( B_{n+1}(1) = (n+1)B_n(x) \), \( n \geq 1 \) and \( B_n(1) = B_n(0) \), \( n \geq 2 \) (see [1]), we have

\[
\int_x^y B_n \left( \frac{y-t}{y-x} \right) dt = \frac{y-x}{n+1} [B_{n+1}(1) - B_{n+1}(0)] = 0,
\]

so \( f \) being a convex function is not a sufficient condition for \( S \) to be Schur-convex. The same holds if \( f \) is a 3-convex function.
**Remark 4.** Consider function $G : I^2 \rightarrow \mathbb{R}$ defined as

$$G(x, y) = \frac{1}{6} f(x) + \frac{2}{3} f\left(\frac{x + y}{2}\right) + \frac{1}{6} f(y).$$

Since

$$(y - x) \left( \frac{\partial G}{\partial y} - \frac{\partial G}{\partial x} \right) = \frac{y - x}{6} [f'(y) - f'(x)],$$

by Lemma 1, $G$ is Schur-convex if and only if $f$ is convex.

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