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On a class of nonlocal elliptic problems via Galerkin method ☆

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Abstract

In this paper we study existence and uniqueness of solutions to some cases of the following nonlocal elliptic problem:

$$-\Delta u = \frac{(g(x,u))^\alpha}{(\int_\Omega f(x,u))^\beta}$$

with zero Dirichlet boundary conditions on a bounded and smooth domain of $\mathbb{R}^N$ and also when $\Omega = \mathbb{R}^N$, where α and β are real constants.

Keywords: Nonlocal elliptic problems; Galerkin method

1. Introduction

In this paper we will study some questions related to the existence and uniqueness of solutions of the nonlocal elliptic problem

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$$-\Delta u = \frac{(g(x,u))^\alpha}{(\int_\Omega f(x,u))^\beta} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where α, β are real constants, $f, g : \Omega \times [0, \infty) \to [0, \infty)$ are Caratheodory functions, whose properties shall be timely introduced and either, in different results, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain or $\Omega = \mathbb{R}^N$. For the last case, boundary conditions are not considered.

As far as applications are concerned, the following class of equations:

$$-\Delta u = \frac{(f(u))^\alpha}{(\int_\Omega f(u))^\beta} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

arises in numerous physical models such as: systems of particles in thermodynamical equilibrium via gravitational (Coulomb) potential, $2 - D$ fully turbulent behavior of real flow, thermal runaway in Ohmic Heating, shear bands in metal deformed under high strain rates, among others. References to these applications may be found in [1].

Mathematically, the presence of the nonlocal term $(\int_\Omega f(x,u))^\beta$ in Eq. (1.1) poses interesting questions and rises some outstanding difficulties in some standard methods for attacking elliptic problems. For instance, variational methods do not work when applied to prove existence results for a large class of these equations.

This kind of problem has been investigated by several authors including Carrillo [5], Tzanetis–Vlamos [6], Stańczy [1], among others. In particular, for $N = 1$, Eq. (1.2) becomes an ordinary equation and it was studied by Stańczy in [1]. In that paper, the author uses theory of fixed point and Green function to prove existence of solutions under the hypotheses that the function f is positive and nondecreasing (hence, bounded from below). The author also stresses that his method works when Ω is an annulus, but cannot be employed when Ω is a ball in $\mathbb{R}^N$, $N \geqslant 3$, even in the radial case.

In this paper we study some classes of Eqs. (1.1) on both bounded and unbounded domains, we deal with (1.2) relaxing the above restrictions and have made substantial improvements in the study of the problem. In part, this is possible thanks to a device explored by Alves–de Figueiredo [8], in [7] and also in [3], which uses Galerkin method to attack a nonvariational elliptic system. We conveniently adapt this technique to our case. Furthermore, we also study the uniqueness question for the equation. In the case $\Omega = \mathbb{R}^N$ we were inspired by Brezis–Kamin [4].

More references on nonlocal elliptic problems may be found in [9–12] among others.

The method we use depends on the following result whose proof may be F in Lions [2, p. 53] and it is a well-known variant of Brouwer's Fixed Point Theorem.

Proposition 1.1. *Suppose that $F : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function such that $\langle F(\xi), \xi \rangle \geqslant 0$ on $|\xi| = r$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^m$ and $|\cdot|$ its related norm. Then, there exists $z_0 \in \overline{B_r}(0)$ such that $F(z_0) = 0$.*

We remark that by a solution of (1.2) we mean a *weak solution*, that is, a function $u \in H_0^1(\Omega)$ such that

$$\left(\int_\Omega f(x,u) \right)^\beta \int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega \left(g(x,u)\right)^\alpha \varphi \quad \text{for all } \varphi \in H_0^1(\Omega).$$

We point out that, depending on the regularity of $f(\cdot,u)$ and $g(\cdot,u)$, a bootstrap argument may be used to show that a weak solution is a classical solution, i.e., a function in $C_0^2(\bar{\Omega})$.

This paper is organized as follows: Section 2 is devoted to the study of problem (1.1) in bounded domains. In Section 3 we state and prove results of existence and uniqueness for (1.1) in bounded domains for $g(x,u) = u^\alpha$ and Ω bounded. Finally, in Section 4, we study a class of this problem for $\Omega = \mathbb{R}^N$.

2. Existence results for bounded Ω

In this section we state and prove the main theorems of existence of solutions to (1.1) when Ω is bounded.

For simplicity, these theorems are stated for $f(x,u) = g(x,u) = f(u)$. We remark that the general case follows if we assume convenient and similar hypotheses on f and g that shall appear in the proofs below. Writing them down would just enlarge the enunciation of the these theorems.

The first theorem deals with a positive bounded function f, and the second one, the function f may vanish and has not to be bounded from below.

Before starting, in what follows, we are going to consider the extension $f(t) = f(0)$, $t < 0$, of the function f to the whole $\mathbb{R}$ and denote it by the same letter f.

Theorem 2.1. *Suppose that*

$$f(t) \geqslant k_0 > 0, \quad \forall t \in [0,\infty), \tag{2.1}$$

$$f(t) < k_\infty, \quad \forall t \in [0,\infty), \tag{2.2}$$

where k_0 and k_∞ are real constants.

Then for any real α and β, problem (1.2) *possesses a positive weak solution.*

Proof. The proof is based on the Galerkin method which works as follows. Let $\{\varphi_1, \dots, \varphi_m, \dots\}$ be an orthonormal basis of the Hilbert space $H_0^1(\Omega)$ endowed, respectively, with the inner product and norm

$$\langle\langle u, v\rangle\rangle = \int_\Omega \nabla u \nabla v, \qquad \| u\|^2 = \int_\Omega |\nabla u|^2.$$

For each fixed $m \in \mathbb{N}$ consider the finite-dimensional Hilbert space

$$\mathbb{V}_m = \operatorname{span}\{\varphi_1, \dots, \varphi_m\}.$$

Since $(\mathbb{V}_m, \|\cdot\|)$ and $(\mathbb{R}^m, |\cdot|)$ are isometric and isomorphic (here, the Euclidean norm in $\mathbb{R}^m$ is $|\cdot|$ and $\langle\cdot,\cdot\rangle$ is its corresponding usual inner product), we shall make the identification

$$u = \sum_{j=1}^m \xi_j \varphi_j \quad \longleftrightarrow \quad \xi = (\xi_1, \dots, \xi_m), \quad \|u\| = |\xi|.$$

Case 1. $\alpha, \beta \geqslant 0$.

Let us rewrite problem (1.2) as

$$-\left(\int_\Omega f(u)\right)^\beta \Delta u = \big(f(u)\big)^\alpha \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

and consider the function $F : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$F(\xi) = \big(F_1(\xi), \ldots, F_m(\xi)\big),$$
$$F_i(\xi) = \left(\int_\Omega f(u)\right)^\beta \xi_i - \int_\Omega \big(f(u)\big)^\alpha \varphi_i,$$

where $i = 1, \ldots, m$ and $u = \sum_{j=1}^m \xi_j \varphi_j$.

So that with the above identifications one has

$$F_i(\xi) = \left(\int_\Omega f(u)\right)^\beta \int_\Omega \nabla u \cdot \nabla \varphi_i - \int_\Omega \big(f(u)\big)^\alpha \varphi_i$$

and

$$\big\langle F(\xi), \xi \big\rangle = \left(\int_\Omega f(u)\right)^\beta \|u\|^2 - \int_\Omega \big(f(u)\big)^\alpha u.$$

Using (2.1), (2.2), Hölder and Poincaré inequalities, we get

$$\big\langle F(\xi), \xi \big\rangle \geqslant k_0^\beta |\Omega|^\beta \|u\|^2 - C k_\infty^\alpha \|u\| > 0,$$

if $\|u\| = r$, for r large enough, independently of m, where $|\Omega|$ is the Lebesgue measure of the set Ω.

Thus, by Proposition 1.1, there is $u_m \in \mathbb{V}_m$, $\|u_m\| \leqslant r$ such that

$$\left(\int_\Omega f(u_m)\right)^\beta \int_\Omega \nabla u \cdot \nabla \varphi_i = \int_\Omega \big(f(u_m)\big)^\alpha \varphi_i, \quad i = 1, \ldots, m,$$

which implies that

$$\left(\int_\Omega f(u_m)\right)^\beta \int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega \big(f(u_m)\big)^\alpha \varphi, \quad \forall \varphi \in H_0^1(\Omega). \tag{2.3}$$

Let us prove that the sequence $(u_m) \subset H_0^1(\Omega)$ has a convergent subsequence which converges to a solution of (1.2). Indeed, since (u_m) is a bounded there exists a subsequence, still denoted by (u_m), such that

$$u_m \rightharpoonup u \quad \text{in } H_0^1(\Omega), \qquad u_m \to u \quad \text{in } L^2(\Omega) \quad \text{and}$$
$$u_m(x) \to u(x) \quad \text{a.e. in } \Omega. \tag{2.4}$$

Hence, using (2.4) and passing to the limit in Eq. (2.3), we get that u is a weak solution of (1.2). The Maximum Principle assures that $u > 0$.

Case 2. $\alpha > 0$, $\beta = -\gamma < 0$.

In this case, problem (1.2) becomes

$$-\Delta u = \big(f(u)\big)^{\alpha} \bigg(\int_{\Omega} f(u) \bigg)^{\gamma} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

The function F is defined as

$$F_i(\xi) = \xi_i - \bigg(\int_{\Omega} f(u) \bigg)^{\gamma} \int_{\Omega} \big(f(u)\big)^{\alpha} \varphi_i$$

and

$$\big\langle F(\xi), \xi \big\rangle = \|u\|^2 - \bigg(\int_{\Omega} f(u) \bigg)^{\gamma} \int_{\Omega} \big(f(u)\big)^{\alpha} u.$$

Again, using (2.1), (2.2), Hölder and Poincaré inequalities, we get

$$\big\langle F(\xi), \xi \big\rangle \geqslant \|u\|^2 - C\|u\| > 0,$$

if $\|u\| = r$, for r large enough, where

$$C = k_{\infty}^{\alpha+\beta} |\Omega|^{\gamma+\frac{1}{2}} \frac{1}{\sqrt{\lambda_1}} > 0.$$

The rest proof follows the previous case and since $f(0) > 0$, we may guarantee a positive solution for (2.5).

Other cases. The proof is similar to the previous cases. $\square$

Remark 2.2. In the first case of Theorem 2.1, with similar proof, assertion (2.1) may be replaced by a more general hypothesis:

$$f(t) \leqslant a|t|^{\sigma} + b,$$

where a and b are real constants, and $\sigma < \frac{1}{\alpha}$.

Now let us state and prove our second theorem.

Theorem 2.3. *Suppose that*

$$0 < \beta < \alpha < \frac{N+2}{2N} < 1, \tag{2.6}$$

$$f(0) > 0 \tag{2.7}$$

and that (2.2) *holds.*

Then problem (1.2) *possesses a positive weak solution.*

Proof. Let us suppose that the hypothesis of Proposition 1.1 does not hold. Then, for a fixed m and for each $r > 0$, there is $u_r \in \mathbb{V}_m$ such that

$$\langle F(u_r), u_r \rangle < 0, \qquad \|u_r\| = r,$$

which implies that

$$\left(\int_\Omega f(u_r)\right)^\beta \int_\Omega \nabla u_r \cdot \nabla \varphi < \int_\Omega \big(f(u_r)\big)^\alpha \varphi, \quad \forall \varphi \in \mathbb{V}_m. \tag{2.8}$$

On the other hand, by Hölder inequality and (2.6), we have

$$\int_\Omega \big(f(u_r)\big)^\alpha u_r \leqslant \left(\int_\Omega f(u_r)\right)^\alpha \|u_r\|_{\frac{1}{1-\alpha}}. \tag{2.9}$$

Therefore, taking $\varphi = u_r$ in (2.8), using (2.9) and Sobolev embeddings, we deduce

$$\left(\int_\Omega f(u_r)\right)^\beta \|u_r\|^2 < C\left(\int_\Omega f(u_r)\right)^\alpha \|u_r\|$$

and consequently, by (2.2) and (2.6),

$$r < C\left(\int_\Omega f(u_r)\right)^{\alpha-\beta} \leqslant C_1$$

for all $r > 0$, which is a contradiction.

Thus, for each m there is $r > 0$ such that

$$\langle F(\xi), \xi \rangle \geqslant 0, \qquad \|u_r\| = r, \quad \xi \in \mathbb{R}^m.$$

By Proposition 1.1, there is $u_m \in \mathbb{V}_m$ such that

$$\left(\int_\Omega f(u_m)\right)^\beta \int_\Omega \nabla u_m \cdot \nabla \varphi = \int_\Omega \big(f(u_m)\big)^\alpha \varphi, \quad \forall \varphi \in \mathbb{V}_m. \tag{2.10}$$

In this case, observe that, a priori, nothing indicates that this r does not depends on m. Nevertheless, we are going to prove that this really occurs.

Picking $\varphi = u_m$ in (2.10) and proceeding as before,

$$\left(\int_\Omega f(u_m)\right)^\beta \|u_m\|^2 = C\left(\int_\Omega f(u_m)\right)^\alpha u_m \leqslant \left(\int_\Omega f(u_m)\right)^\alpha \|u_m\|_{\frac{1}{1-\alpha}}$$

and, as previously, it is forward to conclude that $\|u_m\| \leqslant C$, as desired.

The rest of the proof follows the lines of the proof of Theorem 1. Since (2.7) holds, we have that $u \geqslant 0$, $u \not\equiv 0$ and by the Maximum Principle, $u > 0$ in Ω. $\square$

3. The case $g(x,u) = \rho(x)(u_+)^\alpha$ and Ω bounded

In this section, for $\rho \in C(\bar{\Omega})$, $\rho \geqslant 0$, $\rho \not\equiv 0$, $0 < \alpha < 1$ and Ω a bounded smooth domain, we are going to consider problem (1.1) with $g(x,u) = \rho(x)(u_+)^\alpha$ and study existence and uniqueness of the problem

$$-\left(\int_\Omega f(u)\right)^\beta \Delta u = \rho(x)(u_+)^\alpha \quad \text{in } \Omega,$$
$$u > 0 \quad \text{on } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

where $u_+(x) = \max\{u(x), 0\}$.

The central theorem of this section is

Theorem 3.1. *Suppose that*

$$0 < \alpha < 1, \tag{3.2}$$

$$f \text{ is an increasing function for } t \in [0,\infty] \tag{3.3}$$

and that (2.1) *holds.*

Then problem (3.1) *has a unique solution.*

Proof.

Existence of solution: For this theorem,

$$F_i(\xi) = \left(\int_\Omega f(u)\right)^\beta \xi_i - \int_\Omega \rho(x)(u_+)^\alpha \varphi_i$$

and

$$\langle F(\xi), \xi \rangle = \left(\int_\Omega f(u)\right)^\beta \|u\|^2 - \int_\Omega \rho(x)(u_+)^\alpha u.$$

Hypothesis (3.3) assure the embedding $H_0^1(\Omega) \subset L^{\alpha+1}(\Omega)$ and the estimate $\int_\Omega \rho |u_+|^{\alpha+1} \leqslant C\|\rho\|_\infty \|u\|^{\alpha+1}$.

The rest of the proof is similar to the previous ones.

Uniqueness of solution: Let us suppose that u_1 and u_2 are solutions of Eq. (3.1). After an algebraic manipulation it is readily seen that

$$-\Delta\left(\left(\int_\Omega f(u_i)\right)^{\frac{\beta}{1-\alpha}} u_i\right) = \rho(x)\left(\left(\int_\Omega f(u_i)\right)^{\frac{\beta}{1-\alpha}} u_i\right)^\alpha \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

In this way, both functions $U_i = (\int_\Omega f(u_i))^{\frac{\beta}{1-\alpha}} u_i$ are positive solution of the equation

$$-\Delta U = \rho(x) U^\alpha \quad \text{in } \Omega, \qquad U = 0 \quad \text{on } \partial\Omega,$$

which, by [4] has a unique positive solution.

Therefore, we have

$$\left(\int_\Omega f(u_1)\right)^{\frac{\beta}{1-\alpha}} u_1(x) = \left(\int_\Omega f(u_2)\right)^{\frac{\beta}{1-\alpha}} u_2(x), \quad \forall x \in \Omega. \tag{3.4}$$

If $u_1(x_0) = u_2(x_0)$ for some $x_0 \in \Omega$, Eq. (3.4) yields that

$$\left(\int_\Omega f(u_1)\right)^{\frac{\beta}{1-\alpha}} = \left(\int_\Omega f(u_2)\right)^{\frac{\beta}{1-\alpha}}$$

and consequently $u_1 \equiv u_2$.

If $u_1 \not\equiv u_2$ in Ω, by the afore reasoning, $u_1(x) < u_2(x)$ or $u_1(x) > u_2(x)$, $\forall x \in \Omega$.

But this fact with (3.3) contradicts (3.4). Hence, problem (3.1) has a unique solution. $\square$

4. The case $g(x,u) = u^\alpha$, $f(x,u) = u$ and $\Omega = \mathbb{R}^N$

This section shall be devoted to the study the following problem on the whole $\mathbb{R}^N$:

$$-\Delta u = \rho(x)\left(\int_{\mathbb{R}^N} u\right)^\beta u^\alpha \quad \text{in } \mathbb{R}^N, \qquad u > 0 \quad \text{in } \mathbb{R}^N. \tag{4.1}$$

We prove an existence result for this problem by using a device due to Brezis–Kamin [4].

For that purpose it is important to study the problem in bounded domains. More precisely, we shall consider the problem

$$-\Delta u = \rho(x)\left(\int_\Omega u\right)^\beta u^\alpha \quad \text{in } \Omega, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{4.2}$$

Concerning the above problem, we have the following theorem whose proof is similar to the proof of Theorem 3.1.

Theorem 4.1. *If $\rho \in L^\infty(\Omega)$, $\rho \geqslant 0$, $\rho \not\equiv 0$, and α, β are real constants satisfying $\beta \geqslant 0$, $\alpha > 0$, $\alpha + \beta < 1$, then problem* (4.2) *possesses only a solution.*

Let us go back to the global problem (4.1). We begin with a definition.

Definition. We say that the function $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, $\rho \geqslant 0$, $\rho \not\equiv 0$, satisfies condition (H_1) if the problem

$$-\Delta u = \rho(x) \quad \text{in } \mathbb{R}^N \tag{4.3}$$

possesses a bounded and $L^1(\mathbb{R}^N)$ positive solution.

Theorem 4.2. *Problem* (4.1) *possesses a bounded and* $L^1(\mathbb{R}^N)$ *solution if the function* ρ *satisfies property* (H_1).

A partial converse holds: *if* (4.1) *possesses a bounded and* $L^1(\mathbb{R}^N)$ *solution, then* (4.3) *has a bounded solution.*

Proof. Suppose that ρ satisfies property (H_1). By Theorem 4.1, for each $R > 0$ let $u_R > 0$ be the only positive solution of

$$-\Delta u_R = \rho(x)\Bigl(\int_{B_R} u_R\Bigr)^{\beta} u_R^{\alpha} \quad \text{in } B_R, \qquad u_R > 0 \quad \text{in } B_R \quad \text{and}$$
$$u_R = 0 \quad \text{on } \partial B_R. \tag{4.4}$$

Let us fix R for a moment. For each $R' < R$ one has

$$-\Delta u_{R'} = \rho(x)\Bigl(\int_{B_{R'}} u_{R'}\Bigr)^{\beta} u_{R'}^{\alpha} \quad \text{in } \Omega, \qquad u_{R'} > 0 \quad \text{in } B_{R'} \quad \text{and}$$
$$u_{R'} = 0 \quad \text{on } \partial B_{R'},$$

which implies

$$-\Delta u_{R'} \geqslant \rho(x)\Bigl(\int_{B_R} u_{R'}\Bigr)^{\beta} u_{R'}^{\alpha} \quad \text{in } B_R, \qquad u_{R'} > 0 \quad \text{in } B_R \quad \text{and}$$
$$u_{R'} > 0 \quad \text{on } \partial B_R$$

and so $\bar{u} = u_{R'}$ is a supersolution of the problem (4.4).

Now let us construct a subsolution to problem (4.4). For, take $\varepsilon > 0$, to be chosen later and let λ_1 be the first eigenvalue and $\varphi_1 > 0$ the positive eigenfunction associated to the problem

$$-\Delta\omega = \lambda\rho(x)\omega \quad \text{in } B_R, \qquad \omega = 0 \quad \text{on } \partial B_R.$$

If we take φ_1 normalized as $\int_{B_R} \varphi_1 = 1$, standard calculations show that

$$-\Delta(\varepsilon\varphi_1) \leqslant \rho(x)\Bigl(\int_{B_R} \varepsilon\varphi_1\Bigr)^{\beta} (\varepsilon\varphi_1)^{\alpha} \quad \text{in } B_R, \qquad \varepsilon\varphi_1 = 0 \quad \text{on } \partial B_R,$$

for small ε and it is standard that taking ε positive and small enough one gets $\varepsilon\varphi_1 < u_{R'}$ in B_R. This shows that $\underline{u} = \varepsilon\varphi_1$ is a subsolution of problem (4.4) satisfying

$$\underline{u} = \varepsilon\varphi_1 < \bar{u} = u_{R'}.$$

Thus, since the solutions of (4.4) are between $\underline{u}$ and $\bar{u}$, the only solution u_R of (4.4) must satisfy

$$\underline{u} = \varepsilon\varphi_1 \leqslant u_R \leqslant \bar{u} = u_{R'} \quad \text{in } B_R,$$

that is, u_r increases with respect to R as $R \to \infty$.

Despite the above facts, we have to show that u_R remains bounded. For, let U be a bounded and $L^1(\mathbb{R}^N)$ solution of

$$-\Delta U = \rho(x) \quad \text{in } \mathbb{R}^N.$$

Let $K > 0$ so that

$$K^{1-(\alpha+\beta)} \geqslant \left(\int_{R^N} U \right)^{\beta} \|U\|_\infty^\alpha.$$

This is possible because the right-hand side of the above inequality is finite. For $R > 0$, after some calculations we get

$$-\Delta(KU) \geqslant \rho(x) \left(\int_{B_R} KU \right)^{\beta} (KU)^\alpha \quad \text{in } B_R, \qquad KU > 0 \quad \text{on } \partial B_R.$$

Hence KU is a supersolution of (4.4) and so

$$u_R \leqslant KU \quad \text{in } B_R,$$

for all $R > 0$ and KU does not depend on R. Let $u(x) = \lim_{R\to\infty} u_R(x)$ and since

$$-\Delta u_R = \rho(x) \left(\int_{B_R} u_R \right)^{\beta} u_R^\alpha \quad \text{in } B_R,$$

one gets $u(x) \leqslant KU(x)$ for all $x \in \mathbb{R}^N$ and so, passing to the limit in the last equation when $R \to \infty$, we conclude that

$$-\Delta u = \rho(x) \left(\int_{R^N} u \right)^{\beta} u^\alpha \quad \text{in } \mathbb{R}^N.$$

Then u is a bounded and $L^1(\mathbb{R}^N)$ solution of (4.1) which completes the first part of the proof.

To prove the other part of the theorem, suppose that u is a bounded and $L^1(\mathbb{R}^N)$ solution of (4.1), that is,

$$-\Delta u = \rho(x) \left(\int_{\mathbb{R}^N} u \right)^{\beta} u^\alpha \quad \text{in } \mathbb{R}^N.$$

Set

$$v = \frac{1}{1-\alpha} \left(\int_{\mathbb{R}^N} u \right)^{-\beta} u^{1-\alpha},$$

which implies, after some standard calculations, that

$$-\Delta v = \alpha \left(\int_{\mathbb{R}^N} u \right)^{-\beta} u^{-1-\alpha} |\nabla u|^2 + \rho(x) \geqslant \rho(x).$$

So, the solution ω_R of the problem

$$-\Delta\omega_R = \rho(x) \quad \text{in } B_R, \qquad \omega_R = 0 \quad \text{on } \partial B_R,$$

satisfies $\omega_R \leqslant v$ and so $\omega_R \to \omega$, as $R \to \infty$, where ω is a bounded and $L^1(\mathbb{R}^N)$ solution of

$$-\Delta u = \rho(x) \quad \text{in } \mathbb{R}^N,$$

which completes the proof of the theorem. □

Remark. Condition (H_1) is satisfied, for example, if and only if the convolution $\frac{c}{|x|^{N-2}} * \rho \in L^\infty(\mathbb{R}^N)$, for some real constant c. See [4].

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